MATHEMATICS

A Textbook for Grade 10

M10TB

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Foreword

Liberia, having gone through a period of utmost turmoil till 2003, due to the civil wars, is still reeling under its effect and the added trauma of Ebola in 2014 and effects of the COVID-19 outbreak in 2020. The Liberian government, in the past decade, has made valiant efforts to bring order to the lives of its people. In one such effort, the Ministry of Education (MoE) brought changes to the National Curriculum Framework which are relevant to the present generation, and which would prepare them to meet the challenges of the changing trends of the world. The National Curriculum Framework (NCF) 2018 recommends a change in basic assumptions in the teaching learning process from behaviorist to constructivist approach — moving from hardcore print material to the digital world. Keeping in consideration the sociocultural context and varied experiences of learners as laid down in the Framework, our Teaching Learning Materials are expected to be competent to use multiple methods and techniques like e-learning resources, energized textbooks, and readily available reference material to engage the learners.

As a first initiative, the MoE, through its World Bank-funded Improving Results in Secondary Education (IRISE) project, has adapted textbooks for Grades 10 to 12 in five subjects — English Language and Literature, Mathematics, Biology, Physics and Chemistry.

The National Curriculum Framework, 2018, recommends that children's learning at school is a reflection of their life outside the school and shows them the path to become a responsible citizen who makes knowledge-based choices.This principle marks a departure from the legacy of teacher centered learning to student centered learning. The syllabi and textbooks developed on the basis of the NCF indicate a serious attempt to implement the idea of Activity Base Learning (ABL). We hope these measures will take us ahead in the direction of building a system of education as outlined in the NCF.

Combined with the efforts by the school principals and teachers this will encourage children to reflect on their own learning and to pursue imaginative activities and questions. With this in mind, perhaps for the first time in our country, we are able to provide separate subject specific textbooks accompanied with guides for teachers for 10–12 grades. Not only have these been developed, adapted and modified to the Liberian context, each of the eight Minimum Learning Competencies (MLCs) have been included in each textbook. So as to reach every high school student, for the first time in the country's history we have included the digitized form of the textbook accessible by a Quick Response (QR) code given in each book. Not only does it have the digitized textbook, but it provides additional learning materials for use by students, teachers and interested persons. The links to these e-resources and digitized material is being made available on the MoE's website.

The Textbooks and Teacher Guides have reached the hands of the students after a rigorous quality evaluation by carefully handpicked subject specialists by the MoE, to whom the Ministry expresses gratitude. For the success of this project, I acknowledge the contributions of the IRISE Project Team in the World Bank, and in particular, the Task-Team Leaders; the Project Implementation Team in Liberia headed by its Coordinator Abraham A. Kiazolu II, supported by the Executive Director of the Center of Excellence for Curriculum Development and Textbooks Research, Mrs. Julia K. Sandiman-Gbeyai and her technical working group (TWG), and the International Textbook Consultant and Advisor, Dr Shveta Uppal engaged by the MoE. These notwithstanding would not have been possible without the guidance of the Senior Management Team (SMT) of the Ministry of Education, and in particular, the Deputy Ministers for Instructions, Administration, and Planning, Research and Development, respectively.

Professor Dao Ansu Sonii, Sr.
Minister of Education
Republic of Liberia

Monrovia, Republic of Liberia
January 24, 2023

Acknowledgments

The development of textbooks contributes to the quality of teaching and learning that go on in the classroom.

The Ministry of Education (MoE) has aligned its Curriculum for Grades 10–12 to the National Curriculum Framework (NCF) of 2018. To ensure the provision of Teaching Learning Materials (TLMs) that support the revised curriculum, the Ministry has sought, reviewed and adapted a new set of textbooks and teacher guides along with digitized contents and e-learning resources for the five core subjects taught at the Senior Secondary education level, namely English Language and Literature, Mathematics, Biology, Chemistry and Physics, through an internationally competitive bidding process from the market supported by the World Bank funded Improving Results in Secondary Education (IRISE) Project.

With profound gratitude and honor, we recognize the Senior Management Team of the Ministry, headed by the Coach, Professor D. Ansu Sonii, Sr., for the strategic decision to make teaching learning materials available and accessible to all in the Liberian Senior Secondary School System, and for providing directions through the process of securing these textbooks and other teaching learning materials for our students and teachers. Our special thanks and appreciation to the World Bank for the financial support towards this policy intervention, and its education task-team including Alonso Sanchez, Oni Lusk-Stover and Binta B. Massaquoi for all their technical inputs offered throughout the process to ensure the kind of quality TLMs the Liberian students deserve are made available for improved learning outcomes.

We would like to specifically recognize the invaluable contributions of the 15 subject experts selected by the MoE from across the various education systems and the West African Examinations Council (WAEC) to evaluate, review and sign off on these teaching learning materials. They didn’t just deliver according to our expectations, but also ensured the contextual relevance of the materials to the Liberian Secondary Education Curriculum and its minimum learning competencies (MLCs). These subject experts include Professor Isaac Saye-Lakpoh Zawolo – *Superintendent* of the Monrovia Consolidated School System (MCSS), Mr. Matthew V.Z. Darblo, Sr. – *Mathematics Instructor* at the University of Liberia (UL), Mr. Charles Tieh Bropleh – *Mathematics Specialist* (MoE), Mrs. Linda Y. Dean – *English Specialist*, Mr. Hassan M. Bangura – *English Language and Literature Expert*, Mr. J. Emmanuel Milton – *English Specialist* (MoE), Mr. Moses K.M. Togbah – *Physics Specialist*, Mr. Prince A. Dossen – *Physics Specialist*, Mr. Benjamin Koryah – *Physics Instructor* at the University of Liberia (UL), Mr. Dominic Dugbe Doe – *Chemistry Specialist*, Mr. Patrick A. Anderson, Sr. – *Director o*f the Division of Technical and Vocational Education (MoE), Mr. Kandakai Massaquoi – *Chemistry Specialist*, Ms. Patricia N. Doe – *Head* of Biology Department, African Methodist Episcopal University (AMEU), Mr. Job Carpenter – *Biology Specialist* and Mr. Prince Philip K.A. Aderibigbe – *Biology Specialist*.

The MoE is sincerely grateful to Dr Shveta Uppal, the *International Textbook Consultant* engaged by the IRISE Project to provide technical guidance and quality assurance support to the revising of the Textbooks Management Guidelines (TMG) and the procurement process leading to the provision of textbooks, teacher guides, digital contents and e-learning resources for the Senior Secondary School System in Liberia in accordance with the revised TMG. Heartfelt thanks and appreciations also to the *Executive Director* for the Center of Excellence for Curriculum Development and Textbooks Research, Mrs. Julia K. Sandiman-Gbeyai, and members of her Technical Working Group (TWG) for taking up the responsibility to lead the process of making textbooks and other TLMs available to Liberian students and teachers.

Lastly, we acknowledge the IRISE Project Delivery Team led by Mr. Abraham A. Kiazolu, II – *Project Coordinator*, Mr. Fuseini A. Abu – *International Procurement Specialist* and Mr. Lawrence S. Taylor – *Project Control Specialist* who coordinated the entire process.

We remain grateful to you all!

Hon. Alexander N. Duopu, Sr.,
Deputy Minister for Instruction
Ministry of Education, Republic of Liberia
#The Teacher

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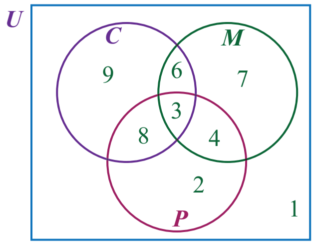

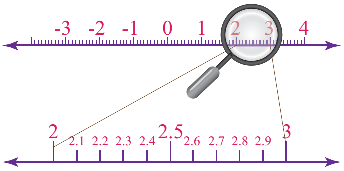
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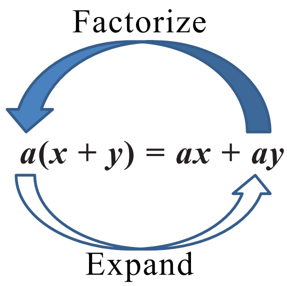
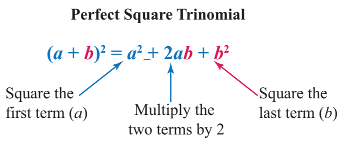
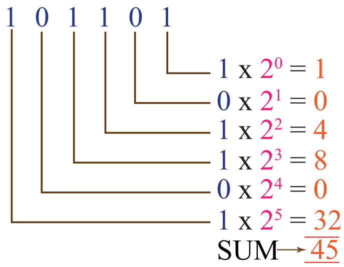
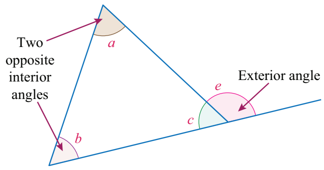
2 2.1 2.2 2.3 2.4 2.5 2.6 2.7 2.8 2.9 3


Factorize
$a(x + y) = ax + ay$
Expand

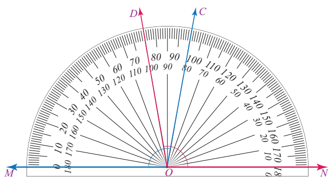

Perfect Square Trinomial
$(a + b)^2 = a^2 + 2ab + b^2$
Square the first term (a)
Multiply the two terms by 2
Square the last term (b)

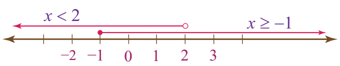


1 0 1 1 0 1
1 x 2^0 = 1
0 x 2^1 = 0
1 x 2^2 = 4
1 x 2^3 = 8
0 x 2^4 = 0
1 x 2^5 = 32
SUM 45



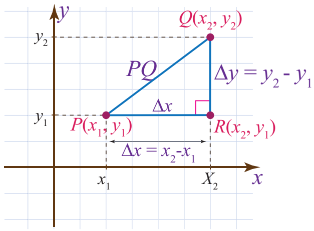
Two opposite interior angles
Exterior angle
a
b
c
e



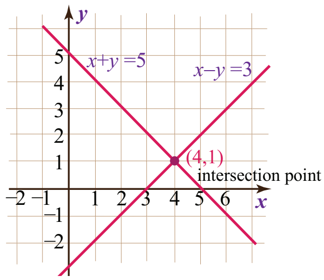


x < 2
x ≥ −1
−2 −1 0 1 2 3

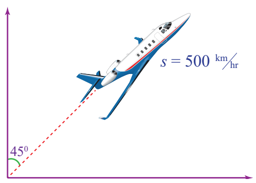
A
B
cA
cB



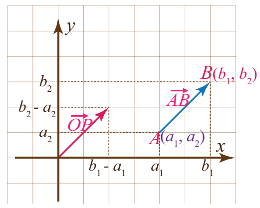
y
Q(x2, y2)
y2
PQ
Δy = y2 - y1
Δx
y1
P(x1, y1)
R(x2, y1)
Δx = x2-x1
x1
X2
x



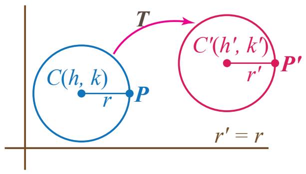
y
x+y =5
x−y =3
(4,1)
intersection point
x

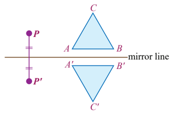


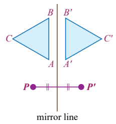
s = 500 km/hr
45⁰

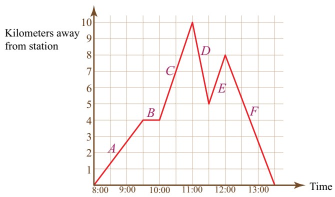


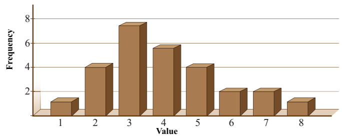
y
B(b1, b2)
AB
OP
A(a1, a2)
b2
b2 - a2
a2
b1 - a1
a1
b1
x


T
C(h, k)
r
P
C′(h′, k′)
r′
P′
r′ = r


mirror line


mirror line


Kilometers away from station
Time
8:00
9:00
10:00
11:00
12:00
13:00

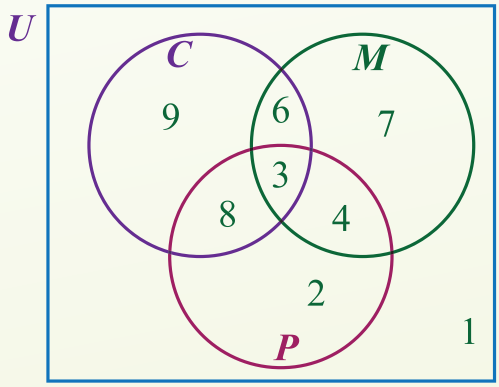

Frequency
Value

M10CH01

CHAPTER 1

SETS AND OPERATIONS ON SETS

Chapter Contents



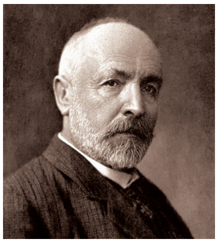
Chapter Outcomes

At the end of this chapter you will be able to:

- define the concept of set;
- identify the different types of sets;
- identify equal and equivalent sets;
- differentiate subsets and proper subsets;
- use Venn diagrams to illustrate intersection, union, disjoint and complement of sets;
- state and discuss the properties of sets;
- use the Venn diagrams to solve two-set and three-set problems.

Introduction

As kids we used to collect a lot of objects like bottle tops, marbles, buttons, toys, pens and pencils, coins, books, etc. We also observe a *flock* of sheep or a *herd* of cattle grazing together, a *class* of students learning together, a *basket* of flowers, a *bunch* of keys and so on. In mathematics, such a collection of things or objects is called a **set**. Sets are valuable in understanding the properties of whole numbers and concepts of geometry. In fact sets help us to understand many topics of mathematics that would be difficult to understand. In this unit therefore, you will be introduced with the main concepts of sets and their types. Besides, you will learn about some relations among sets and basic set operations like union and intersection and their representations using Venn diagrams.



Georg Cantor
(1845 – 1918)

Georg Cantor was a Russian-born Mathematician who can be considered as the founder of set theory and introduced the concept of infinite numbers with his discovery of cardinal numbers. He also advanced the study of trigonometric series.

Definition

A set is a collection of well-defined objects. The objects in a set are called elements or members of the set.

Activity 1

	Are you a member of the following collections? Show your decision by putting a "✓" mark.	YES	NO	NOT SURE
1	A collection of girls in your class			
2	A collection of football players in your school			
3	A collection of students in your school with three ears			
4	A collection of all short students in your class.			
5	A collection of smart students in your school			
6	A collection of beautiful people in your city			

Note: When we say a set is ***well-defined***, we mean it is possible to identify without doubt whether or not a given object belongs to that specific collection.

Example 1

(a) A collection of boys in your class

(b) A collection of Toyota cars in a city

(c) A collection of football players in your school

(d) A collection of whole numbers less than 100

(e) A collection of students in your school with two noses

All the collections in the example 1 given above are sets because they are well defined and it is possible to identify without doubt whether or not a given object belongs to that specific collection.

However, collections like the ones given in the above activity under numbers 4, 5 and 6, are not sets because they are not well defined.

Can you tell whether or not you belong to the collection of *all short students in your class*? No you can't because the height limit to call a person short or tall is not defined.

Similarly, the collection of smart students in your school and the collection of beautiful people in your city, do not define a set since we may not agree on who is "smart" and who is not, and, who is "beautiful" and who is not.

Example 2

(a) The collection of bad boys in your school

(b) The collection of tall people in the world

(c) The collection of best African football players

Collections in example 2 given above are all not well defined. Hence they are not sets.

Exercises

Identify whether the following collections define a set or not

1. The collection of the continents of the world ____________
2. The collection of intelligent teachers in your school. ____________
3. The collection of students in your class who are less than 1.60 meters tall. ______________________________

4. The collection of rich people in the world. _______
5. The collection of all even natural numbers. _______
6. The collection of best schools in your country. __________

Set notation

Braces or curly brackets like { } are used to indicate a set. Sets are usually named or denoted by capital letters like A, B, C, X, W, Z and so on.

Consider for example, "The set of natural numbers less than 5". If we denote the set by A, then we can rewrite the above statement using mathematical symbol (notation) as follows

$$A = \{1, 2, 3, 4\}$$

Note that the numbers 1, 2, 3 and 4 inside the braces are called **elements** or **members** of set A.

The members of A are separated by a comma (,).

The relationship between the elements of a set and the set itself is expressed in the form: **is an element of** or **is a member of**

Note: $x \in A$ means x is an element of set A.

$x \notin A$ means x is not an element of set A.

Thus, in $A = \{1, 2, 3, 4\}$ you see that $2 \in A$ and $5 \notin A$

The symbol $\in$ is a Greek letter and here we use it to show when an element belongs to a set.

Example 3

Let B be the set of vowels in the English alphabet. List all the elements of B.

Solution

$B = \{a, e, i, o, u\}$. So $a \in B$ $e \in B$ $i \in B$ $o \in B$ $u \in B$

B has five elements.

But $b \notin B$ $c \notin B$ $d \notin B$ $f \notin B$ and so on.

Example 4

Let M be the set of letters in the word "**mathematics**". List all the elements of M.

Solution

$M = \{\mathbf{m}, \mathbf{a}, \mathbf{t}, \mathbf{h}, \mathbf{e}, \mathbf{i}, \mathbf{c}, \mathbf{s}\}$, so $\mathbf{m} \in M$ $\mathbf{a} \in M$ $\mathbf{t} \in M$

$\mathbf{e} \in M$ $\mathbf{i} \in M$ $\mathbf{c} \in M$ $\mathbf{s} \in M$. M has eight elements.

Remember that if the same element is repeatedly written, we take only one of them as a member. For example, $A = \{a, b, b, c\}$ is the same set as $A = \{a, b, c\}$

Example 5

Let E be the set of students in your class with three eyes. List all the elements of E. How many elements has E?

Solution

Since there are no students in your class with three eyes, the set E has no members or elements. It is called an **Empty set**. So set E is represented as $E = \{\ \}$

Empty set

Definition

A set with no elements or members in it is called an empty set or null set.

An empty set is represented by the symbol $\{\ \}$ or Ø.

Example 6

The following sets are all empty sets:

- The set of whole numbers less than 0.
- The set of students in your class who are 50 years old.
- The set of people who can fly.
- The set of days in a week starting with the letter D.
- The set of months with 33 days.

Example 7

Which of the following symbols represent the empty set?

(a) Ø (b) {Ø} (c) { } (d) 0 (e) { 0 }

Solution

Only "a" and "c" represent the empty set. {Ø} is not an empty set. It is a set with one element.

Methods or ways of describing Sets

A set is described by any of the following three methods (ways):

- Verbal (word description) method
- Listing (tabulation) method
- Set - builder method

A. Verbal (word description) method

Verbal method is describing a set verbally or in words.

Example 8

- C is the set of countries of Africa.
- T is the set of teachers in your school.
- W is the set of whole numbers less than 100.

B. Listing (tabulation) method

Listing (tabulation) method is describing a set by listing all or part of the elements

(i) Complete listing method

We use complete listing method when it is possible to list all the elements of the set.

Example 9

Use complete listing method to describe the following set

A is the set of all even natural numbers less than thirteen".

Solution

We describe this set by *completely listing all the elements* of A as

$A = \{2, 4, 6, 8, 10, 12\}$

(ii) Partial listing method

When using partial listing method, we describe the set by listing *only few element of the set.*

Example 10

Use partial listing method to describe the following set

(a) O is the set of all odd natural numbers.

Here no matter how many odd numbers we might list there are always odd numbers not included in our list. We therefore list a few of the first odd numbers and then put three dots after the last number listed to indicate that the set of all odd numbers goes on and on.

Thus we write $O = \{1, 3, 5, 7, 9, \ldots\}$

(b) Similarly the set of all natural numbers can be written as

$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

(c) The set of Whole numbers less than 100, $W = \{0, 1, 2, 3, . . . , 99\}$

The three dots after the element 3 (called *ellipsis*) indicate that the elements in the set continue in that manner up to and including the last element 99.

C. Set Builder Method

Sometimes sets have so many elements that it is tedious or difficult or even impossible to list their elements. The difficulty of listing the elements of a set is minimized by using the method of **set-builder notation**. The set-builder notation states a property or a rule which its elements must satisfy.

EXAMPLE 11

Let *A represents the set of all even natural numbers*; then we use a letter, usually *x*, (or any other lower case letter) to represent any element

$A = \{x \mid x \text{ is an even natural number}\}$

This notation is read as "*A* is the set of natural numbers *x* such that *x* is even". We call this the **set-builder notation** of a set. The vertical line "|" is read "such that".

Note that some books use the colon ":" instead of the vertical line "|"

$A = \{x : x \text{ is an even natural number}\}.$

EXAMPLE 12

Use set-builder notation to denote

(a) The set of natural numbers less than 10.

(b) The people living on the earth.

(c) $D = \{1, 3, 5, 7, 9\}$.

(d) $E = \{0, 3, 6, 9, . . . \}$.

Solution

(a) $\{x|x\in \mathbb{N}, x < 10\}$ $A = \{x|x \in \mathbb{N}, x < 10\}$

We read this as

"the set of all elements *x* such that *x* is a natural number less than 10.

(b) $B = \{x|x \text{ is a person living on the earth}\}$

We read this as

"the set of all elements *x* such that *x* is a person living on the earth".

(c) $D = \{x|x$ is an odd natural number less than $10\}$
(d) $E = \{x|x$ is a whole number which is a multiple of $3\}$

Exercises

1. Write your answers in the blank spaces.
   (a) How many elements has the set of odd natural numbers less than 11?
   ____
   (b) How many elements has the set of all integers? ____________
   (c) How many elements has the set of students in your class with three ears? ______________________________
2. Rewrite the following statements using mathematical symbols
   (a) y is an element of set A. ____________________
   (b) 3 is not an element of set B. ______________________
3. Write True or False for each of the following
   (a) $1 \in \{1, 2, 3, 4\}$ ________________________________
   (b) $0 \in \{2, 4, 6, 8, 10\}$ ______________________________
   (c) $3 \notin \{6, 9, 12, 15\}$ ______________________________
   (d) $7 \in$ the set of prime numbers _____________________
   (e) $5 \notin \{1, 2, 3, \ldots, 8\}$ ______________________________
   (f) $0 \in \{\ \}$ ____

Multiple choice items

4. Which verbal or word description best describes the given set.
   (i) $A = \{2, 4, 6, 8, 10\}$
      (a) A is the set of even numbers.
      (b) A is the set of some even numbers.
      (c) A is the set of even natural numbers less than 12.
   (ii) $B = \{5, 10, 15, 20, 25, 30\}$
      (a) B is the set of natural numbers less than 35.
      (b) B is the set of the first six natural numbers that are exactly divisible by five.
      (c) B is the set of all numbers obtained by counting by fives.

(iii) $C = \{a, b, c, \ldots, x, y, z\}$

(a) C is the set of all the English alphabets.

(b) C is the set of the first and the last three English alphabets.

(c) C is the set of consonants in the English alphabets.

(iv) $R = \{x | x \in \mathbb{Z}, -1 < x < 1\}$

(a) R is the set of all integers greater than −1.

(b) R is the set of all integers between −1 and 1.

(c) R is the set of all integers greater than 1.

5. Match the sets described by the verbal (word description) method in column A with the corresponding sets described by the set – builder notation or listing method given in column B.

Column A

1. The set of natural numbers less than 6.
2. The set of letters in the word "mandela".
3. The set of odd whole numbers less than 20.
4. The set of whole numbers between 88 and 89. [88 and 89 not included].
5. The set of all natural number greater than 11.
6. The set of all whole numbers between 0 and 5 not including 0 and 5.

Column B

(a) Ø

(b) $\{x | x \in \mathbb{W} \text{ and } 0 < x < 5\}$

(c) $\{x | x \in \mathbb{N} \text{ and } x > 11\}$

(d) $\{a, l, e, d, n, m\}$

(e) $\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$

(f) $\{1, 2, 3, 4, 5\}$

Finite and Infinite sets

A set may have limited number of elements, or unlimited number of elements. For example, the set of natural numbers less than 5 has a limited number of elements, 1, 2, 3, and 4. The set of natural numbers greater than 5 has unlimited number of elements.

How many members does each set have?

1. $\{a, e, i, o, u\}$.
2. The set of even whole numbers less than 10.
3. The set of natural numbers between 9 and 29.
4. The set of odd natural numbers.
5. The set of fractions between 0 and 1.

DEFINITION

A ***finite set*** is a set that has a fixed or limited number of elements.

An **infinite *set*** is a set which is not finite.

Notation: If a given finite set A has k elements, then we write this as $n(A) = k$.
$n(A) = k$ is read as the cardinal number of elements in the set A is k.

EXAMPLE 13

(a) $A = \{0, 1, 2, 3, \ldots, 9\}$ is a finite set; set A has 10 members that is $n(A) = 10$.

(b) $B = \{x | x$ is a natural number between 2 and 15$\}$ is a finite set; B has 12 members that is $n(B) = 12$. [Remember 2 and 15 are not included]

(c) $D = \emptyset$ is a finite set: Set D has no members, that is $n(D) = 0$.

(d) $\mathbb{N} = \{1, 2, 3, \ldots\}$ is an infinite set. It has an infinite number of elements.

(e) The set of all fractions between 0 and 1 is an infinite set.

EXERCISES

1. How many members does each of the following sets have?

   (a) $A = \{ 0 \}$, $n(A) =$ ____________

   (b) $D = \{0, 2, 4, 6, \ldots, 30\}$, $n(D) =$ __________________

   (c) $C = \{x | x$ is a consonant in English alphabet$\}$, $n(C) =$ _____________

   (d) $E = \emptyset$, $n(E) =$ _______________

   (e) $Q = \{x | x$ is an even whole number less than 20$\}$, $n(Q) =$ ______________

   (f) $R = \{ \emptyset \}$, $n(R) =$ ____________

   (g) $F = \{0, \emptyset, \{\emptyset\}\}$, $n(F) =$ ____________

2. Which of the following sets are finite and which are infinite?
   (a) $A = \{x|x$ is a student in your class$\}$.
   (b) $B = \{x|x$ is a natural number greater than 20$\}$.
   (c) $P = \{x|x$ is a whole number less than 1,000,000$\}$.
   (d) W is the set of cars in the world.
   (e) H is the set of people living in China.
   (f) $E = \{x|x$ is an even natural number between 2 and 4$\}$.
   (g) F is the set of fractions between 1 and 2.

Equal and equivalent sets

A. Equal sets

Definition

If A and B are sets such that every element of A is an element of B and every element of B is an element of A, then A and B are called **equal sets**. It is denoted by $\boldsymbol{A = B}$ and is read as ***A* is equal to *B***.

Example 14

(a) If $A = \{x|x$ is a natural number less than 7$\}$, and $B = \{1, 2, 3, 4, 5, 6\}$, then $A = B$, because all element of A are elements of B and all element of B are also elements of A.

(b) If $Q = \{x|x \in W$ and $x < 0\}$, $R = \emptyset$, then $Q = R$, because every element of Q is an element of R and every element of R is an element of Q.

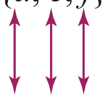
B. Equivalent sets

Look carefully at the two sets A and B given below.

$A = \{d, e, f\}$ $\qquad\qquad$ $B = \{6, 8, 9\}$

Even though these two sets are not equal, they have the same number of elements. So, for each member of set B we can find a partner in set A.

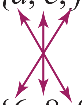
$A = \{d, e, f\}$



$B = \{6, 8, 9\}$

The double arrow shows how each element of a set is matched with an element of another set. This matching could be done in different ways, for example:

$A = \{d, e, f\}$



$B = \{6, 8, 9\}$

No matter which way we match the sets, each element of A is matched with exactly one element of B and each element of B is matched with exactly one element of A. We say that there is a one–to–one correspondence between A and B.

Definition

Two sets A and B are said to be **equivalent**, written as $\boldsymbol{A \leftrightarrow B}$**,** if there is a one to one correspondence between them.

If any two finite sets A and B have equal number of elements, then A and B are called equivalent sets. If $n(A) = n(B)$, then A is equivalent to B and is denoted by $A \leftrightarrow B$.

Remember that for two finite sets to be equivalent it is not necessary that they have same elements, or they are a subset of each other.

Example 15

let $A = \{2, 4, c\}$, $B = \{5, 8, 9\}$ and $C = \{9, 8, 5\}$. Then

$A \leftrightarrow B,\ A \leftrightarrow C,\ B \leftrightarrow C,\ B = C.$

If any two sets A and B are equal, then they are also equivalent. However any two equivalent sets may not necessarily be equal.

Exercises

1. Which of the following sets are equal? Give reasons.

   $A = \{0, 2, 4, 6, 8\}$;

   $B =$ The set of even natural numbers less than 10;

   $C = \{4, 8, 0, 6, 2\}$;

   $D = \{2, 4, 8, 6, 4\}$.

2. Let $A = \{1, 3, 5, 7, 9\}$ and $B =$ the set of odd natural numbers less than 10.

   (a) Is $A = B$? Why? (b) Is $A \leftrightarrow B$? Why?

3. $A = \{x | x \in \mathbb{N} \text{ and } 2 < x < 10\}$, $B = \{x | x \in \mathbb{W} \text{ and } 3 \leq x \leq 9\}$

   (a) Is $A = B$? Why? (b) Is $A \leftrightarrow B$? Why?

4. Which statement is always true?
   (a) If two sets A and B are equal, then they are equivalent.
   (b) If two sets A and B are equivalent, then they are equal.

Subsets and proper subsets

DEFINITION

If A and B are sets and every element of A is also an element of B, then we say A is a **subset** of B. This relation is denoted by $A \subseteq B$ and is read as 'A is a subset of B'.

From the above definition, it follows that if there exists a single element of A that is not an element of set B, then A is not a subset of B (written as $A \nsubseteq B$).

Note that,

The empty set is a subset of every set and any set A is a subset of itself.

That is, if A is any set, then $\emptyset \subseteq A$ and $A \subseteq A$.

EXAMPLE 16

Let $A = \{a, b\}$. List all the subsets of A.

Solution

$\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$ are the subsets of A. Remember that $\emptyset$ and the set itself, i.*e*, $\{a, b\}$ are subsets. A has 4 subsets.

EXAMPLE 17

Let $B = \{1, 2, 3\}$. List all the subsets of B.

Solution

$\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$ are the subsets of B. Remember that $\emptyset$ and the set itself, i.*e*, $\{1, 2, 3\}$ are subsets. B has 8 subsets.

EXAMPLE 18

Let $C = \{0\}$. List all the subsets of C.

Solution

$\emptyset$, $\{0\}$ are the subsets of C. C has 2 subsets.

Example 19

Let $D = \varnothing$. List all the subsets of D,

Solution

$\varnothing$, that is, the set itself, is the only subset. D has 1 subset.

Example 20

If $A = \{x, y, z\}$ and $B = \{y, z, x\}$, then $A \subseteq B$ and $B \subseteq A$.

Remember sets A and B are equal. If two sets are equal one is the subset of the other, that is, if $A = B$, then $A \subseteq B$ and $B \subseteq A$.

Activity 2

1. Complete the following table

Set	Subsets	Number of elements (n)	Number of subsets
{ }		0	1
{ 5 }		1	2
{ 0, 1 }			
{3, 4 , 5 }			
{ 6,7, 8, 9}			

How many subsets has a set with 5 elements? ___________

How many subsets has a set with 6 elements? ___________

How many subsets has a set with 7 elements? ___________

How many subsets has a set with n elements? ___________

If a given finite set has n number of elements, then it has 2^n subsets.

Example 21

How many subsets does the set $\{a, b, c, d, e\}$ with 5 members have?

Solution

The set has 5 members, $n = 5$

There are $2^5 = 32$ subsets. Can you list down all the 32 subsets?

Example 22

Look carefully at the two sets A and B given below:

$A = \{a, b, c\}$ $B = \{a, b, c, d, e\}$

Is set A a subset of set B? Is B a subset of A? Why?

Solution

Since all elements of A are also elements of B, we say, A is a subset of B, i.e., $A \subseteq B$

But all elements of B are not in A, So B is not a subset of A. $B \nsubseteq A$

Definition

If A and B are sets such that A is a subset of B, but B is not a subset of A, then A is also called a **proper subset** of B. It is denoted by $A \subset B$ and is read as '*A is a proper subset of B*'.

Example 23

If $A = \{7, 8, 9\}$. List all the proper subsets of A.

Solution

The proper subsets of A are Ø, $\{7\}$, $\{8\}$, $\{9\}$, $\{7, 8\}$, $\{7, 9\}$, $\{8, 9\}$. There are 7 proper subsets.

A set is not a proper subset of itself. If a set has n number of elements, then it will have $2^n - 1$ *proper subsets.*

Example 24

(a) If $B = \{1, 2, 3, 4, 5\}$ and $A = \{3, 4\}$, then $A \subset B$ because $A \subseteq B$, but $B \not\subseteq A$.

(b) Let $R = \{a, b, \mathbf{c}, d\}$ and $S = \{a, b, c\}$.

Then, $R \not\subset S$ because $R \not\subseteq S \longrightarrow$ (R has extra members which are not S)

Exercises

1. Below is given the set of the days of the week:

   {Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday}

   Write the subset of the days of the week that

   (a) begin with letter M. ___________

   (b) begin with the letter S. ___________

   (c) end with the letter y. ___________

   (d) begin with the letter Z. ___________

2. Make three subsets from the set $\{x, y, z, w\}$
   (i) ________________ (iii) ______________
   (ii) ________________
3. How many subsets have the following sets?
   (i) $A = \{0, 1, 2, 3, 4\}$ _________________
   (ii) $B = \{5, 9\}$ _________________________
   (iii) $C = \{a, b, c, d, e, f\}$ _______________
4. If $A = \{2, k, 3, m, 4\}$, how many proper subsets has set A? _____
5. Put the appropriate symbols $\subset$ or $\not\subset$ between the sets K and L.
   (a) $K = \{x \in \mathbb{N} : x \text{ is prime}\}$: $L = \{2, 3, 5, 7\}$, then L _________ K.
   (b) $K = \{l, m, n, o, p\}$: $L = \{l, o, p, m\}$, then L _________ K.
   (c) $K = \{t, a, c, t, i, c\}$: $L = \{c, a, t\}$, then L ___________ K.
   (d) $K = \{x \in \mathbb{N}: x \text{ is less than } 20\}$: $L = \{x \in \mathbb{N}: x \text{ is less than } 10\}$, then L ________ K.
6. (a) Is the set of female students in your class a proper subset of the set of all students in your class? Explain. ________________
   (b) Is the set of students in your class a proper subset of the set of all students in your school? Explain. ____________

In previous sections, you have learned how notations and symbols are used to describe sets and relations between sets. In this section, you will learn how sets, relationships between sets and operations on sets are represented using diagrams called Venn diagrams (named in honor of *John Venn*, an English philosopher, (1834 – 1923).

In a Venn diagram, the sets are represented by shapes, usually circles and ovals. The elements of a set are labeled within the circle or oval.

Example 25

Given the set P of odd numbers less than 9 and $Q = \{3, 7, 8, 9, 10\}$.

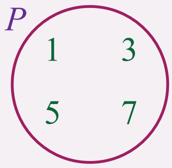
Draw a Venn diagrams to represent the sets P and Q and indicate all the elements of P and Q in the Venn diagrams.

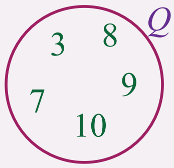
Solution

$P = \{1, 3, 5, 7\}$

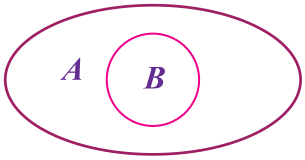
$Q = \{3, 7, 8, 9, 10\}$

Draw a circle or oval. Label P. Put the elements in P as shown below





There could be several ways to describe the relationships of sets using Venn diagrams. If you know, for example, that all members of B also belong to A or $B \subseteq A$, then you can draw B within A as shown below



The Venn diagram represents

$B \subseteq A$

In arithmetic, you have the basic operations of addition, subtraction, multiplication, and division. Each operation gives a third number for any two arbitrary numbers. Similarly, there are operations on sets which give a third set for any two arbitrary sets. In the following sections you will learn about two main operations on sets called *intersection* and *union* of sets

ACTIVITY 3

Consider the following two sets:

$A = \{r, s, t, u, v\}$ and
$B = \{a, r, b, s, c\}$.

Are there elements which are common to both sets? ________

List the elements which belong to both sets A and B? ________

Let C be the set which includes elements which are common to both A and B. Describe set C using listing method

____

The Intersection of Sets

In $A = \{r, s, t, u, v\}$ and $B = \{a, r, b, s, c\}$ obviously elements r and s belong to both A and B. That is $\{r, s\}$ is the set that contains the common elements of sets A and B. It is called the intersection of sets A and B.

DEFINITION

The intersection of two sets A and B is the set of all elements that belong to both A and B. It is denoted by $A \cap B$ and is read as 'A intersection B'.

Example 26

(a) In $A = \{r, s, t, u, v\}$ and $B = \{a, r, b, s, c\}$, $A \cap B = \{r, s\}$.

(b) In $C = \{1, 2, 3\}$ and $D = \{2, 3, 4\}$, $C \cap D = \{2, 3\}$.

(c) In $A = \{x | x \in \mathbb{N} \text{ and } 0 < x < 1\}$ and $B = \{0, 1\}$, $A \cap B = \{\ \}$.

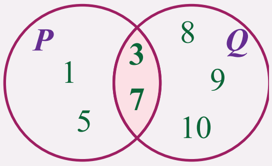
If two sets A and B have common elements, then they are called intersecting sets. If two sets A and B have no any common elements, then they are called disjoint sets. if $A \cap B = Ø$, then A and B are disjoint.

Example 27

Let $P = \{1, 3, 5, 7\}$

$Q = \{3, 7, 8, 9, 10\}$ $\qquad P \cap Q = \{3, 7\}$

The shaded part of the following Venn diagram represents $P \cap Q$



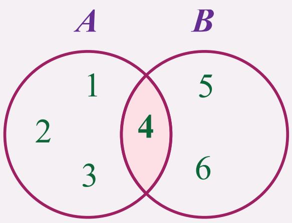
The shaded part represents the intersection of both sets

$P \cap Q = \{3, 7\}$

Example 28

(a) If $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6\}$, then represent the sets A and B using a Venn diagram to show the intersection of the two sets.

Solution



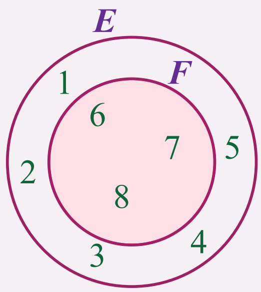
$A \cap B = \{4\}$.

(b) If $C = \{2, 3, 4, 5\}$ and $D = \{1, 6, 7, 8\}$, then

$C \cap D = Ø$

The two sets have no common members. So nothing is shaded as an intersection

(c) If $E = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $F = \{6, 7, 8\}$, then
$E \cap F = \{6, 7, 8\}$.
$E \cap F$ is represented by the shaded portion of the following Venn diagram:



Set F is a subset and proper subset of set E

Since $F \subseteq E$, $F \cap E = F$

$E \cap F = F$

Exercises

1. Find the intersection of the following sets.
   (a) $A = \{x \in N: x \text{ is a prime number less than } 10\}$
   (b) $B = \{y \in N: y \text{ is an odd number less than } 10\}$
   $A \cap B =$ ____________________
   (c) $A = \{\text{a}, b, c, d, e\}$ $\quad$ $B = \{\text{a}, e, i, o, \text{u}\}$
   $A \cap B =$ ____________________
   (d) $A =$ {lion, mongoose, hippopotamus, elephant}; $B =$ {cat, zebra, hyena}
   $A \cap B =$ ____________________
2. If $A = \{2, 4, 6, 8\}$, $B = \{1, 4, 6\}$, and $C = \{1, 6, 9\}$, then find
   (a) $A \cap B$ and $B \cap A$. Is $A \cap B = B \cap A$? ______________
   (b) $(A \cap B) \cap C$ and $A \cap (B \cap C)$. Is $(A \cap B) \cap C = A \cap (B \cap C)$? _____
3. Which of the following statement is true for any sets A and B?
   (a) If $A \subseteq B$. then $A \cap B = A$ __________________
   (b) If $A \subset B$, then $A \cap B = A$ __________________
   (c) $A \cap \emptyset = A$ ________________________

Like the intersection, union is another basic operation on sets. As in the expressions "workers union" or "students union", the term union, as applied to sets, describes "joining or uniting of elements".

Set E in the activity 4 is the set which contains all the elements that belong to either A or B or both A and B. Collecting or uniting all the elements of the two sets together, we get a new set E which includes all the English alphabets, that is $E = \{a, b, c, d, e, . . . , z\}$

Activity 4

Consider the following two sets:

$A = \{a, e, i, o, u\}$ and B = the set of all consonants in the English alphabet.

Create a new set E that includes or unites or combines all the elements of A and also all the elements of B together. So E = ______________

Definition

The union of two sets A and B is the set of all elements, which are in A, or in B, or in both A and B. It is denoted by $A \cup B$ and is read as '***A* union *B***'.

Example 29

If $A = \{a, e, i, o, u\}$

B = The set of all consonants in the English alphabet

Then $A \cup B = \{a, b, c, d, e, . . . , z\}$ which is the set of all the English alphabets.

Example 30

(a) If $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6\}$, then $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$.

(b) If $C = \{a, b, c\}$ and $D = \{d, e, f, g\}$ then $C \cup D = \{a, b, c, d, e, f, g\}$.

Activity 5

1. Let $A = \{a, b\}$ and $B = \{5, 6, 7\}$
   (a) How many members does set A have? $n(A)$ = _________
   (b) How many members does set B have? $n(B)$ = _________
   (c) How many members does set $A \cap B$ have? $n(A \cap B)$ = _________
   (d) How many members does set $A \cup B$ have? $n(A \cup B)$ = _________
   (e) Is it always true that $n(A \cup B) = n(A) + n(B)$? Why or why not? _________
2. Let $A = \{3, 4, 5, 6\}$ and $B = \{5, 6, 7\}$
   (a) How many members does set A have? $n(A)$ = _________
   (b) How many members does set B have? $n(B)$ = _________
   (c) How many members does set $A \cap B$ have? $n(A \cap B)$ = _________
   (d) How many members does set $A \cup B$ have? $n(A \cup B)$ = _________
   (e) Is it always true that $n(A \cup B) = n(A) + n(B)$? Why or why not? _________
   (f) What is always true for $n(A \cup B)$? _________

Example 31

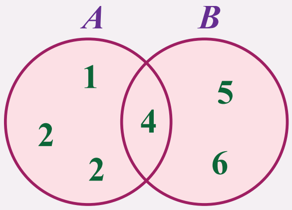
(a) If $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6\}$, then represent the union of A and B using Venn diagram

Solution

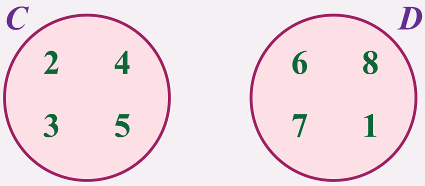
$A \cup B = \{1, 2, 3, 4, 5, 6\}$

In $A \cup B$, everything of A and everything of B is shaded

$A \cup B$



(b) If $C = \{2, 3, 4, 5\}$ and $D = \{1, 6, 7, 8\}$, then

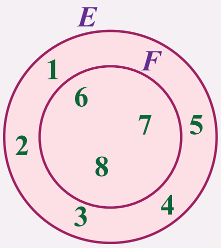


$C \cup D = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$C \cup D$ is represented by the shaded portions of the following Venn diagram:

Since the two sets have no common members, the two circles are not intersecting

(c) If $E = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $F = \{6, 7, 8\}$, then



$E \cup F = \{1, 2, 3, 4, 5, 6\ 7, 8\}$

$E \cup F$ is represented by the shaded portions of the following Venn diagram:

Set F is a proper subset of set E

Since $F \subseteq E$, $E \cup F = E$

$E \cup F = E$

Example 32

If $C = \{1, 3, 5, 7, 9\}$, and $D = \{2, 4, 6, 8\}$, then

$C \cup D = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. You also find that:

$n(C) = 5$; $n(D) = 4$; $n(C \cup D) = 9$;

Thus, $n(C \cup D) = n(C) + n(D) = 5 + 4 = 9$

The number of members of the union of two sets is equal to the sum of the members of each set <u>if the two sets are disjoint</u>.

Example 33

If $A = \{3, 4, 5, 6\}$ and $B = \{5, 6, 7\}$ then $A \cup B = \{3, 4, 5, 6, 7\}$ and $A \cap B = \{5, 6\}$

$n(A) = 4$: $n(B) = 3$; $n(A \cup B) = 5$; That means $n(A \cup B) \neq 4 + 3$

If the sets A and B are not disjoint, the number of elements in $A \cup B$ is $n(A) + n(B)$ minus the number of members in the intersection.

For any two intersecting sets A and B; $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

EXAMPLE 34

Let $A = \{2, 4, a, b, 6\}$, and $B = \{c, d, 2, r, 4\}$.

Then, $A \cup B = \{2, 4, a, b, 6, c, d, r\}$ and $A \cap B = \{2, 4\}$

$n(A) = 5$; $n(B) = 5$; $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 5 + 5 - 2 = 8$

EXERCISES

1. Find the union of the following sets.
   (a) $A = \{x \in \mathbb{N}$: x is a prime number less than 10$\}$
   $B = \{y \in \mathbb{N}$: y is an odd number less than 10$\}$
   $A \cup B =$ ____________________
   (b) $A = \{a, b, c, d, e\}$
   $B = \{a, e, i, o, \mathrm{u}\}$
   $A \cup B =$ ____________________
   (c) $A = \{$lion, mongoose, hippopotamus, elephant$\}$;
   $B = \{$cat, zebra, hyena$\}$
   $A \cup B =$ ____________________
2. If $A = \{2, 4, 6, 8\}$, $B = \{1, 3, 5\}$, and $C = \{a, b, c, d\}$, then find
   (a) $A \cup B$ and $B \cup A$. Is $A \cup B = B \cup A$? ______________
   (b) $(A \cup B) \cup C$ and $A \cup (B \cup C)$. Is $(A \cup B) \cup C = A \cup (B \cup C)$? ______
3. If $n(X) = 22$, $n(Y) = 9$ and $n(X \cap Y) = 4$, then, find $n(X \cup Y)$. ___________
4. Given $n() = a$, $n(Y) = b$, and $n(X \cap Y) = c$, then $n(X \cup Y) =$ _____________
5. If $n(A \cup B) = 10$, $n(A) = 5$, $n(B) = 6$, then, find $n(A \cap B)$. ________
6. Which of the following statement is true for any sets A and B?
   (a) If $A \cap B = \emptyset$, $n(A \cup B) = n(A) + n(B)$ ____________________
   (b) If $A \cap B \neq \emptyset$, $n(A \cup B) = n(A) + n(B)$ ____________________
   (c) If $A \cup B = A \cap B$, then $A = B$. ___________________________
   (d) If $n(A) = n(B)$ then $A = B$ _________________________
   (e) If $A = B$ then $n(A) = n(B)$ _________________________

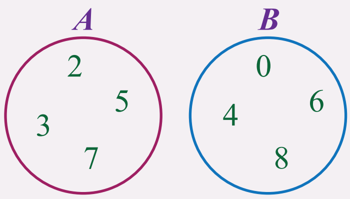
1.7 Venn Diagram to show Disjoint Sets and Complement of a Set

Disjoint sets

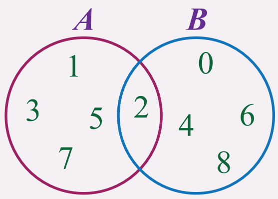
Note that if two sets A and B have no any common elements, then they are called disjoint sets. if $A \cap B = Ø$, then A and B are disjoint.

Example 35

If $A = \{2, 3, 5, 7\}$ and $B = \{0, 4, 6, 8\}$, the $A \cap B = Ø$. So we say A and B are non-intersecting or disjoint sets. Disjoint sets are represented in a Venn diagram like the following:



Disjoint sets



Not disjoint sets

Relative complement

Definition

The *relative complement* of B with respect to A (or the difference between A and B) which is denote as $A - B$ or $A \setminus B$ is defined as the *set of all elements in A that are not in B*

That is , $A \setminus B = \{x | x \in A \text{ and } x \notin B\}$. $A \setminus B$ is read as $\boldsymbol{A}$ **less** $\boldsymbol{B}$

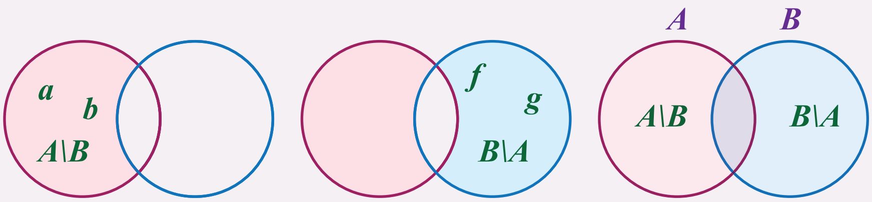
Example 36

If $A = \{a, b, c, d , e\}$

$B = \{c, d, e, f, g\}$, Find $A \setminus B$ and $B \setminus A$ and represent them in a Venn diagram

Solution

$A \setminus B = \{a, b\}$ and $B \setminus A = \{f, g\}$



Universal Set

Let A = the set of all grade 10 students in your school

B = the set of all girls in your school

C = the set of all grade 11 students in your school

U = the set of all students in your school

Each set A, B and C are subsets of the bigger set U. This bigger set U which includes all other elements of the set under our consideration is called a **Universal Set.**

DEFINITION

A **universal set** (usually denoted by U) is a set which has elements of all the related sets, without any repetition of elements.

EXAMPLE 37

Let $O = \{x \mid x \text{ is odd natural number and } x < 100\}$

$E = \{y \mid y \text{ is even natural number and } y < 100\}$

Choose a universal set U for O and T.

Solution

One possible universal set is U = the set of natural numbers less than 100.

EXAMPLE 38

$A = \{y : y \text{ is a female student in your class}\}$

$B = \{z : z \text{ is a female student in your class below 13 years old}\}$

$C = \{w: w \text{ is a female student in your class who are 13 years old and above}\}$

Find a possible universal set for A, B and C and represent them in a Venn diagram

Solution

One possible universal set is $U = \{x : x \text{ is a female student in your class}\}$ which is equal to A

The other possible universal set is $U = \{x: x \text{ is a student in your class}\}$

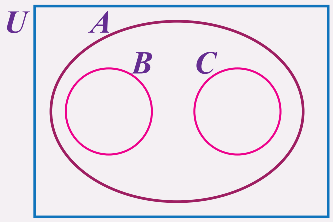
The Venn diagram representation looks like the following

$A \subseteq U$

$B \subseteq A$

$C \subseteq A$ and

$B \cap C = \varnothing$ (B and C are disjoint)



Example 39

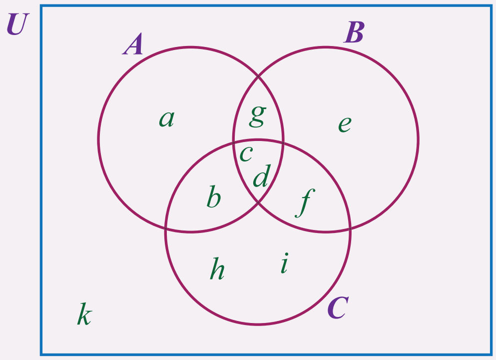
Let $U = \{a, b, c, d, e, f, g, h, i, k\}$

$A = \{a, b, c, d, g\}$

$B = \{c, d, e, f, g\}$

$C = \{b, c, d, f, h, i\}$

Represent all the above sets in a Venn diagram



Solution

$A \cap B = \{c, d, g\}$ $A \cap C = \{b, c, d\}$ $B \cap C = \{c, d, f\}$

$A \cap B \cap C = \{c, d\}$

Complement

Let U = {All students in your class} and A = {All boys in your class}. Can you tell the set which is not in A but in U? Of course the set which is out of A but in U is {All girls in your class}. This set is called the complement A or the absolute complement of A.

Definition

Let A be a subset of a universal set U. The complement of A, denoted by A', is defined to be the set of all elements of U that are not in A.

That is, $A' = \{x | x \in U \text{ and } x \notin A\}$. It is $U \setminus A$.

Example 40

If $U = \{x | x \in \mathbb{N} \text{ and } x < 10\}$

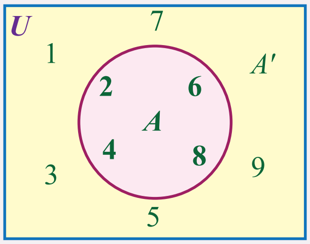
$A = \{2, 4, 6, 8\}$

$B = \{1, 2, 3, 4, 5, 6\}$, Find A' and B' and represent them in a Venn diagram

Solution

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

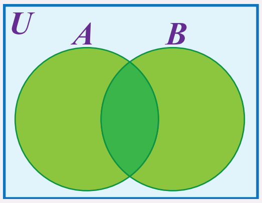
$A' = \{1, 3, 5, 7, 9\}$



The region colored yellow is A'

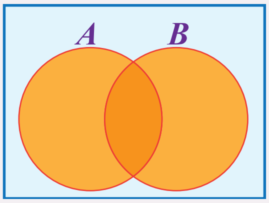
Example 41

In the Venn diagrams given below shade $(A \cup B)'$ and $(A \cap B)'$



The green shaded part is $(A \cap B)'$

The white unshaded part is $A \cap B$



The shaded part in blue is $(A \cup B)'$

The shaded part in orange is $A \cup B$

Exercises

1. Let $A = \{0, 2, 3, 5, 7\}$, $B = \{0, 2, 4, 6, 8\}$ and $C = \{1, 2, 3, 6\}$, Find:
   (a) $A \setminus B$ (b) $B \setminus A$ (c) $(A \setminus B) \setminus C$ (d) $A \setminus (B \setminus C)$ (e) $A \cap B'$
2. Let $U = \{a, e, i, o, u\}$ $A = \{a, e\}$ $B = \{o, u\}$, Find
   (a) A' (b) B' (c) $A' \cup B'$ (d) $A' \cap B'$ (e) $(A \cup B)'$ (f) $(A \cap B)'$
3. From your answer in number 2 above, what relationship exists between
   (i) $(A \cup B)'$ and $A' \cap B'$ (ii) $(A \cap B)'$ and $A' \cup B'$
4. Show by shading in a Venn diagram that $A \setminus B = A \cap B'$

Do you remember the commutative, associative, distributive and identity properties you learned while dealing with operations on numbers? Just in a similar way, union, intersection and complements of sets also satisfy these properties. Observe the following comparisons:

	For three number *a*, *b* and *c*	For any three sets *A*, *B* and *C*	Property
1	$a + b = b + a$ $a \times b = b \times a$	$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Property
2	$(a + b) + c = a + (b + c)$ $(a \times b) \times c = a \times (b \times c)$	$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$	Associative Property

	For three number *a*, *b* and *c*	For any three sets *A*, *B* and *C*	Property
3	$a \times (b + c) = (a \times b) + (a \times c)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive Property
4	$a + 0 = a$ $a \times 1 = a$	$A \cup \varnothing = A$ $A \cap U = A$	Identity property
5		$A \cup A' = U$ $A \cap A' = \varnothing$	Complement property
6		$A \cup A = A$ $A \cap A = A$	Idempotent property

Example 42

U = $\{a, b, c, d, e, f, g, h, k, m, n, o\}$

Let $A = \{a, b, c, d, e\}$

$B = \{c, d, e, f, g\}$

$C = \{d, e, h, k, m\}$

1. Use the sets given above and show that
   (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (c) $A \backslash B = A \cap B'$

Solution

(a) $B \cup C = \{c, d, e, f, g, h, k, m\}$

$A \cap (B \cup C) = \{c, d, e\}$

$A \cap B = \{c, d, e\}$ $A \cap C = \{d, e\}$

$(A \cap B) \cup (A \cap C) = \{c, d, e\}$

Therefore, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Intersection of sets distributed over union)

(b) $B \cap C = \{d, e\}$ $A \cup (B \cap C) = \{a, b, c, d, e\}$

$A \cap B = \{a, b, c, d, e, f, g\}$

$A \cup C = \{a, b, c, d, e, h, k, m\}$

$(A \cup B) \cap (A \cup C) = \{a, b, c, d, e\}$

Therefore, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Union of sets distributed over intersection)

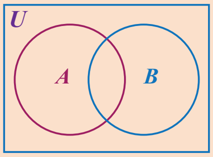
(c) $A \backslash B = \{a, b\}$ $B' = \{a, b, h, k, m, n, o\}$ $A \cap B' = \{a, b\}$

Therefore, $A \backslash B = A \cap B'$

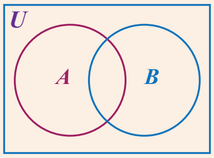
Activity 6

1. Consider the sets given above, that is,
   U = $\{a, b, c, d, e, f, g, h, k, m, n, o\}$
   Let $A = \{a, b, c, d, e\}$
   $B = \{c, d, e, f, g\}$
   $C = \{d, e, h, k, m\}$
   Show that
   (a) $(A \cup B)' = A' \cap B'$
   (b) $(A \cap B)' = A' \cup B'$
2. Show by shading the Venn diagrams given below that
   (a) $(A \cup B)' = A' \cap B'$
   (b) $(A \cap B)' = A' \cup B'$

Shade $(A \cap B)'$

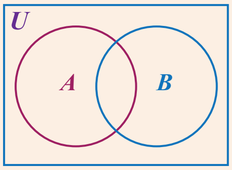


Shade $A' \cup B'$

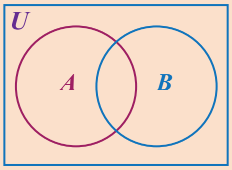


Are the shaded regions in the two of the figures the same?
If yes,
$(A \cap B)' = A' \cup B'$

Shade $(A \cup B)'$



Shade $A' \cap B'$



Are the shaded regions in the two of the figures the same?
If yes,
$(A \cup B)' = A' \cap B'$

Historical Note

De Morgan's laws

For any two sets A and B

1. $(A \cap B)' = A' \cup B'$
2. $(A \cup B)' = A' \cap B'$

Augustus De Morgan (1806-1871)

Augustus De Morgan was the first professor of mathematics at London University (University College London) and a co-founder of the London Mathematical Society.

De Morgan formulated his laws during his study of symbolic logic. De Morgan's laws have applications in the areas of set theory, mathematical logic and the design of electrical circuits.

Exercises

1. Determine whether each of the following statements is true or false:
   (a) For any set B, $B \cup \varnothing = B$ ________________
   (b) For any set B, $B \cap \varnothing = B$ ________________
   (c) For any set B and a universal set U, $B \cup \mathrm{U} = B$ ____________
   (d) For any set B and a universal set U, $B \cap \mathrm{U} = B$ ____________
   (e) For any set B and a universal set U, $B \cap B' = \varnothing$ ____________
   (f) For any set B and a universal set U, $B \cup B' = U$ ____________
2. For any two sets P and Q, which of the following is false?
   (a) $(P')' = P$
   (b) $P \backslash Q = P \cap Q'$
   (c) $P' \cap Q' = (P \cup Q)'$
   (d) $P' \cup Q' = (P \cap Q)'$
   (e) $(P \backslash Q) \cup (Q \backslash P) = (P \cup Q) \backslash (P \cap Q)$
   (f) $(P \backslash Q) \cap (Q \backslash P) = (P \cap Q)$

So far you have learned about the concepts of a set, the relationships between sets and operations like union, intersection, relative and absolute complements on sets and their properties. In this sub section, you will learn how to use Venn diagrams in solving real life application problems involving two or three sets.

Example 43

1. Of 50 students in a class, 25 of them are members of a mathematics club and 20 are members of a physics club. If 8 students are members of both clubs, then
   (a) How many students are members of Math club only?
   (b) How many students are members of Physics but not math club?
   (c) How many students are members neither mathematics nor Physics club?

To solve such word problems involving sets one has to pay attention to the terminologies used in the problem. Study the terminologies and their meaning given below:

- Mathematics only means Mathematics but not Physics; which is the same as $M \setminus P$
- Physics but not Mathematics; means $P \setminus M$
- Neither Mathematics nor Physics means; not Mathematics and also not Physics; which is the same as $M' \cap P' = (M \cup P)'$ **(De Morgan's Law)**
- Both Mathematics and Physics means $M \cap P$
- Mathematic or Physics (either Mathematics or Physics) means $M \cup P$

Solution

Let M and P represent the group of students who are in Mathematics and Physics club respectively.

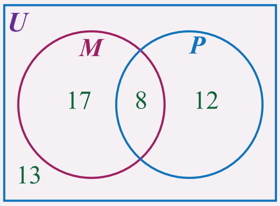
Given:

25 of them are members of a Mathematics club, that is $n(M) = 25$

20 are members of a Physics club, that is $n(M) = 20$

8 students are members of both clubs, that is $n(M \cap P) = 8$

The information given in the word problem can be represented in the Venn diagram like the following:



Put first the intersection of both in the Venn diagram, i. e, $n(M \cap P) = 8$

Then subtract the intersection from $n(M)$ and $n(P)$ to get $n(M \setminus P)$ and $n(P \setminus M)$. The remaining which is out of the union is $n(M \cup P)' = 13$

(a) How many students are members of Math club only?

Members of Math club only = $n(M \setminus P) = 17$ from the Venn diagram

You can also calculate it as $n(M \setminus P) = n(M) - n(M \cap P) = 25 - 8 = 17$

(b) How many students are members of Physics but not Math club?

Members of Physics but not math club = $n(P \setminus M) = 12$ from the Venn diagram

You can also calculate it as $n(P \setminus M) = n(P) - n(M \cap P) = 20 - 8 = 12$

(c) How many students are neither Mathematics nor Physics club members?

Neither Mathematics nor Physics club members = $M' \cap P' = (M \cup P)'$

$(M \cup P)'$ is outside of the union but in U. As seen on the Venn diagram, there are 13 students outside of the union of the two. These are students who are members of neither of the two clubs

You can also calculate it as $n\,(M \cup P)' = n(U) - n(M \cup P)$

$= n(U) - [n(M) + n(P) - n(M \cap P)]$

$= 50 - [25 + 20 - 8]$

$= 50 - 37$

$= 13$ (who are members of neither of the two clubs)

Example 44

40 science students were asked about their preferences of the three subjects: Chemistry, Mathematics, and Physics. Their preferences are listed in the table below

Subject Preference of students	Number of students
Chemistry	26
Math	20
Physics	17
All three	3
None of the subjects	1
Both Chemistry and Physics	11
Both Math and Physics	7
Chemistry Math and Physics	9

Draw a Venn diagram and answer the following questions

(a) How many students prefer Chemistry or Math?

(b) How many students prefer Chemistry or Math but not both?

(c) How many students prefer either Chemistry or Math or Physics?

(d) How many students prefer either Chemistry or Math or Physics but not all the three?

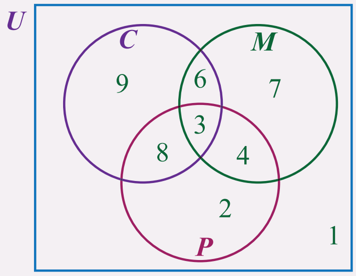
(e) How many students prefer Math but not Physics?

Solution

Let C, M and P represent the group of students who prefer Chemistry, Mathematics and Physics respectively.

Given:

Subject Preference of students	Number of students
Chemistry	$n(C) = 26$
Math	$n(M) = 20$
Physics	$n(P) = 17$
All three	$n(C \cap M \cap P) = 3$
None of the subjects	$n(C \cup M \cup P)' = 1$
Both Chemistry and Physics	$n(C \cap P) = 11$
Both Math and Physics	$n(M \cap P) = 7$
Chemistry Math and Physics	$n(C \cap M) = 9$



From the above Venn diagram, it is clear to see that students who preferred

(a) Chemistry or Math $= n(C \cup M) = n(C) + n(M) - n(C \cap M) = 26 + 20 - 9 = 37.$

(b) Chemistry or Math but not both $= n[(C \cup M) \setminus (C \cap M)] = 37 - 9 = 28.$

(c) Either Chemistry or Math or Physics $= n(C \cup M \cup P) = 39.$

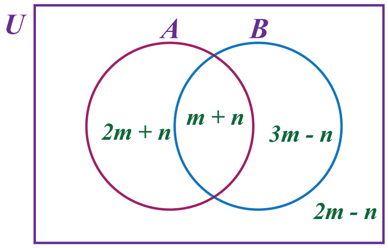
(d) Either Chemistry or Math or Physics but not all the three

$= n[(C \cup M \cup P) \setminus (C \cap M \cap P) = 39 - 3 = 36.$

Exercises

1. Suppose A and B are sets such that $n(A) = 10$, $n(B) = 23$ and $n(A \cap B) = 4$, then find:
   (a) $n(A \cup B)$
   (b) $n(A \setminus B)$
   (c) $n(B \setminus A)$
   (d) $n[(A \setminus B) \cup B \setminus A)]$

2. The following Venn diagram shows two sets A and B. If $n(A) = 14$, $n(B) = 8$, then find:



   (a) $n(A \cup B)$
   (b) $n(U)$
   (c) $n(B \setminus A)$
   (d) $n(A \cap B')$
3. In a group of 60 people, 27 like tea and 42 like coffee and each person likes at least one of the two drinks.
   (a) How many like both coffee and tea?
   (b) How many like tea but not coffee?
   (c) How many like coffee only?

   [**Hint:** *each person likes at least one of the two is the union of the two)*
4. In a competition, a school awarded medals in different categories.36 medals in Dance, 12 medals in Painting and 18 medals in music. 5 persons got medals both in Dance and Painting, 9 got medals in Dance and Music, 5 got medals in Painting and Music. If these medals went to a total of 51 persons and only 4 persons got medals in all the three categories,
   (a) Draw a Venn diagram to represent the given information
   (b) How many people received medals in either of the three but not in all of the three categories?
   (c) How many received medals in only Dance?
   (d) How many received medals in Dance or Music but not in both?

Key Terms

- Disjoint sets
- Empty (Null) set
- Equal sets
- Equivalent sets
- Element
- Finite set
- Intersection
- Infinite set
- Member
- Proper subset
- Set
- Subset
- Set - builder
- Union
- Verbal (word description)
- Venn diagram
- Well - defined

SUMMARY

- A set is a well – defined collection of objects.
- The objects in a set are called elements or members of that set. $\in$ is the symbol for membership, and $\notin$ is the symbol for non – membership.
- An empty set is a set with no elements; it is denoted by Ø or { }.
- A finite set is a set that has a limited number of elements.
- An infinite set is a set which is not finite.
- A set A is a subset of set B if every element of A is also an element of B. The symbol for subset is $\subseteq$. If A is not a subset of B, we write $A \not\subseteq B$.
- Disjoint sets are sets that have no members in common.
- A set A is a proper subset of set B if $A \subseteq B$ but $B \not\subseteq A$. The symbol for proper subset is $\subseteq$.
- Equal sets are sets that contain the same members. $A = B$ if $A \subseteq B$ and $B \subseteq A$.
- Equivalent sets are sets which match one – to – one or sets which have equal number of members. The symbol for equivalence is $\leftrightarrow$.
- The intersection of two sets A and B is a set which contains common members of both A and B. The symbol for intersection is $\cap$

  $A \cap B$ is read as "A intersection B."
- The union of two sets A and B is a set which contains all members of A or B or both. The symbol for union is $\cup$. $A \cup B$ is read as "A union B."
- A Venn diagram is a pictorial representation used to visualize relationships between sets and operations on sets.
- The Relative complement of set B with respect to A is denoted by $A \setminus B$. It includes elements which are in A but not in B.
- A Universal set usually denoted by U is a larger set which includes all elements of the sets under consideration.
- The complement of A is written as A'. It includes elements which are in a universal set U but not in A.
- Union and Intersection of sets are commutative: $A \cup B = B \cup A$; $A \cap B = B \cap A$.
- Union and Intersection of sets are associative:

  $(A \cup B) \cup C = A \cap (B \cap C)$;

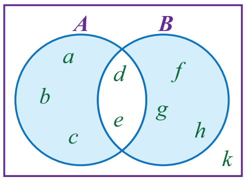
  $(A \cap B) \cap C = A \cap (B \cap C)$.

- Union is distributive over intersection of sets: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
- Intersection is distributive over union of sets: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

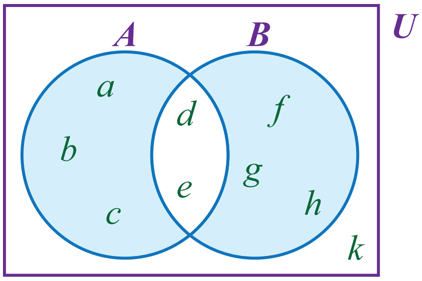
Exercises

1. Circle the letter which represents a well-defined collection or a set.
   (a) The group of handsome boys in your school.
   (b) The set of the last ten letters in the English alphabet.
   (c) The set of even integers.
   (d) The group of pretty girls in your school.
2. Circle the letter which represents a finite set
   (a) $A = \{1, 3, 5, 7, \ldots\}$.
   (b) B = the set of coffee beans in a sack.
   (c) C = the set of natural numbers which are greater than 9.
   (d) D = the set of people in the world.
3. Which of the following pair of sets are not equivalent?
   (a) $\{a, b\}$ and $\{2, 4\}$.
   (b) $\{\varnothing\}$ and $\varnothing$.
   (c) $\{a, \{b, c\}\}$ and $\{a, b, c\}$.
   (d) $\{2, 4, 6\}$ and $\{x | x$ is composite and $x < 8\}$.
   (e) Only b and c.
4. Which of the following is false?
   (a) The set $\{1, 2\}$ is a proper subset of $\{1, 2, 3\}$.
   (b) Any set is a subset of itself.
   (c) The set $\{1, 2, 3\}$ is a proper subset of $\{1, 2, 3\}$.
   (d) The empty set is a subset of any given set A.
   (e) The empty set is a proper subset of any given set A.
5. Which one is <u>false</u> about the set $A = \{\varnothing, \{\varnothing\}\}$
   (a) $\varnothing \subseteq A$
   (b) $\varnothing \in A$
   (c) $\{\varnothing\} \in A$
   (d) $\{\varnothing\} \subseteq A$
   (e) None of the above

6. Which of the following statements is true?
   (a) If $x \in A$ and $x \notin B$, then $x \notin (A \cup B)$.
   (b) If $x \notin A$ and $x \notin B$, then $x \notin (A \cup B)$.
   (c) If $A \subseteq B$, then $A \cup B = A$.
   (d) If $A \cap B = \emptyset$, then $A \setminus B = \emptyset$.
7. If $A \cup B = \emptyset$, then which one is false about A and B?
   (a) $A = \emptyset$ and $B = \emptyset$.
   (b) $A \cap B = \emptyset$.
   (c) $(A \setminus B) \cup (B \setminus A) = \emptyset$.
   (d) $n(A) = n(B) = 0$.
   (e) none of the above.
8. Use the Venn diagram to answer the following
   The shaded part indicates



   (a) $A \cup B$
   (b) $A \setminus B$
   (c) $B \setminus A$
   (d) $A \cap B$
   (e) $(A \cup B) \setminus (A \cap B)$
   (f) $(A \setminus B) \cup (B \setminus A)$
   (g) both (e) and (f)
9. Which of the following statements is false?
   (a) If $x \in A$ and $x \notin B$, then $x \in B \setminus A$
   (b) If $x \notin A$ then $x \notin A \setminus B$
   (c) If $A \setminus B = \emptyset$, then $A \subseteq B$
   (d) If $A \cap B = \emptyset$, then $n(A \cup B) = n(A) + n(B)$
10. Refer to the Venn diagram given below to answer the following question



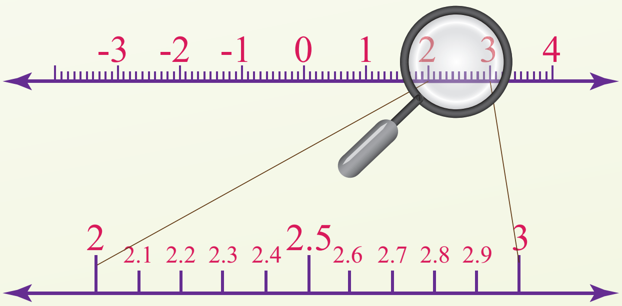
   Which set includes elements which are neither in A nor in B?
   (a) $\{d, e\}$
   (b) $\{\text{a}, b, c\}$
   (c) $\{k\}$
   (d) $\{f, g, h\}$

M10CH02

CHAPTER 2

RATIONAL NUMBERS

Chapter Contents



Chapter Outcomes

At the end of this chapter you will be able to:

- add and subtract any given rational number;
- multiply and divide any rational number;
- state closure, commutative, associative, identity and inverse properties on rational numbers;
- define the set of real numbers;
- solve problems involving real numbers.

Chapter Introduction

In this chapter your knowledge of integers and fractions will be extended to the set of rational numbers. The set of integers which includes the set of natural and whole numbers as its subset will be revised first. Then the four operations, that is, addition, subtraction, multiplication and division with rational numbers and the decimal representations of rational numbers will be discussed. Finally you will be introduced with the set of real numbers and their properties.

Do you remember the set of natural and whole numbers? Can you describe the set of integers? The following activity helps you revise the sets of natural numbers, whole numbers and integers:

Activity 1

1. To Which set of numbers does each of the following numbers belong? Complete the table given below by putting a" ✓ " mark.

Number	Set it belongs Natural Number	Whole number	Integer
4			
−4			
0			
2.5			
$\frac{2}{4}$			
2.555...			

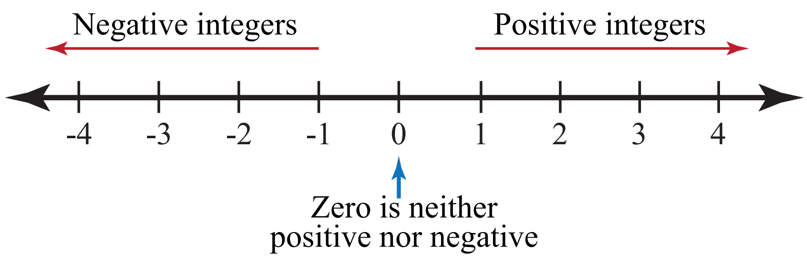
2. Define the sets of Natural numbers, Whole numbers and Integers and tell the relationship that exists among them.
3. Write true if the statement is true and false if it is false
   (a) The set { −2, 0, 2 } is a sub set of the set of whole numbers
   (b) The set { 0, 2, 4, 6 } is a sub set of the set of natural numbers
   (c) The set { 1, 2, 3 . . . } describes the set of natural numbers
   (d) The set { 0, 1, 2, 3 . . . } describes the set of whole numbers
   (e) The set { −2, −1, 0, 1, 2, . . . } describes the set of natural numbers

Addition and subtraction of rational numbers

Addition and Subtraction of rational numbers are carried out in the same way as the arithmetic operations like addition and subtraction on integers and fractions. Remember that the set of integers is defined as

$$\mathbb{Z} = \{\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$$

Integer Number Line



From this it is evident that the set of natural and whole numbers are sub sets of integers, that is, $\mathbb{N} \subseteq \mathbb{Z}$ and $\mathbb{W} \subseteq \mathbb{Z}$.

Using the set of integers, we define the set of rational numbers as follows:

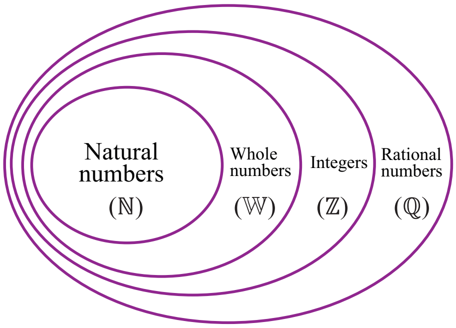
DEFINITION

A rational number is a number that can be expressed in the form $\frac{p}{q}$, where: p and q are integers, $q \neq 0$. The set of rational numbers, denoted by $\mathbb{Q}$ is the set described by

$$\mathbb{Q} = \left\{\frac{p}{q} : p, q \text{ are integers and } q \neq 0\right\}$$

Do not confuse rational numbers with that of fractions. Fractions include only positive numbers, while rational numbers include both positive and negative numbers. Fractions are a part of rational numbers, while rational numbers are a broad category that includes other types of numbers. So we say that the set of fractions is a sub set of the set of rational numbers. The following are some examples of rational numbers: $\frac{1}{2}$, $\frac{-2}{5}$, 0.4 or $\left(\frac{4}{10}\right)$, -0.9 or $\left(-\frac{9}{10}\right)$ etc.

Note that integers are included in the set of rational numbers. This is because all integers such as 5, 0 and –9 can be expressed as $\frac{5}{1}$, $\frac{0}{1}$, and $\frac{-9}{1}$.



0 is a special rational number.

It can be written as $0 = \frac{0}{q}$, where $q \neq 0$

$\mathbb{N} \subseteq \mathbb{W} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$

The set of rational numbers also includes terminating and repeating decimal numbers because terminating and repeating decimals can be expressed as fractions.

Example 1

(a) 0.3 can be written as $\frac{3}{10}$ [0.3 is a terminating decimal number and it is rational]

(b) 0.31 can be written as $\frac{31}{100}$ [0.31 is a terminating decimal number and it is rational]

(c) $0.333\ldots = 0.\bar{3}$ can be written as $\frac{1}{3}$ [$0.\bar{3}$ is a repeating decimal number since 3 repeats indefinitely; and it is rational]

Mixed numbers are also included in the set of rational numbers because any mixed number can be written as an improper fraction

Example 2

$$3\frac{2}{5} = \frac{(3 \times 5) + 2}{5} = \frac{17}{5}.$$

Addition and subtraction of rational numbers

Adding and subtracting rational numbers can be done in the same way as adding and subtracting fractions.

Case i) Adding or subtracting rational numbers with same denominators

To add or subtract two or more rational numbers with same denominators, we simply add or subtract all the numerators and write one of the common denominator.

Activity 2

1. Are there rational numbers between 0 and 1? __________
2. If yes, how many rational numbers are there between 0 and 1? ___________
3. Can you give just five rational numbers between 0 and 1? _________________

Example 3

Find the sum of

(a) $\frac{-2}{5} + \frac{3}{5}$.

Solution

Since the denominator is the same, you can simply add the numerators and take one of the common denominator, that is, $\frac{-2}{5} + \frac{3}{5} = \frac{-2+3}{5} = \frac{1}{5}$.

(b) Find the difference of $\frac{7}{8} - \frac{-9}{8}$.

Solution

$$\frac{7}{8} - \frac{-9}{8} = \frac{7-(-9)}{8} = \frac{7+9}{8} = \frac{16}{8} = 2.$$

If $\frac{c}{b}$ and $\frac{c}{b}$ are two rational numbers having the **same denominator** b ($b \neq 0$), then

- $\frac{a}{b} + \frac{c}{b} = \frac{a+b}{b}$
- $\frac{a}{b} - \frac{c}{b} = \frac{a-b}{b}$

Case ii) Adding or subtracting rational numbers with different denominators.

When rational numbers have different denominators, you can use any of the following methods to add or subtract them.

Method 1: *Make their denominators the same by finding an equivalent fraction.*

Example 4

Find the sum of

(a) $\frac{-5}{7} + \frac{2}{5}$

Solution

$$\frac{-5}{7} + \frac{2}{5} = \frac{-25}{35} + \frac{14}{35}$$

$\left[\text{Remember that } \frac{-5}{7} = \frac{-5 \times 5}{7 \times 5} = \frac{-25}{35} \text{ and } \frac{2}{5} = \frac{2 \times 7}{5 \times 7} = \frac{14}{35}\right]$

Since they have the same denominators now, you can apply the method used in case i) above. That is:

$$\frac{-5}{7} + \frac{2}{5} = \frac{-25}{35} + \frac{14}{35} = \frac{-25+14}{35} = \frac{-11}{8}.$$

(b) Find the difference of $\frac{23}{25} - \frac{1}{5}$.

Solution

$$\frac{23}{25} - \frac{1}{5} = \frac{23}{25} - \frac{5}{25} = \frac{23 - 5}{25} = \frac{18}{25}.$$

$$\left[\text{Note that } \frac{1}{5} = \frac{1 \times 5}{5 \times 5} = \frac{5}{25}\right]$$

Method 2: *Use cross products of the numerators and denominators like in the following examples.*

Example 5

Find the sum of

(a) $\frac{4}{6} + \frac{3}{7}$.

Solution

$$\frac{4}{6} + \frac{3}{7} = \frac{(4 \times 7) + (6 \times 3)}{6 \times 7} = \frac{28+18}{42} = \frac{46}{42} = \frac{23}{21}.$$

(b) Find the difference $\frac{3}{11} - \frac{2}{10}$.

Solution

$$\frac{3}{11} - \frac{2}{10} = \frac{(3 \times 10) - (11 \times 2)}{11 \times 10} = \frac{30 - 22}{110} = \frac{8}{100} = \frac{4 \times 2}{55 \times 2} = \frac{4}{55}.$$

[2 *is cancelled out by* 2]

In general

If $\frac{a}{b}$ and $\frac{c}{d}$ are any two rational numbers with $b \neq 0$ and $d \neq 0$, then

1. $\frac{a}{b} + \frac{c}{d} = \frac{(a \times d) + (b \times c)}{(b \times d)}$

2. $\frac{a}{b} - \frac{c}{d} = \frac{(a \times d) - (b \times c)}{(b \times d)}$

Example 6

Find the sum and difference of the following:

(a) $\frac{3}{4} + \frac{-3}{5} =$ ______

(b) $\frac{-32}{13} - \frac{-6}{5} =$ ______

(c) $\frac{-7}{27} + \frac{11}{18} + \frac{36}{243} =$ ______

(d) $\frac{-4}{7} - (-7) + \frac{-3}{7} =$ ______

(e) $-\left(2\frac{1}{2}\right) + -\left(1\frac{2}{3}\right) =$ ______

(f) $\frac{-3n}{8} + \frac{2n}{8} + \frac{n}{8} =$ ______ $(n \in \mathbb{W})$

Solution

(a) $\frac{3}{4} + \frac{-3}{5} = \frac{(3\times5)+(4\times-3)}{4\times5} = \frac{15+(-12)}{20} = \frac{3}{20}$.

(b) $\frac{-32}{13} - \frac{-6}{5} = \frac{(-32\times5)-(13\times-6)}{13\times5} = \frac{160-78}{65} = \frac{-238}{65}$.

(c) $\frac{-7}{27} + \frac{11}{18} + \frac{36}{243} = \frac{(-7\times18)+(27\times11)}{27\times18} + \frac{36}{243} = \frac{126+297}{486} + \frac{36}{243} = \frac{171}{486} + \frac{36}{243}$

$= \frac{171}{486} + \frac{72}{486} = \frac{171 + 72}{486} = \frac{243}{486} = \frac{1}{2}$.

(d) $\frac{-4}{7} - (-7) + \frac{-3}{7} = \frac{-4}{7} - \frac{-7}{1} + \frac{-3}{7} = \frac{(4\times1)-(7\times-7)}{7\times1} + \frac{-3}{7} = \frac{-4-(-49)}{7} + \frac{-3}{7}$

$= \frac{-4 + 49}{7} + \frac{-3}{7} = \frac{45}{7} + \frac{-3}{7} = \frac{45 - 3}{7} = \frac{42}{7} = 6$.

(e) $-\left(2\frac{1}{2}\right) + -\left(1\frac{2}{3}\right) = -\frac{5}{2} + -\frac{5}{3} = \frac{(-5\times3)+(2\times-5)}{2\times3} = \frac{(-15)+(-10)}{6} = \frac{-15-10}{6} = \frac{-25}{6}$.

(f) $\frac{-3n}{8} + \frac{2n}{8} + \frac{n}{8} = \frac{3n + 2n + n}{8} = \frac{3n + 3n}{8} = \frac{0}{8} = 0$.

A given rational number is in its simplest form if there is no any other number, except 1, that divides both the numerator and denominator.

That is $\frac{a}{b}$ $(b \neq 0)$ is in its simplest form if GCF $(a, b) = 1$

The simplest form of $\frac{20}{60} = \frac{(2 \times 2 \times 5)}{(2 \times 2 \times 3 \times 5)} = \frac{1}{3}$ [By factorization and cancellation]

Exercises

1. Add or subtract the following rational numbers with same denominators and write your answer in simplest form

(a) $\frac{5}{8}+\frac{-3}{8}=$ ____________

(b) $\frac{5}{4}-\frac{1}{4}=$ ____________

(c) $\frac{-3}{5}+\frac{3}{5}=$ ____________

(d) $\frac{2}{3}-\frac{-7}{3}=$ ____________

(e) $\frac{-6}{10}+\frac{-1}{10}+\frac{-3}{10}=$ ____________

(f) $\frac{22}{15}-1-\frac{2}{15}=$ ____________

(g) $\frac{31}{-5}+\frac{-5}{-5}+\frac{-6}{-5}=$ ____________

(h) $\frac{-15}{3}-\frac{-25}{3}-\frac{-2}{3}=$ ____________

(i) $\frac{1}{4}+\frac{1}{3}+\frac{3}{4}+\frac{-2}{6}=$ ____________

(j) $\frac{2}{5}+\frac{-3}{2}-\frac{3}{4}+\frac{1}{10}=$ ____________

2. Add and subtract the following rational numbers having different denominators and write your answer in simplest form

(a) $\frac{1}{3}+\frac{3}{4}=$ ____________

(b) $\frac{-6}{5}-\frac{-32}{13}=$ ____________

(c) $\frac{2}{3}+\frac{-5}{6}=$ ____________

(d) $\frac{-3}{5}-\frac{-8}{9}=$ ____________

(e) $\frac{31}{-4}+\frac{-5}{8}=$ ____________

(f) $\frac{-4}{7}-(-7)=$ ____________

(g) $-3+\frac{3}{5}+\frac{-5}{7}=$ ____________

(h) $0-\frac{-13}{9}-\frac{7}{10}=$ ____________

(i) $\frac{5m}{8}+\frac{-2m}{3}+\frac{m}{6}=$ ____ $(m \in \mathbb{W})$

(j) $\frac{2d}{3}+\frac{-2d}{6}-\frac{5d}{2}=$ ____ $(d \in \mathbb{W})$

3. Solve the following word problems

(a) The sum of two rational numbers is –2. If one of the numbers is $\frac{-14}{5}$, find the other number.

(b) The sum of two rational numbers is $\frac{-1}{2}$. If one of the numbers is $\frac{5}{6}$, find the other number.

(c) What number should be subtracted from $\frac{-2}{3}$ to get $\frac{-1}{6}$?

Properties of addition and subtraction on rational numbers

ACTIVITY 3

Discuss and answer the following questions:

1. Take any two rational numbers and add them. Is the sum always a rational number?
2. Take any two rational numbers and subtract one from the other. Is the difference always a rational number? _________
3. For any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$; is it always true that $\frac{a}{b}+\frac{c}{d}=\frac{c}{d}+\frac{a}{b}$? Is addition of rational numbers commutative? ___________________
4. For any three rational numbers $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$; is it always true that $\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f}=\frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right)$? Is subtraction of rational numbers associative? _______
5. What is the sum of any rational number and zero? ____________________
6. Given a rational number $\frac{a}{b}$, can you find another rational number that gives you zero when added with $\frac{a}{b}$? ___________

Properties of addition and subtraction on rational numbers

1. Closure property

(a) The sum of any two rational numbers is a gain a rational number. That is, for any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$ is a rational number. So we say, the set of rational numbers is closed under addition.

For example, $\frac{3}{4}$ and $\frac{-3}{5}$ are rational numbers. Their sum $\frac{3}{4}+\frac{-3}{5}=\frac{3}{20}$ is also a rational number.

(b) The difference of any two rational numbers is a gain a rational number. That is, for any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, $\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$ is a rational number. So we say, the set of rational numbers is closed under subtraction.

For example, $\frac{3}{4}$ and $\frac{-3}{5}$ are rational numbers. Their difference $\frac{3}{4}-\frac{-3}{5}=\frac{27}{20}$ is also a rational number.

2. Commutative property

For any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$; it is always true that $\frac{a}{b}+\frac{c}{d}=\frac{c}{d}+\frac{a}{b}$
Addition is commutative on the set of rational numbers.

Forexample, $\frac{45}{7}$ and $\frac{-3}{7}$ arerationalnumbers.Theirsum $\frac{45}{7}+\frac{-3}{7}=\frac{-3}{7}+\frac{45}{7}=\frac{42}{7}$. Commutative property of addition says; changing the order of addends does not change the sum.

Activity 4

Is subtraction commutative on the set of rational numbers? Why or Why not?

Subtraction is not commutative on the set of rational numbers. This is because for any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$;

$\frac{a}{b}-\frac{c}{d}\neq\frac{c}{d}-\frac{a}{b}$. For example,

$\frac{45}{7}-\frac{-3}{7}=\frac{48}{7}$ but $\frac{-3}{7}-\frac{45}{7}=\frac{48}{7}$. Since $\frac{48}{7}\neq\frac{-48}{7}$, subtraction is not commutative.

3. Associative property

For any three rational numbers $\frac{a}{b},\frac{c}{d}$ and $\frac{e}{f}$; it is always true that:
$\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f}=\frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right)$ [you can group the addends in any order and add them.]

Addition is associative on the set of rational numbers. For example, $\left(\frac{11}{18}+\frac{36}{243}\right)=\left(\frac{-7}{27}+\frac{11}{18}\right)+\frac{36}{243}=\frac{1}{2}$.

Is subtraction associative on the set of rational numbers? Why or why not?

Is $\frac{-3}{4}-\left(-\frac{5}{6}-\frac{2}{3}\right)=\left(\frac{-3}{4}-\frac{5}{6}\right)-\frac{2}{3}$?

$$\frac{-3}{4}-\left(-\frac{5}{6}-\frac{2}{3}\right)=\left[\frac{-3}{4}-\frac{(-5\times3)-(6\times2)}{6\times3}\right]=\frac{-3}{4}-\frac{(-15)-(12)}{18}=\frac{-3}{4}-\frac{-27}{18}$$

$$=\frac{(-3\times18)-(4\times-27)}{4\times18}=\frac{(-54)-(-108)}{72}=\frac{(-54)+108}{72}=\frac{54}{72}=\frac{6}{8}=\frac{3}{4}.$$

But $\left[\frac{-3}{4}-\left(-\frac{5}{6}\right)-\frac{2}{3}\right]=\left[\frac{(-3\times6)-(4\times-5)}{4\times6}-\frac{2}{3}\right]=\left[\frac{(-18)-(-20)}{24}-\frac{2}{3}\right]$

$$=\left[\frac{(-18)+20}{24}-\frac{2}{3}\right]=\frac{2}{24}-\frac{2}{3}=\frac{6-48}{72}=\frac{-42}{72}=\frac{-7}{12}.$$

Since $\frac{3}{4}\neq\frac{-7}{12}$, Subtraction is not associative on the set of rational numbers because the grouping will affect the result of the problem.

4. Additive Identity property

For any rational number $\frac{a}{b}, \frac{a}{b}+0=\frac{a}{b}$; When we add 0 to any rational number, the sum is always the number itself. So we say, 0 is the additive identity element in $\mathbb{Q}$.

5. Additive inverse property

For any rational number $\frac{a}{b}$, there is always another rational number $-\frac{a}{b}$, so that $\frac{a}{b}+\left(-\frac{a}{b}\right)=0$.

$-\frac{a}{b}$ ***is called the additive inverse of*** $\frac{a}{b}$.

For example,

The additive inverse of $\frac{-7}{27}$ is $\frac{7}{27}$ because $\frac{-7}{27}+\frac{7}{27}=0$.

The additive inverse of $\frac{1}{2}$ is $\frac{-1}{2}$ because $\frac{1}{2}+\frac{-1}{2}=0$.

The additive inverse of 5 is –5 because 5 + –5 = 0.

The additive inverse of 0 is 0 because 0 + 0 = 0. [0 is its own additive inverse]

The sum of a rational number and its additive inverse is always 0.

Exercises

1. Study the following arithmetic

   $(4-2)-3 \neq 4-(2-3)$

   $2-3 \neq 4-(-1)$

   $-1 \neq 5$

   What property of subtraction does the arithmetic in the left demonstrate?

   (a) That subtraction is not commutative
   (b) That subtraction is not associative
   (c) That subtraction is commutative

2. What property of addition is demonstrated by $\frac{3}{5}+\frac{-5}{7}=\frac{-5}{7}+\frac{3}{5}$?

   (a) The additive inverse property
   (b) The associative property
   (c) The additive identity property
   (d) The commutative property
   (e) The closure property

3. What property of addition is demonstrated by $\frac{3}{5}+0=\frac{3}{5}$?

   (a) The additive inverse property
   (b) The associative property
   (c) The additive identity property
   (d) The commutative property
   (e) The closure property

4. Which equation shows the associative property of addition?

   (a) $\frac{3}{5}+\frac{-3}{5}=0$
   (b) $\frac{3}{5}+\frac{-3}{7}=\frac{-5}{7}+\frac{3}{5}$
   (c) $\left(-3+\frac{3}{5}\right)+\frac{-5}{7}=-3+\left(\frac{3}{5}+\frac{-5}{7}\right)$
   (d) $0+\frac{-5}{7}=\frac{-5}{7}$

5. What property states that the sum and difference of any two rational numbers is always a rational number?

   (a) The closure property
   (b) The commutative property
   (c) The associative property
   (d) The additive identity property
   (e) The additive inverse property

6. Which of the following is the additive inverse of $\frac{2}{9}$?

(a) $\frac{9}{2}$ (b) $\frac{-2}{9}$ (c) $\frac{-9}{2}$ (d) 0

7. For any two rational numbers m and n, if $n + m = 0$, which of the following is always true?
   (a) m and n are additive inverses to each other
   (b) m and n are both equal to 0
   (c) One of them (either m or n) is equal to 0
   (d) n is the additive inverses of m but m is not the additive inverse of n.

Multiplication of rational numbers is similar to how we multiply fractions. To multiply any two rational numbers, say $\frac{a}{b}$ and $\frac{c}{d}$, we have to follow the following three simple steps.

- Multiply the numerators a and c
- Multiply the denominators b and d
- Reduce the resulting number to its lowest term

For any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$,

$$\frac{a}{b}\times\frac{c}{d}=\frac{a\times c}{b\times d}=\frac{ac}{bd}=\frac{\text{product of numerators}}{\text{product of denominators}}$$

Example 7

Find the product of the following

(a) $\frac{1}{3}\times\frac{3}{4}$

(b) $\frac{-4}{7}\times(-7)$

(c) $\frac{2}{3}\times\frac{-5}{6}$

(d) $\frac{-3}{5}\times\frac{8}{-9}$

(e) $-3\times\frac{3}{5}\times\frac{-5}{7}$

(f) $\frac{-2m}{3}\times\frac{m}{6}\times\frac{9}{m}$

Solution

(a) $\frac{1}{3}\times\frac{3}{4}=\frac{1\times 3}{3\times 4}=\frac{3}{12}=\frac{1}{4}$.

(b) $\frac{-4}{7}\times(-7)=\frac{-4}{7}\times\frac{-7}{1}=\frac{-4\times -7}{7\times 1}=\frac{28}{7}=4$.

(c) $\frac{2}{3}\times\frac{-5}{6}=\frac{2\times -5}{3\times 6}=\frac{-10}{18}=\frac{-5}{9}$.

(d) $\frac{-3}{5}\times\frac{8}{-9}=\frac{-3\times 8}{5\times -9}=\frac{-24}{-45}=\frac{8}{15}$.

(e) $-3\times\frac{3}{5}\times\frac{-5}{7}=\frac{-3}{1}\times\frac{3}{5}\times\frac{-5}{7}=\frac{-3\times 3\times -5}{1\times 5\times 7}=\frac{45}{35}=\frac{9}{7}$.

(f) $\frac{-2m}{3}\times\frac{m}{6}\times\frac{9}{m}=\frac{-2m\times m\times 9}{3\times 6\times m}=\frac{-18m^2}{18m}=\frac{-m}{1}=-m$. $(m \in \mathbb{N})$

NOTE: The Cancellation Method

Sometimes, it would be easier to reduce or simplify first before directly multiplying the numerators and denominators to avoid getting bigger numbers. Simplifying or reducing the numerators and denominators using their common factors is called the cancellation method.

$$\frac{27}{45}\times\frac{45}{1}=\frac{27}{\cancel{45}_{1}}\times\frac{\cancel{45}^{1}}{1}=\frac{27}{1}=27.$$

Example 8

Find the product using simplification method

(a) $\frac{35}{40}\times\frac{100}{1000}$ (b) $\frac{7}{6}\times\frac{3}{9}$ (c) $\frac{14}{25}\times\frac{10}{9}\times\frac{5}{6}$

Solution

(a) $\frac{35}{40}\times\frac{100}{1000}=\frac{\cancel{35}^{7}}{\cancel{40}_{8}}\times\frac{\cancel{100}^{1}}{\cancel{1000}_{10}}=\frac{7}{8}\times\frac{1}{10}=\frac{7}{80}$.

(b) $\frac{7}{6}\times\frac{3}{9}=\frac{7}{\cancel{6}_2}\times\frac{\cancel{3}^1}{9}=\frac{7}{18}$.

(c) $\frac{14}{25}\times\frac{10}{9}\times\frac{5}{6}=\frac{\cancel{14}^7}{\cancel{25}_5}\times\frac{\cancel{10}^2}{9}\times\frac{5}{\cancel{6}_3}=\frac{7}{\cancel{5}_1}\times\frac{2}{9}\times\frac{\cancel{5}^1}{3}=\frac{14}{27}$.

Exercises

1. Find the product of the following

(a) $\frac{2}{5}\times\frac{1}{8}=$ __________

(b) $\frac{-5}{9}\times(-9)=$ __________

(c) $\frac{4}{6}\times\frac{-7}{2}\times\frac{-6}{7}=$ __________

(d) $\frac{-3}{5}\times\frac{5}{-9}\times\frac{6}{2}=$ __________

(e) $\frac{-24}{5}\times\frac{15}{-16}\times\frac{-12}{-82}=$ _____

2. Find the product using simplification method

(a) $\frac{9}{10}\times\frac{2}{3}=$ __________

(b) $\frac{4}{5}\times\frac{2}{12}\times\frac{3}{6}=$ __________

(c) $\frac{6}{7}\times\frac{14}{5}\times\frac{15}{4}=$ __________

(d) $\frac{9}{16}\times\frac{20}{63}=$ __________

(e) $\frac{9}{16}\times\frac{20}{63}\times\frac{28}{5}=$ __________

(f) $\frac{26}{7}\times\frac{12}{20}\times\frac{9}{13}=$ __________

(g) $\frac{6}{18}\times\frac{9}{42}=$ __________

(h) $\frac{54}{50}\times\frac{2}{18}\times\frac{5}{3}=$ __________

(i) $\frac{28}{13}\times\frac{9}{6}\times\frac{13}{20}=$ __________

(j) $\frac{33}{7}\times\frac{14}{21}=$ __________

(k) $\frac{45}{6}\times\frac{3}{21}\times\frac{7}{3}=$ __________

(l) $\frac{9}{5}\times\frac{2}{8}\times\frac{15}{18}=$ __________

(m) $\frac{22}{15}\times\frac{45}{4}=$ __________

(n) $\frac{48}{45}\times\frac{15}{6}\times\frac{2}{27}=$ __________

(o) $\frac{7}{12}\times\frac{30}{11}\times\frac{22}{7}=$ ____________

3. Is it always true that for any three rational numbers $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$, $\frac{a}{b}\times\frac{c}{d}\times\frac{e}{f}=\frac{a\times c\times e}{b\times d\times f}=\frac{ace}{bdf}$? ——— [Remember! if numbers are rational, then their denominators are all non-zero]

Do you remember the properties of addition and subtraction on rational numbers? They are *closure*, *commutative*, *associative*, *identity* and existence of *inverse* properties. In this section, you will see if multiplication of rational numbers is valid on these properties or not. In addition you will also see one additional property which is called the distributive property.

1. Closure property

The product of any two rational numbers is again a rational number. That is, for any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$ is a rational number. So we say, the set of rational numbers is closed under multiplication.

For example, $\frac{-3}{4}$ and $\frac{5}{6}$ are rational numbers. Their product $\frac{-3}{4}\times\frac{5}{6}=\frac{-15}{24}$ is also a rational number.

2. Commutative property

For any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$; it is always true that $\frac{a}{b}\times\frac{c}{d}=\frac{c}{d}\times\frac{a}{b}$. Multiplication is commutative on the set of rational numbers.

For example, $\frac{-3}{4}$ and $\frac{5}{6}$ are rational numbers. Their product $\frac{-3}{4}\times\frac{5}{6}=\frac{5}{6}\times\frac{-3}{64}=\frac{-15}{24}$.

Changing the order of the factors does not change the product.

3. Associative property

For any three rational numbers $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$; it is always true that

$\left(\frac{a}{b}\times\frac{c}{d}\right)\times\frac{e}{f}=\frac{a}{b}\times\left(\frac{c}{d}\times\frac{e}{f}\right)$ [you can group the factors in any order and multiply them.]

Multiplication is associative on the set of rational numbers.

For example, $\frac{-1}{2}\times\left(\frac{3}{4}\times\frac{2}{5}\right)=\left(\frac{-1}{2}\times\frac{3}{4}\right)\times\frac{2}{5}=\frac{-6}{40}$.

4. Multiplicative Identity property

For any rational number $\frac{a}{b}$, $\frac{a}{b}\times1=\frac{a}{b}$; When we multiply any rational number by one, the product is always the number itself. So we say, 1 is the multiplicative identity element in $\mathbb{Q}$.

5. Multiplicative inverse property

For any non-zero rational number $\frac{a}{b}$, there is always another rational number $\frac{b}{a}$, so that $\frac{a}{b}\times\frac{b}{a}=1$. $\frac{b}{a}$ is called the additive inverse of $\frac{a}{b}$.

To find the multiplicative inverse of a non-zero rational number, we simply invert the fraction, that is, flip it over and get its reciprocal.

For example,

The multiplicative inverse of $\frac{7}{27}$ is $\frac{27}{7}$ because $\frac{7}{27}\times\frac{27}{7}=1$.

The multiplicative inverse of $\frac{1}{2}$ is $\frac{2}{1}$ because $\frac{1}{2}\times\frac{2}{1}=1$.

The multiplicative inverse of -5 is $\frac{-1}{5}$ because $-5\times\frac{-1}{5}=1$ [Remember! $\frac{-1}{5}=\frac{1}{-5}=-\frac{1}{5}$]

The multiplicative inverse of 1 is 1 because 1 × 1 = 1 [1 is its own multiplicative inverse]

What is the multiplicative inverse of 0?

0 has no multiplicative inverse because 0 multiplied by any number never gives you 1.

The product of a rational number and its multiplicative inverse is always 1.

6. Distributive Property

For any three rational numbers $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$; it is always true that

$\frac{a}{b}\times\left(\frac{c}{d}+\frac{e}{f}\right)=\left(\frac{a}{b}\times\frac{c}{d}\right)+\left(\frac{a}{b}\times\frac{e}{f}\right)$.

Multiplication is distributive over addition in the set of rational numbers.

For example, $\frac{-1}{2}\times\left(\frac{3}{4}+\frac{2}{5}\right)=\frac{-1}{2}\times\frac{15\times8}{20}=\frac{-1}{2}\times\frac{23}{20}=\frac{23}{40}$.

$\left(\frac{-1}{2}\times\frac{3}{4}\right)+\left(\frac{-1}{2}\times\frac{2}{5}\right)=\frac{-3}{8}+\frac{-2}{10}=\frac{(-30+(-16))}{80}=\frac{-46}{80}=\frac{23}{40}$.

Therefore, $\frac{-1}{2}\left(\frac{3}{4}+\frac{2}{5}\right)=\left(\frac{-1}{2}\times\frac{3}{4}\right)+\left(\frac{-1}{2}\times\frac{2}{5}\right)$.

7. Property of zero

The product of any rational number and zero is always zero.

If $\frac{m}{n}$ is any rational number, then $\frac{m}{n}\times0=0\times\frac{m}{n}=0$.

Exercises

1. What property is demonstrated in the arithmetic $\frac{2}{3}\times\frac{3}{2}=1$?
   - (a) The commutative property
   - (b) The associative property
   - (c) The distributive property
   - (d) The inverse property
2. What property is demonstrated in the arithmetic $\frac{2}{3}\times1=\frac{2}{3}$?
   - (a) The associative property
   - (b) The identity property
   - (c) The inverse property
   - (d) Property of zero
3. What property of multiplication is demonstrated by $\frac{3}{5}\times0=0$?
   - (a) The inverse property
   - (b) The associative property
   - (c) The identity property
   - (d) Property of zero
4. Which equation shows the associative property of multiplication?
   - (a) $\frac{3}{5}\times\frac{5}{3}=1$
   - (b) $\frac{3}{5}\times\frac{-5}{7}=\frac{-5}{7}\times\frac{3}{5}$
   - (c) $\left(-3\times\frac{3}{5}\right)\times\frac{-5}{7}=-3\times\left(\frac{3}{5}\times\frac{-5}{7}\right)$
   - (d) $1\times\frac{-5}{7}=\frac{-5}{7}$

5. What property states that the product of any two rational numbers is always a rational number?
   (a) The closure property
   (b) The commutative property
   (c) The associative property
   (d) The additive identity property
   (e) The additive inverse property
6. Which of the following is the multiplicative inverse of $\frac{7}{10}$?
   (a) $\frac{7}{10}$ (b) $\frac{-7}{10}$ (c) $\frac{10}{7}$ (d) $\frac{-10}{7}$
7. For any two rational numbers m and n, if $n \times m = 1$, which of the following is always true?
   (a) Either m or n is 1
   (b) m and n are multiplicative inverses to each other
   (c) m and n are both equal to 1
   (d) m and n have opposite signs
8. Which equation shows the distributive property of multiplication over addition?
   (a) $\frac{3}{5}\times\left(\frac{3}{4}+\frac{-5}{7}\right)=\left(\frac{3}{5}\times\frac{3}{4}\right)+\left(\frac{3}{5}\times\frac{-5}{7}\right)$
   (b) $\frac{3}{5}\times\left(\frac{3}{4}\times\frac{-5}{7}\right)=\left(\frac{3}{5}\times\frac{3}{4}\right)\times\left(\frac{3}{5}+\frac{-5}{7}\right)$
   (c) $\frac{3}{5}\times\left(\frac{3}{4}\times\frac{-5}{7}\right)=\left(\frac{3}{5}\times\frac{3}{4}\right)\times\frac{-5}{7}$
   (d) $\frac{3}{4}\times\frac{-5}{7}=\frac{-5}{7}\times\frac{3}{4}$

ACTIVITY 5

Answer the following:

1. $10 \div 2 = 10 \div \frac{2}{1} =$ ___ and $10 \times \frac{1}{2} =$ ___
2. $20 \div 4 = 10 \div \frac{4}{1} =$ ___ and $20 \times \frac{1}{4} =$ ___
3. $10 \div 5 = 10 \div \frac{5}{1} =$ ___ and $10 \times \frac{1}{5} =$ ___
4. $60 \div \frac{6}{2} =$ ___ and $60 \times \frac{2}{6} =$ ___
5. $\frac{m}{n} \div \frac{r}{k} =$ ___ $(n, r, k \neq 0)$

To divide a rational number by another rational number, we first recall that division is defined to be the inverse operation of multiplication. For example, dividing 60 by 3 is the same as multiplying 60 by the reciprocal of the denominator 3, that is $\frac{1}{3}$.

$60 \div 3 = 60 \div \frac{3}{1} = 60 \times \frac{1}{3} = \frac{60}{3} = \frac{20 \times 3}{3} = 20$.

For any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{a} = \frac{ad}{bc}$.

To divide $\frac{a}{b}$ by $\frac{c}{d}$, multiply $\frac{a}{b}$ by $\frac{d}{c}$, which is the reciprocal of $\frac{c}{d}$.

Remember! If a number is rational, its denominator is non-zero number.

Example 9

Find the quotient of the following:

(a) $\frac{5}{6} \div \frac{-3}{2} = \frac{5}{-6} \times \frac{2}{-3} = \frac{10}{18} = \frac{5}{9}$

(b) $\frac{3}{13} \div \frac{-4}{39} = \frac{3}{13} \times \frac{39}{-4} = \frac{9}{-4} = -\frac{9}{4}$

(c) $\frac{15}{50} \div \frac{15}{50} = \frac{15}{50} \times \frac{50}{15} = 1$

(d) $0 \div \frac{15}{50} = 0 \times \frac{50}{15} = 0$

(e) $\frac{5}{7} \div 1 = \frac{5}{7} \times \frac{1}{1} = \frac{5}{7}$

(f) $\frac{9}{8} \div \square = \frac{-3}{2}$, Find the unknown number.

Solution

Let the unknown rational number be $\frac{m}{n}$.

$$\frac{9}{8} \div \frac{m}{n} = \frac{-3}{2}$$

$$\frac{9}{8} \times \frac{n}{m} = \frac{9n}{8m} = \frac{-3}{2}$$

$$\frac{9n}{8m} \times \frac{8}{9} = \frac{-3}{2} \times \frac{8}{9}$$ *[Multiplying both sides by the same number]*

$$\frac{n}{m} = \frac{-3}{2} \times \frac{8}{9} = \frac{-24}{18}$$

Therefore, the unknown number $\frac{m}{n} = \square = -\frac{18}{24}$

Exercises

1. Find the quotient of the following:

(a) $\frac{-8}{25} \div \frac{16}{-5} =$ ______

(b) $\left(\frac{4}{5} \surd \frac{12}{2}\right) \surd \frac{6}{3} =$ ______

(c) $\frac{3}{10} \surd \frac{6}{7} =$ ______

(d) $\frac{7}{6} \surd \left(\frac{5}{14} \surd \frac{15}{4}\right) =$ ______

(e) $\frac{5}{11} \surd \frac{7}{5} =$ ______

(f) $\left(\frac{26}{7} \surd \frac{13}{20}\right) \surd \frac{10}{9} =$ ______

(g) $\frac{-5}{8} \div \left(-\frac{5}{16}\right) =$ ______

(h) If $\frac{2}{3} \div k = \frac{-5}{9}$, then $k =$ ___

(i) $-6 \div \left(\frac{-3}{4}\right) =$ ______

(j) If $m \div \frac{8}{-9} = \frac{8}{15}$, then $m =$ ___

Properties of division of rational numbers

Activity 6

1. If we divide any rational number by any other rational number, the result is always a rational number [True or False?] ________
2. Is $\frac{3}{8} \div \frac{1}{4} = \frac{1}{4} \div \frac{3}{8}$ true? ________ Is division commutative on the set of rational numbers? ________
3. Is $\left(\frac{4}{5} \div \frac{12}{2}\right) \div \frac{6}{3} = \frac{4}{5} \div \left(\frac{12}{2} \div \frac{6}{3}\right)$ true? ________ Is division associative ________?
4. If you divide any number by 1, the result is ________
5. If you divide any non-zero number by itself, the result is ________

NOTE

1. **The set of rational numbers is <u>not closed</u> under division**. This is true only because of the rational number 0. Since $\frac{n}{0}$ is not defined for any number n, we cannot safely say that the quotient of any two rational numbers is always rational.

   But if we exclude 0 from the set of rational numbers, we get a set $\mathbb{Q}\backslash\{0\}$. This set includes all rational numbers except 0. So $\mathbb{Q}\backslash\{0\}$ is closed under division.

The collection of non-zero rational numbers is closed under division.

2. **Division is not commutative** because for any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$;

$$\frac{a}{b} \div \frac{c}{d} \neq \frac{c}{d} \div \frac{a}{b}$$

EXAMPLE 10

$$\frac{3}{8} \div \frac{1}{4} = \frac{3}{8} \times \frac{4}{1} = \frac{3}{2}$$

But $\frac{1}{4} \div \frac{3}{8} = \frac{1}{4} \times \frac{8}{3} = \frac{2}{3}$ Since $\frac{3}{2} \neq \frac{2}{3}, \frac{3}{8} \div \frac{1}{4} \neq \frac{1}{4} \div \frac{3}{8}$

3. **Division is not associative** because for any three rational numbers $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$

$$\left(\frac{a}{b} \div \frac{c}{d}\right) \div \frac{e}{f} \neq \frac{a}{b} \div \left(\frac{c}{d} \div \frac{e}{f}\right)$$

For example, $\frac{-1}{2} \div \left(\frac{3}{4} \div \frac{2}{5}\right) = \left(\frac{-1}{2}\right) \div \left(\frac{3}{4} \times \frac{5}{2}\right) = \frac{-1}{2} \div \frac{15}{8} = \frac{-1}{2} \times \frac{8}{15} = \frac{-4}{15}$

But $\left(\frac{-1}{2} \div \frac{3}{4}\right) \div \frac{2}{5} = \left(\frac{-1}{2} \times \frac{4}{3}\right) \div \frac{2}{5} = \frac{-4}{6} \times \frac{5}{2} = \frac{-10}{6} = \frac{-5}{3}$

Since $\frac{-4}{15} \neq \frac{-5}{3}$, $\frac{-1}{2} \div \left(\frac{3}{4} \div \frac{2}{5}\right) \neq \left(\frac{-1}{2} \div \frac{3}{4}\right) \div \frac{2}{5}$

4. **Property of 1 on Division of Rational Numbers**

(a) For every rational number $\frac{m}{n}$, $\frac{m}{n} \div 1 = \frac{m}{n}$, [Any rational number divided by 1 is the number itself]

(b) For every non zero rational number $\frac{m}{n} \surd \left(\frac{m}{n}\right) \surd \frac{m}{n}$, = 1 [Any non-zero rational number divided by itself is equal to 1]

What are decimal numbers?

Decimals numbers are numbers lying between two integers on a number line. They are just other ways of representing rational numbers in mathematics. With the help of decimals, we can write more precise values of measurable quantities like length, weight, distance, money, etc. For example, we may say

- A full marathon distance is equal to 42.195 km.
- The body weight/mass of a person is 78.5 kg.
- Half of 1 in a decimal number = 0.5

Such numbers are called **decimal numbers**. The **dot** presented between the integer to the left of the dot and the fractional part to the right of the dot is called the **decimal point**.

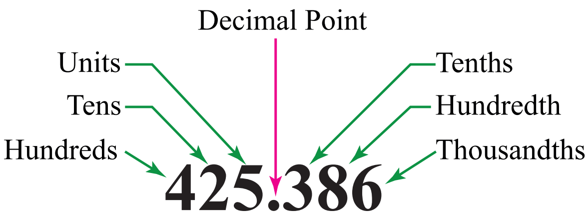
For example, 42**.**195 is a decimal number. Here, 42 is an integer part and 195 is the fractional part which is always less than 1. [**.**195 = 0**.**195]

"**.**" is the decimal point.

Place value of decimals

Study the place values of each digit in the number **425.386**

When we are going towards the left of the decimal point, each place is ten times greater than the previous place value. So, to the right of the decimal point, we have **tenths** $\left(\frac{1}{10}\right)$, to the right of tenths, we have **hundredths** $\left(\frac{1}{100}\right)$ and to the right of hundredths, we have **thousandths** $\left(\frac{1}{1000}\right)$ and so on.



- 4 is in the **hundreds** place. Its value is *4 hundred* = 400
- 2 is in the **tens** place. Its value is *2 ten* = 20
- 5 is in the **ones** place. Its value is *5 ones* = 5
- 3 is in the ***tenths*** place. Its value is *3 tenths* = $\frac{3}{10}$ = 0.3

- 8 is in the ***hundredths*** place. Its value is *8 hundredths* = $\frac{8}{100}$ = 0.08
- 6 is in the ***thousandths*** place. Its value is *6 thousandths* = $\frac{6}{1000}$ = 0.006

So $425.386 = 425 + \frac{3}{10} + \frac{8}{100} + \frac{6}{1000}$

$= 425 + 0.3 + 0.08 + 0.006$

$$\begin{array}{r} 425.000 \\ 0.300 \\ 0.080 \\ \underline{0.006} \\ \underline{\underline{425.386}} \end{array}$$

Reading Decimal Numbers

The decimal number **425.386** is read as

- *Four hundred twenty five point three-eight-six or*
- *Four hundred twenty five and three hundred eighty six* ***thousandths***

Addition and subtraction of decimals

To add or subtract decimals:

1. Line up the decimal points vertically. (Fill in any 0's as a place holder where necessary)
2. Add or subtract the numbers as if they were whole numbers.
3. Place the decimal point in the sum or difference so that it lines up vertically with the numbers being added or subtracted

EXAMPLE 11

A Father and his daughter recently got weighed. He weighed 97.9 kilograms. His daughter weighed 46. 80 kilograms

(a) How much did both weigh?

(b) How many less kilograms does the daughter weigh?

Solution

(a) Add the weights of the two. 97.9 + 46.80 = ____________

$$\begin{array}{r} 97.6 \\ +46.80 \\ \hline \end{array} \longrightarrow \begin{array}{r} \overset{1}{9}\overset{1}{7}.60 \\ +46.80 \\ \hline 144.40 \\ \hline\hline \end{array}$$

- Note that 97.6 = 97.60; so you can add **0** as a place holder.
- The decimal points are correctly arranged or lined up

So both the father and the daughter weigh 144.4kg

(b) Subtract the weight of the daughter from that of the father.
97.9 – 46.80 = __________

$$\begin{array}{r} 97.6 \\ -46.80 \\ \hline \end{array} \longrightarrow \begin{array}{r} 9\overset{6}{\not 7}.\not 6\overset{16}{0} \\ -46.80 \\ \hline 50.80 \\ \hline\hline \end{array}$$

To add or subtract decimals just use the same method of adding or subtracting whole numbers. Do not forget to put the decimal point in the answer

Example 12

(a) Find the sum of the following decimals.

(i) 0.6 + 0.2 = ___ (ii) 1.5 + 0.4 = ___ (iii) 2.4 + 1.04 = ___

Solution

(i) $\begin{array}{r} 0.6 \\ +0.2 \\ \hline 0.8 \\ \hline\hline \end{array}$ (ii) $\begin{array}{r} 1.5 \\ +0.4 \\ \hline 1.9 \\ \hline\hline \end{array}$ (iii) $\begin{array}{r} 2.40 \\ +1.04 \\ \hline 3.44 \\ \hline\hline \end{array}$

(b) Find the difference of the following decimals.

(i) 0.9 – 0.3 = ___ (ii) 5.7 – 1.2 = ___ (iii) 15.35 – 2 = ___

Solution

(i) $\begin{array}{r} 0.9 \\ -0.3 \\ \hline 0.6 \\ \hline\hline \end{array}$ (ii) $\begin{array}{r} 5.7 \\ -1.2 \\ \hline 4.5 \\ \hline\hline \end{array}$ (iii) $\begin{array}{r} 15.35 \\ -02.00 \\ \hline 13.35 \\ \hline\hline \end{array}$

Example 13

Add the following decimals with carrying.

(i) 12.6 + 3.75 = ___ (iii) 45.7 + 15.345 = ___

(ii) 125.87 + 15.5 = ___

Solution

(i)
$$\begin{array}{r} 12.60 \\ +\underline{03.75} \\ \underline{\underline{16.35}} \end{array}$$

(ii)
$$\begin{array}{r} 125.87 \\ +\underline{015.50} \\ \underline{\underline{141.37}} \end{array}$$

(iii)
$$\begin{array}{r} 45.700 \\ +\underline{15.345} \\ \underline{\underline{61.045}} \end{array}$$

Example 14

Subtract the following decimals with borrowing.

(i) 7.5 – 0.9 = ____

(ii) 23.4 – 1.75 = ____

(iii) 45.25 – 35.5 = ____

Solution

(i)
$$\begin{array}{r} \overset{6}{\not7}.\overset{15}{\not5} \\ -\underline{0.9} \\ \underline{\underline{6.6}} \end{array}$$

(ii)
$$\begin{array}{r} 2\overset{2}{\not3}.\overset{13}{\not3}\overset{10}{\not0} \\ -\underline{01.75} \\ \underline{\underline{21.65}} \end{array}$$

(iii)
$$\begin{array}{r} \overset{3}{\not4}\overset{14}{\not5}.\overset{12}{\not2}5 \\ -\underline{35.50} \\ \underline{\underline{09.75}} \end{array}$$

Exercises

1. Add the following decimals:

(a) 0.1 + 0.3 = ____________

(b) 0.7 + 0.2 = ____________

(c) 2.2 + 1.2 = ____________

(d) 3.05 + 1.50 = ____________

(e) 3.21 + 1.5 = ____________

(f) 5.45 + 0.23 = ____________

(g) 2.54 + 1.3 = ____________

(h) 20.05 + 1.5 = ____________

(i) 0.047 + 2.99 = ____________

(j) 6.77 + 1.66 = ____________

(k) 5.645 + 1.95 = ____________

(l) 1.09 + 35.963 = ____________

2. Subtract the following decimals:

(a) 0.5 – 0.3 = ____________

(b) 0.7 – 0.2 = ____________

(c) 2.2 – 1.2 = ____________

(d) 3.35 – 1.20 = ____________

(e) 3.51 – 1.5 = ____________

(f) 5.45 – 0.23 = ____________

(g) 2.54 – 1.3 = ____________

(h) 20.05 – 1 = ____________

(i) 5.047 – 2.99 = ____________

(j) 6.33 – 1.66 = ____________

(k) 5.645 – 1.95 = ____________

(l) 27.02 – 18.963 = ____________

3. Solve the following word problems:
   (a) Three pens cost 4.50, 7.25 and 23.5 LRD respectively. How much money do all the three cost? ____________________
   (b) A man walked 4.5 km on the first day, 5.2 km on the second day, and 6.2 km on the third day. How many km did the man walk in the three days? ____________________
   (c) You have 2.6 grams of yogurt in your bowl and you add another spoonful of 1.3 grams. How much yogurt do you have in total? ________

Multiplication and division of decimals

A. Multiplication of decimals

The technique of multiplying decimals is simple. It has the same procedure as that of multiplication of whole numbers, except the placement of the decimal point in the product. We multiply the given decimal numbers just considering them as whole numbers and then put the decimal point in the appropriate place in the answer. The product will have as many decimal places as the decimal places of the two original numbers combined.

Activity 7

1. Imagine going out with your friend for lunch. You both order a lunch costing L$6.75 each. How much should you pay for both of you? Calculate the total cost of the lunch by multiply L$6.75 by 2.
2. The length and width of a rectangular table is 1.25 m and 0.8 m, respectively. What is the area of the top surface of the table?

To multiply decimals:

1. Ignore the decimal point and perform the multiplication as usual. (Just multiply like whole numbers)
2. Count number of decimal places after the decimal point in both of the given decimal numbers.
3. Put the decimal point in the product after counting from right to left the same number of decimal places as in the given.

Example 15

Find the product of the following

(a) $4.6 \times 10 =$ ________________

(b) $4.65 \times 100 =$ ______________

(c) $4.653 \times 1000 =$ ____________

(d) 2.5 × 3.4 = ________________

(e) 3.12 × 1.5 = _______________

(f) 4.003 × 1.05 = _____________

Solution

(a) 4.6 × 10 [Ignore the decimal point and multiply 46 by 10]

46 × 10 = 460

4.6 has only one decimal place after the decimal point and 10 has no decimal place. So count one decimal place from right to left in the product and put the decimal point. That is, 460 will be 46.0, which is the same as 46.

Therefore 4.6 × 10 = 46.0 = 46

(b) 4.65 × 100 [Ignore the decimal point and multiply 465 by 100]

465 × 100 = 46500

4.65 has two decimal places after the decimal point and 100 has no decimal place. So count two decimal places from right to left in the product and put the decimal point. That is, 46500 will be 465.00, which is the same as 465

Therefore 4.65 × 100 = 465.00 = 465.

(c) 4.653 × 1000 [Ignore the decimal point and multiply 4653 by 1000]

4653 × 1000 = 4653000

4.653 has three decimal places after the decimal point and 1000 has no decimal place. So count three decimal places from right to left in the product and put the decimal point. That is, 4653000 will be 4653.000, which is the same as 4653.

Therefore 4.653 × 1000 = 4653.000 = 4653.

(d) 2.5 × 3.4 [Ignore the decimal points and multiply 25 by 34]

25 × 34 = 850

2.5 has one decimal place and 3.4 has also one decimal place after the decimal point. So count two decimal places from right to left in the product and put the decimal point. That is, 850 will be 8.50.

Therefore 2.5 × 3.4 = 8.50.

(e) 3.12 × 1.5 [Ignore the decimal points and multiply 312 by 15]

312 × 15 = 4680

3.12 has two decimal places and 1.5 has has one decimal place after the decimal point. So count three decimal places from right to left in the product and put the decimal point. That is, 4680 will be 4.680.

Therefore 3.12 × 1.5 = 4.680.

(f) 4.003 × 1.05 [Ignore the decimal points and multiply 4003 by 105]

4003 × 105 = 420,315

4.003 has three decimal places and 1.05 has two decimal place after the decimal point. So count five decimal places from right to left in the product and put the decimal point. That is, 420315 will be 4.20315.

Therefore 4.003 × 1.05 = 4.20315.

Exercises

1. Find the product of the following decimals
   (a) 0.1 × 0.3 = ________
   (b) 0.7 × 0.2 = ________
   (c) 2.2 × 1.2 = ________
   (d) 3.05 × 1.50 = ________
   (e) 3.21 × 1.5 = ________
   (f) 5.45 × 0.23 = ________
   (g) 2.54 × 1.3 = ________
   (h) 20.05 × 1.5 = ________
2. Suppose you plan to gift your mother a flower bouquet on her birthday. Each flower costs L$0.75 and you buy a total of 6 flowers. Find the total cost of the bouquet. ________
3. A girl went to the grocery store with her mother. Her mother bought 15 apples costing L$1.25 each. Help the girl to calculate the amount her mother needs to pay by using the concept of multiplying decimals.
4. A woman went to the grocery shop to buy flour for preparing cookies. She bought 17.75 lbs of flour. If price for 1 lb is L$19.68, calculate the total amount she paid to the shopkeeper.

B. Division of Decimals

The process of dividing decimals is similar to the normal division process, but we just need to keep in mind the decimal point which should be correctly placed in the quotient. Since we are referring to decimals, the following steps will help us to understand the process of dividing decimals.

Activity 8

1. 15.75 kg of potatoes are needed for making 7 batches of chips. How much kg of potato are needed just to make one batch of chips?
2. The weight of one marble is 0.8 grams. How many marbles weigh a total of 376 grams?

1. **To divide a decimal number by a whole number,**
   - Perform the division exactly the same way as in the whole numbers ignoring the decimal point.

- Put the decimal point in the quotient in the same position as in the dividend.

Example 16

Divide 18.2 by 2

Solution

Ignore the decimal point and divide 182 by 2

$182 \div 2 = 91$

Since the dividend 18.2 has one decimal place after the decimal point, the answer will also have only one decimal place after decimal point. That is, $18.2 \div 2 = 9.1$

Example 17

Find the quotient $78.92 \div 4$

Solution

Ignore the decimal point and perform the division $7892 \div 4$

$7892 \div 4 = 1973$

Since the dividend 78.92 has two decimal places after the decimal point, the answer will also have only two decimal places after decimal point, that is, $78.92 \div 4 = 19.73$

2. **To divide a decimal number by a decimal number,**
   - Multiply both the divisor and the dividend by 10 or 100 or 1000 or by any multiple of these to convert both the divisor and the dividend to a whole number.
   - Divide the converted whole numbers using long division until you get the answer.

Example 18

Find the quotient of $2.5 \div 0.5$

Solution

Divide both decimals by 10 to convert them to whole numbers

$2.5 \times 10 = 25$

$0.5 \times 10 = 5$

Now divide the two whole numbers, that is $25 \div 5 = 5$

Therefore, $2.5 \div 0.5 = 25 \div 5 = \mathbf{5}$

Example 19

Find the quotient of 15.75 ÷ 3.5

Solution

15.75 × 10 = 157.5 and 3.5 × 10 = 35. So multiplying both by 10 will not change 15.75 to a whole number. Therefore we have to multiply both by 100.

If we multiply 15.75 by 100, we get 15.75 × 100 = 1575 which is a whole number.

If we multiply 3.5 by 100, we get 3.5 × 100 = 350 which is a whole number.

Now divide the two whole numbers.

```
        4.5
350 )1575
    −1400
      1750
    − 1750
      0000
```

Therefore, 15.75 ÷ 3.5 = 1575 ÷ 350 = 4.5

Example 20

Find the quotient of 7.205 ÷ 0.05

Solution

Multiply both decimals by 1000 to change them to whole numbers

7.205 × 1000 = 7205.000 = 7205

0.05 × 1000 = 50

Now divide the two whole numbers using long division: 7205 ÷ 50 = 144.1

Therefore, 7.205 ÷ 0.05 = 7205 ÷ 50 = 144.1

Exercises

1. Find the quotient of the following decimals
   (a) 0.42 ÷ 7 = ____________
   (b) 5.6 ÷ 8 = ____________
   (c) 23.95 ÷ 5 = ____________
   (d) 1.437 ÷ 2 = ____________
   (e) 25.5 ÷ 1.25 = ____________
   (f) 77.4 ÷ 0.03 = ____________
   (g) 69.27 ÷ 0.3 = ____________
   (h) 18.69 ÷ 0.005 = ____________
   (i) 1.329 ÷ 0.06 = ____________

2. Three friends want to share equally a lunch bill of L$66.24. How much does each person need to contribute? ___________
3. The area of a rectangular plot of land having a length of 42.5 m is 658.75 m^2. What is the width of the land? __________________

C. Converting fractions to decimals

To convert a fraction of the form $\frac{a}{b}$, where $b \neq 0$, in to a decimal, just divide the numerator by the denominator using long division method.

Example 21

$$
\begin{array}{r}
0.375 \\
8\overline{)3.000} \\
-24 \\
\hline
60 \\
-56 \\
\hline
40 \\
-40 \\
\hline
0
\end{array}
\qquad\qquad
\begin{array}{r}
0.333... \\
3\overline{)1.0000} \\
9 \\
\hline
10 \\
9 \\
\hline
10 \\
9 \\
\hline
1
\end{array}
$$

Therefore, $\frac{3}{8} = 0.375$ and $\frac{1}{3} = 0.333\ldots$

Observe that the process of dividing 3 by 8 terminated three digits after the decimal point when the remainder becomes 0. Such a decimal number like 0.375 is called a terminating decimal.

The fraction $\frac{1}{3} = 0.333\ldots$ The process of dividing 1 by 3 doesn't stop or terminate and the digit 3 after the decimal point repeats itself indefinitely. Such decimal numbers like 0.333 . . . is called a repeating decimal.

To show a repeating digit or a block of repeating digits in a repeating decimal number, we put a bar " – "above the repeating digit (or block of digits). For example 0.333 . . . can be written as $0.\overline{3}$ And $1.\overline{25}$ Means the two digits 2 and 5 repeat without end

D. Converting decimals to fractions of the form $\frac{a}{b}$, where $b \neq 0$

To convert a terminating decimal,

- Write down the decimal divided by 1, like this: $\frac{decimal}{1}$
- If the decimal number has one or two or three etc., digits after the decimal point, multiply the numerator and the denominator by 10 or 100 or 1000 etc, respectively.
- Simplify (or reduce) the fraction.

EXAMPLE 22

Convert the following terminating decimals to fraction

(a) 1.5 (b) 0.75 (c) 0.625 (d) 0.2319

Solution

(a) First write 1.5 as $\frac{1.5}{1}$

Since 1.5 has one decimal place after the decimal point, multiply both the numerator and denominator by 10

$\frac{1.5 \times 10}{1 \times 10} = \frac{15}{10}$ So $1.5 = \frac{15}{10}$

(b) First write 0.75 as $\frac{0.75}{1}$

Since 0.75 has two decimal places after the decimal point, multiply both the numerator and denominator by 100

$\frac{0.75 \times 100}{1 \times 100} = \frac{75}{100}$ So $0.75 = \frac{75}{100}$

(c) First write 0.625 as $\frac{0.625}{1}$

Since 0.625 has three decimal places after the decimal point, multiply both the numerator and denominator by 1000

$\frac{0.625 \times 1000}{1 \times 1000} = \frac{625}{1000}$ So $0.625 = \frac{625}{1000}$

(d) First write 0.2319 as $\frac{0.2319}{1}$

Since 0.2319 has four decimal places after the decimal point, multiply both the numerator and denominator by 10,000

$\frac{0.2319 \times 10{,}000}{1 \times 10{,}000} = \frac{2319}{10{,}000}$ So $0.2319 = \frac{2319}{10{,}000}$

If d is a terminating decimal having k terminating digits after the decimal point, then the fractional form of d is given by the formula

$$d = \frac{d \times 10^k}{1 \times 10^k}$$, the right side is the fractional part of the decimal d.

EXAMPLE 23

If $d = 0.301$, then $k = 3$

Use the formula $d = \frac{d \times 10^k}{1 \times 10^k}$ $\quad 0.301 = \frac{0.30 \times 10^3}{1 \times 10^3} = \frac{0.301 \times 1000}{1 \times 1000} = \frac{301}{1000}$

EXERCISES

Concert the following terminating decimals to fraction

(a) 0.3
(b) 1. 4
(c) 0.34
(d) 3.56
(e) 1.001
(f) 0.010
(g) 4. 4444
(h) 0.9876
(i) 5.50502
(j) 90.31042

Converting repeating decimals to fractions of the form $\frac{a}{b}$, Where $b \neq 0$

Decimal numbers like $0.\overline{3}, 1.\overline{25}, 0.2\overline{375}$ are called repeating decimals. They are all rational numbers. That means they can be converted to a fraction of the form $\frac{a}{b}$, where $b \neq 0$

EXAMPLE 24

Convert the decimal $0.\overline{3}$ to a fraction

Solution

Let $d = 0.333\ldots = 0.\overline{3}$

$10d = 3.333\ldots$ [Since only one digit, that is 3, repeats we multiply both sides by 10]

$\underline{d = 0.333}\ldots$ [Subtract d from $10d$]

$9d = 3$

$\frac{9d}{9} = \frac{3}{9}$ [Divide both sides by 9 to solve for d]

$d = \frac{3}{9} = \frac{1}{3}$ Therefore $0.\overline{3} = \frac{1}{3}$ [Note that $\frac{1}{3} \in \mathbb{Q}$ implies $0.\overline{3} \in \mathbb{Q}$]

EXAMPLE 25

Convert the decimal $1.\overline{25}$ to a fraction

Solution

Let $d = 1.252525\ldots = 1.\overline{25}$

$100d = 125.2525\ldots$ [Since two digits, that is 2 and 5, repeat we multiply both sides by 100]

$\underline{d\ =\ \ \ \ 1.2525}\ldots$ [Subtract d from $100d$]

$99d = 124$

$$\frac{99d}{99} = \frac{124}{99}$$ [Divide both sides by 99 to solve for d]

$d = \frac{124}{99}$ Therefore $1.\overline{25} = \frac{124}{99}$

EXAMPLE 26

Convert the decimal $02.\overline{375}$ to a fraction

Solution

Let $d = 0.2375375\ldots = 02.\overline{375}$

Note that in $02.\overline{375}$, the digit 2 is not repeating and 3, 7 and 5 are repeating. Therefore there is 1 non repeating and 3 repeating, totally 4 digits after the decimal point]

$10{,}000d = 2375.375\ldots$ [Since there are 4 digits in total we multiply both sides by 10000]

$\underline{10d =\ \ \ \ 2.375}\ldots$ [Subtract d from $100d$]

$9990d = 2373$

$$\frac{9990d}{9990} = \frac{2373}{9990}$$ [Divide both sides by 9990 to solve for d]

$d = \frac{2373}{9990}$ Therefore $02.\overline{375} = \frac{2373}{9990}$

If d is a repeating decimal having k terminating and p repeating digits after the decimal point, then the fractional form of d is given by the formula

$d = \dfrac{d(10^{k+p} - 10^{k})}{(10^{k+p} - 10^{k})}$, the right side is the fractional part of the repeating decimal d.

EXAMPLE 27

Convert the decimal $2.4\overline{01}$ to a fraction

Solution

$d = 2.4\overline{01}$ $\quad k = 1$, $\quad p = 2$, $\quad k + n = 1 + 2 = 3$

Use the formula $d = \dfrac{d(10^{k+p} - 10^{k})}{(10^{k+p} - 10^{k})}$

$$2.4\overline{01} = \frac{2.4\overline{01}(10^{1+2} - 10^{1})}{(10^{1+2} - 10^{1})} = \frac{2.4\overline{01}(10^{3} - 10^{1})}{(10^{3} - 10^{1})} = \frac{2.4\overline{01}(1000 - 10)}{(1000 - 10)}$$

$$= \frac{(2.4\overline{01} \times 1000) - (2.4\overline{01} \times 10)}{990} = \frac{(2.401.0101...) - (24.0101...)}{990} = \frac{2377}{990}.$$

Our discussion of decimals so far can be summarized as follows

- Every rational number can be expressed as either terminating or repeating decimal
- Every terminating or repeating decimal is a rational number

EXERCISES

Concert the following repeating decimals to fraction

(a) $0.\overline{5}$	(c) $0.\overline{43}$	(e) $1.3\overline{03}$	(g) $0.25\overline{47}$
(b) $1.\overline{2}$	(d) $1.3\overline{18}$	(f) $1.25\overline{1}$	(h) $0.16\overline{029}$

In your discussion of decimals, you observed that every rational number is either a terminating decimal or a repeating decimal. Also, any terminating or repeating decimal is a rational number. However there are decimal numbers which are not terminating and which are also not repeating. That means there are decimal numbers which are neither terminating nor repeating.

Carefully study the digits of the following decimal number: $2.030030003000030000003\ldots$

Which digits regularly repeat themselves?

Can we rewrite it as $2.\overline{03}$? No because $2.\overline{03} = 2.030303\ldots$ [It is repeating]

Can we rewrite it as $2.\overline{030}$? No because $2.\overline{030} = 2.030030030\ldots$ [It is repeating]

Can we rewrite it as $2.\overline{0300}$? No because $2.\overline{0300} = 2.03000300\ldots$ [It is repeating]

Therefore, the number 2.03003000300003000003. . . is not repeating and also not terminating. Such a decimal number is NOT rational. It is called an irrational number.

The square roots of square numbers like $\sqrt{4}, \sqrt{9}, \sqrt{16}, \sqrt{25}, \sqrt{0.04}, \sqrt{0.09}$ etc are all rational numbers since $\sqrt{4}=2$ $\sqrt{9}=3$ $\sqrt{16}=4$ $\sqrt{25}=5$ $\sqrt{0.04}=0.2$ $\sqrt{0.09}=0.3$

But square root of non-square numbers like $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}$ etc are not rational. They are irrational numbers

The well-known number π which is the ratio of the circumference of any circle to its diameter is also an irrational number. $\pi = 3.1415926536\ldots$

In general, if n is a natural number that is not a perfect square, then $\sqrt{n}$ is an irrational number

DEFINITION

An irrational number is a number that cannot be expressed in the form of $\frac{a}{b}$, Where ***a*** and b are integers and $b \neq 0$

Any number that is not rational is irrational

The set of irrational numbers is denoted by $\mathbb{Q}'$

Remember that $\mathbb{Q}$ and $\mathbb{Q}'$ are disjoint or non-intersecting sets, that is, there is no any number which is both rational and irrational. $\mathbb{Q} \cap \mathbb{Q}' = \varnothing$

Real numbers

DEFINITION

A number is called a real number, if and only if it is either a rational number or an irrational number

The set of real numbers denoted by the symbol $\mathbb{R}$ is the union of the sets of rational and irrational numbers

$\mathbb{R} = \{ x | x$ is a rational numbers or an irrational number$\}$

$\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$

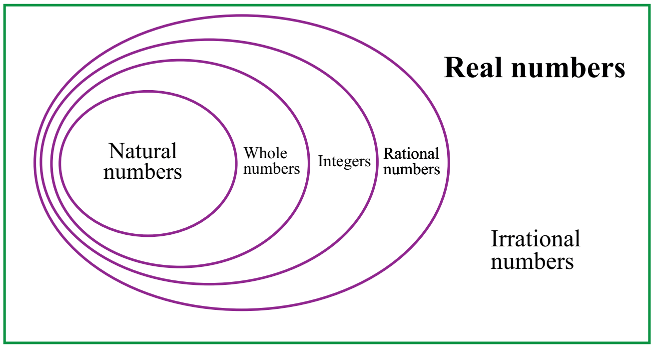
The set of **real numbers** is simply the set of all rational numbers combined with the set of all irrational numbers. Therefore, all the numbers defined so far are subsets of the set of real numbers

In summary, the relationship among the different sets of numbers is seen in the Venn diagram below

$$\mathbb{N} \subseteq \mathbb{W} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

$$\mathbb{Q}' \subseteq \mathbb{R}$$

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$$

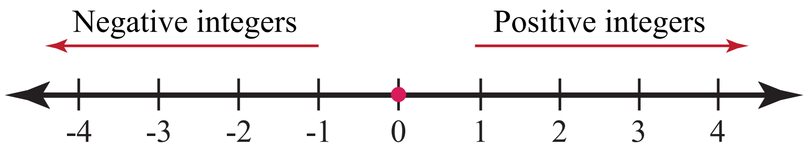


Exercises

1. Which of the following statement is false?
   (a) All integers are real numbers
   (b) All irrational numbers are real numbers
   (c) All rational numbers are real numbers
   (d) All irrational numbers are rational
2. Which statement is true?
   (a) $\mathbb{Z} \cup \mathbb{Q} = \mathbb{R}$
   (b) $\mathbb{Z} \cup \mathbb{Q}' = \mathbb{R}$
   (c) $\mathbb{W} \cup \mathbb{Z} \cup \mathbb{Q} = \mathbb{R}$
   (d) $\mathbb{Q} \cup \mathbb{Q}' = \mathbb{R}$
3. Which statement is false?
   (a) $\mathbb{Q} \cap \mathbb{Q}' = \emptyset$
   (b) $\mathbb{R} \setminus \mathbb{Q}' = \mathbb{Q}$
   (c) $\mathbb{R} \setminus \mathbb{Q} = \mathbb{Q}'$
   (d) $\mathbb{Q} \cap \mathbb{Q}' = \mathbb{R}$
4. Which statement is true?
   (a) All natural numbers are real numbers
   (b) All integers are real numbers
   (c) All rational numbers are real numbers
   (d) All irrational numbers are real numbers
   (e) All of the above are true

The real number is defined as the union of a set of rational numbers and the set of irrational numbers. The real numbers include natural numbers, whole numbers, integer, rational and irrational numbers. We can represent all these types of numbers in the number line.

A real number line simply called a number line represents real numbers with unique points associated with each number on the line. In the number line, the number 0 is called the origin. All the positive numbers or integers are represented on the right side of the origin, and the negative numbers or integers are represented on the left side of the origin.



Locating the Real Numbers on the Number Line

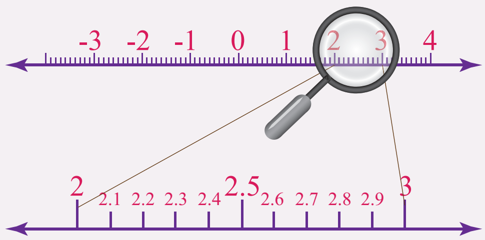
The idea of locating the real numbers on the number Line is based on the principle that every decimal number, be it rational or irrational, can be associated with a unique point on the number line and conversely that every point on the number line can be associated with a unique number, either rational or irrational. This is usually expressed by saying that there exists a one-to-one correspondence between the set of points on the number line and the set of real numbers.

EXAMPLE 28

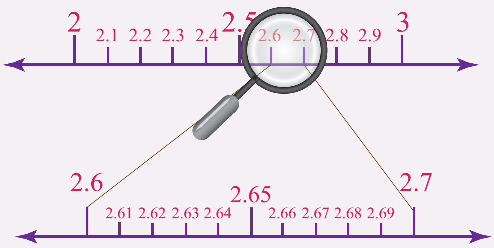
Locate the point that corresponds to the number 2.665 on the number line.

Solution

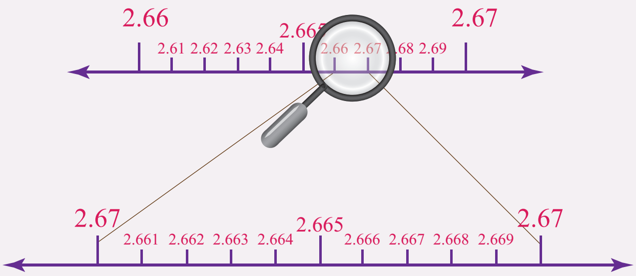
The number 2.665 lies between 2 and 3 on the number line



In between 2 and 3, there are 10 equal parts, say 2.1, 2,2, 2.3, and so on. In order to locate the 2.665 exactly, again focus on the points between 2.6 and 2.7, as 2.665 is located in between



Since 2.665 is located between 2.66 and 2.67, again focus on these points.



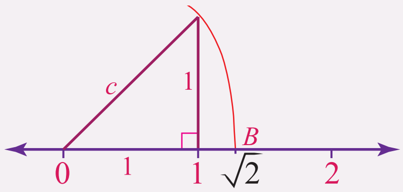
In the above example we have seen how to represent or locate a number on the number line using the process of successive magnification. So, with the help of this method, we can locate a number by a sufficient successive magnification process to visualize the representation of real numbers (rational and irrational numbers) on the number line.

Example 29

Locate $\sqrt{2}$ on the number line

Solution

$\sqrt{2}$ is an irrational number. There must be a point on the number line that is associated with this number. To find that exact location we do the following:



Draw a number line as shown in the figure. At the point corresponding to 1 on the number line, construct a perpendicular line segment 1 unit long. Draw a line segment from the point corresponding to 0 to the top of the 1 unit segment and label it as c. by Pythagorean Theorem $c = \sqrt{2}$ unit long. Open the compass to the length of c.

With the tip of the compass at the point corresponding to 0, draw an arc that intersects the number line at B. The distance from the point corresponding to 0 to B is $\sqrt{2}$ units. So the exact location of $\sqrt{2}$ on the number line is at point B.

Exercises

Plot the following real numbers on the number line:

(a) 2.5 (c) 5.37 (e) $\sqrt{3}$

(b) 2.67 (d) $-\sqrt{2}$ (f) $\sqrt{5}$

Since the set of rational numbers is a sub set of the set of real numbers, all the properties which are valid in the set of rational numbers will also be valid in the set of real numbers

Study the following summarized properties of real numbers

For any three real numbers ***a***, ***b*** and ***c***	
Closure Property	$a + b \in \mathbb{R}$, $a - b \in \mathbb{R}$, $a \times b \in \mathbb{R}$ and $a \div b \in \mathbb{R}$ $(b \neq 0)$
Commutative Property	$a + b = b + a$ and $a \times b = b \times a$
Associative Property	$a + (b + c) = (a + b) + c$ and $a \times (b \times c) = (a \times b) \times c$
Distributive Property	$a \times (b + c) = (a \times b) + (a \times c)$ Remember! $a + (b \times c) \neq (a + b) \times (a + c)$
Identity property	$a + 0 = 0 + a = a$ and $a \times 1 = 1 \times a = a$ 0 is the additive identity element and 1 is the multiplicative identity element
Inverse property	$a + -a = 0$ and $a \times \frac{1}{a} = 1$ $(a \neq 0)$ $-a$ is the additive inverse of ***a*** and vice versa and $\frac{1}{a}$ is the multiplicative inverse of ***a*** and vice versa
Multiplicative property of zero	$0 \times a = 0$ [0 times any number is equal to 0]

Exercises

Match the appropriate property from column A with the correct example in column B

Column A	Column B
1. Additive identity	(a) $4 + 5 = 5 + 4$
2. Commutative property of addition	(b) $a \times (0) = 0$
3. associative property of addition	(c) $2 \times (½) = 1$
4. multiplicative inverse	(d) $(a) \times (d) = (d) \times (a)$
5. multiplicative identity	(e) $(8 + 2) + 4 = 8 + (2 + 4)$
6. commutative property of multiplication	(f) $8 \times (1) = 8$
7. distributive property	(g) $42 - 6a = 6(7 - a)$
8. associative property of multiplication	(h) $a + 0 = a$
9. multiplicative property of zero	(i) $(9 \times 8) \times 7 = 9 \times (8 \times 7)$

An approximation is anything that is similar, but not exactly equal, to something else. A number can be approximated by rounding. A calculation can be approximated by rounding the values within it before performing the operations.

Rounding numbers to tens, hundreds, thousands, ten thousands and hundred thousands

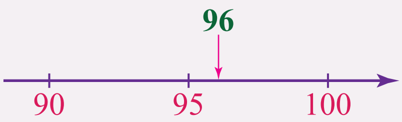
Activity 9

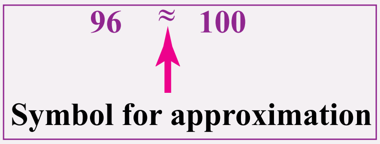
1. To which number is 24 closer? To 20 or to 30?
2. To which number is 176 closer? To 100 or to 200?
3. To which number is 7535 closer? To 7000 or to 8000?

Example 30

(a) To which number is 96 closer? To 90 or 100? Round it to its nearest multiple of *ten*

Solution

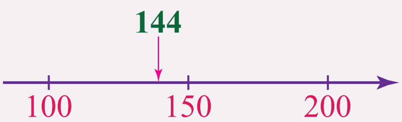




From the number line we can see that 96 is greater than 95. It is a little bit closer to 100 than 90.

So we say, If we round off 96 to its nearest **ten**, it will be 100. $\mathbf{96 \approx 100}$

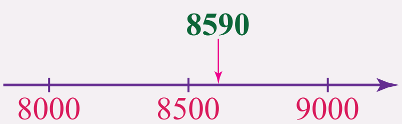
(b) To which number is 144 closer? To 100 or 200? Round it to its nearest multiple of 100.



From the number line we can see that 144 is smaller than 150. It is a little bit closer to 100 than 200.

Therefore, 144 rounded to its nearest hundred is 100. $\mathbf{144 \approx 100}$

(c) To which number is 8590 closer? To 8000 or 9000?



From the number line we can see that 8590 is greater than 8500. It is a little bit closer to 9000 than 8000.

Therefore, 8590 rounded to its nearest thousand is 9000. $\mathbf{8590 \approx 9000}$

Steps to round off a number to a given place value

Step1: Identify the rounding digit

- If you want to round it off to ten; the rounding digit is the tens place.

- If you want to round it off to hundred; the rounding digit is the hundreds place.
- If you want to round it off to thousand; the rounding digit is the thousands place, etc.

Step 2: Mark the digit to the right of the rounding digit.

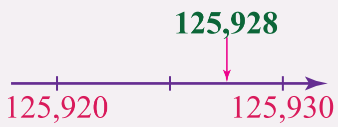
1. If this digit is < 5, change it to 0 and change all other digits to its right also to 0.
2. If this digit is ≥ 5, add 1 to the rounding digit found in the left and change all other digits to the right of the rounding digit to 0.

EXAMPLE 31

(a) Round the number 125,928 to the nearest **tens**.

Solution

- Step 1: The rounding digit is the *tens* place, that is 2.
- Step 2: The digit to the right is 8.
- Since 8 ≥ 5, add 1 to the rounding digit. That means raise 2 to 3 and change the digit to its right side to 0.



Therefore, 125,928 ≈ **125,930** [It *is just rounding the last two digits 28 to 30*]

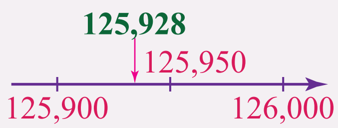
125,928 is closer to 125,930 than 125,920.

125,928 ≈ 125,930

(b) Round the number 125,928 to the nearest hundreds.

Solution

- Step 1: The rounding digit is the *hundreds* place, that is 9.
- Step 2: The digit to its right is 2.
- Since 2 < 5, keep the rounding digit 9 as it is(without raising it) and change all other digits to its right to 0



Therefore, 125,928 ≈ **125, 900** [*It is just rounding the last three digits 928 to 900*]

125,928 is closer to 125,900 than 126,000.

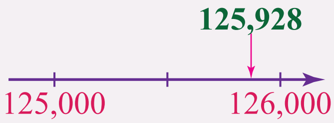
125,928 ≈ 125,900

(c) Round the number 125,928 to the nearest **thousands**.

Solution

- Step 1: The rounding digit is the *thousands* place, that is 5.
- Step 2: The digit to its right is 9.

- Since 9 ≥ 5, add 1 to the rounding digit. That means raise 5 to 6 and make all others to its right 0.



Therefore, **125,928 ≈ 126,000** [It *is just rounding the last four digits 5,928 to 6000*]

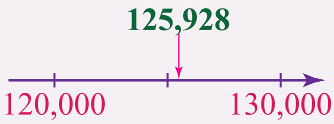
125,928 is closer to 126,000 than 125,000.

125,928 ≈ 126,000

(d) Round the number 125,928 to the nearest **ten thousands**.

Solution

- Step 1: The rounding digit is the *ten thousands* place that is 2.
- Step 2: The digit to its right is 5.
- Since 5 ≥ 5, add 1 to the rounding digit. That means, raise 2 to 3 and make all others to its right 0.



Therefore, **125,928 ≈ 130,000** [It *is just rounding the last five digits 25,928 to 30,000*]

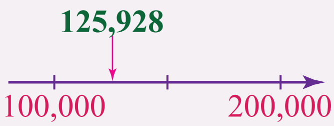
125,928 is closer to 130,000 than 120,000.

125,928 ≈ 130,000

(e) Round the number 125,928 to the nearest **hundred thousands**

Solution

- Step 1: The rounding place is the *hundred thousands* place that is 1.
- Step 2: The digit to its right is 2.
- Since 2 < 5, keep the rounding digit 1 as it is(without raising it) and change all other digits to its right to 0



Therefore, **125,928 ≈ 100,000**

125,928 is closer to 100,000 than 200,000.

125,928 ≈ 100,000

Exercises

1. Round the following numbers to the nearest **tens, hundreds, thousands, ten thousands** and **hundred thousands** and complete the table:

Round the numbers to the nearest					
Number	**Hundred thousands**	**Ten thousands**	**thousands**	**hundreds**	**tens**
65	0	0	0	100	70
723	0	0	1000	700	720

Round the numbers to the nearest					
Number	Hundred thousands	Ten thousands	thousands	hundreds	tens
8651					
18,708					
37,291					
764,510					
806,365	800,000				806,370

Pay attention to the following two statements:

- The Earth is about 4500000000 years old
- The Earth is about 4.5 billion years old

Which statement do you think is easier to understand?

Reading and calculation with very big or small numbers can be made easier by converting numbers to a **standard form**

A **standard form** is a form of writing a given number in a form that follows certain rules.

It is based on using powers of 10 to express how big or small a number is.

Standard form is written in the form of $a \times 10^{n}$, where $\boldsymbol{a}$ is a number bigger than or equal to 1 and less than 10. $\boldsymbol{n}$ can be any positive or negative integer.

Example 1: The standard form of 4,500,000,000 = 4.5×10^9 [$a = 4.5$ and $n = 9$]

Scientists use standard form when working with the speed of light and distances between galaxies, which can be very big. The size of bacteria or atoms may also be referred to in standard form as they are so small. For example,

The distance between the Sun and Mars is 141,700,000 miles or 228,000,000 km.

This distance can easily be written in standard form as: 1.417×10^8 miles or 2.28×10^8 km.

Atoms are tiny units of matter composed of three fundamental particles — proton, neutron, and electron. A proton and a neutron weigh equally, 1.67×10^{-27} kg. Many other quantities such as the size of planets, the size of microorganisms, the size of microchips, and the country's population are all expressed in standard form.

Because of this, Standard form is also sometimes referred to as **scientific notation**

Example 32

Write 3890 in standard form or scientific notation.

Solution

Step 1: Write the first digit 3

Step 2: Put a decimal point after this number and write the remaining non zero numbers like 3.89

Step 3: Now count the number of digits after the first digit 3. There are 3 digits. This 3 will be the power of ten while writing the given number in standard form

So the standard form of **3890 is 3.89** × **10^3**

Note: In writing a number in standard form

Numbers greater than 1 use a **positive powers of 10**

Numbers less than 1 use a negative power of 10.

Example 33

Write 0.0451 in standard form

Solution

The given number is less than 1. So it will have a negative power of 10

Step 1: Start from the decimal point and go to the right until you get the first non-zero number and write it. The first non-zero number as we go from left to right is 4.

Step 2: Put a decimal point after this number and write the remaining numbers like 4.51

Step 3: Count the number of digits that exist after the decimal point up to this non-zero number including the non-zero number itself. There are 2 digits including the digit 4. This 2 will be the **negative power of ten** while writing the given number in standard form

So the standard form of 0.0451 **is 4.51** × **10^{-2}** [The exponent is negative because the given number is less than 1]

Example 34

Write the number 81 900 000 000 000 in standard form

Step 1: Write the first number **8**.

Step 2: Add a decimal point after this and write the remaining non-zero numbers: **8.19**

Step 3: Now count the number of digits after **8.** There are **13** digits. This 13 will be the power of 10 while writing the given number in standard form.

Step 4: So, in standard form: 81 900 000 000 000 is $\mathbf{8.19 \times 10^{13}}$

EXAMPLE 35

The weight of an electron is 0.000000000000000000000000000000911kilograms

Write it in standard form.

Solution

Count the number of digits after the decimal point until you reach 9. There are 30 zeros and when you count the digit 9 to it, it will be 31.

So the standard form will be 9.11×10^{-31} kg.

EXERCISES

1. Convert the following numbers in standard form or scientific notation
   - (a) 3000
   - (b) 123000000
   - (c) 537000000000
   - (d) 0.0305
   - (e) 0.00240000
   - (f) 0.00015
2. Write the number for the following standard forms
   - (a) 1.5×10^4
   - (b) 4.12×10^7
   - (c) 1.23×10^{-5}
   - (d) 3.5×10^{-9}

So far we have seen the four basic operations namely, addition, subtraction, multiplication and division on the set of real numbers. These are the four basic binary operations on real numbers.

Let us look at the binary operation of addition "+" on a set of real numbers. We have seen that whenever any two real numbers are added, we still get a new real number. In such cases, we say that the set of real numbers is closed under the binary operation addition "+".

Similarly if $A = \{1, 2, 3, 4\}$ and the usual binary operation addition "+" is defined on A, then A is not closed under addition "+" because for the two elements 3 and 4 of A, $3 + 4 = 7$ and $7 \notin A$.

Let us define the binary operation subtraction "–" on the set of Natural numbers $\mathbb{N}$. Is $\mathbb{N}$ closed under "–"? The answer is no because for any two natural numbers m

and n, $m - n$ may not be a natural number. Example: 5 and 6 are natural numbers, but $5 - 6 = -1$ is not a natural number.

Binary Operation: Any operation or sign that combines any two elements of a given set according to some clearly defined rule.

Binary operations on a given set are calculations that combine two elements of the set to produce another element of the same set.

The word "binary" means composed of **two** pieces.

A binary operation is simply a rule for combining two values to create a new value.

In addition to the four usual basic binary operations, it is normal in mathematics to define a new binary operation with a clearly defined rule on a given set.

Example 36

A new binary operation, using the symbol Φ, is defined to be $a \Phi b = 3a + b$, where a and b are real numbers. Then

(a) Evaluates $8 \Phi 3$?

(b) Is $\mathbb{R}$ closed under Φ

(c) Is Φ commutative?

(d) Is Φ associative?

Solution

	Question	Explanation
1	What is $8 \Phi 3$?	Substitute the values of a and b into the right-hand side of the definition, namely $3a + b$. $8 \Phi 3 = 3 \times 8 + 3 = 24 + 3 = 27$
2	Is $\mathbb{R}$ closed under Φ	3a is a real number because $3 \times a$ any real number a is a real number b is also a real number. So $3a + b$ will be a real number because $\mathbb{R}$ is closed under addition Therefore, $\mathbb{R}$ is closed under Φ because for any a and $b \in \mathbb{R}$, $a \Phi b = 3a + b \in \mathbb{R}$.
3	Is Φ commutative?	Is $a \Phi b = b \Phi a$ for all possible values a and b? $a \Phi b = 3a + b$ but $b \Phi a = 3b + a$ $3a + b = 3b + a$? This is not true for all real numbers. If $a = 8$ and $b = 2$; $3 \times 8 + 2 \neq 3 \times 2 + 8$; $26 \neq 14$. The operation Φ is not commutative for real numbers.
4	Is Φ associative?	Is $a \Phi (b \Phi c) = (a \Phi b) \Phi c$? $a \Phi (3b + c) = (3a + b) \Phi c$? $3a + (3b + c) = 3(3a + b) + c$? $3a + 3b + c = 9a + 3b + c$? Not true for all real numbers If $a = 2, b = 3, c = 4$; $3 \times 2 + (3 \times 3 + 4) \neq 3(3 \times 2 + 3) + 4$; $6 + 13 \neq 3(9) + 4$; $19 \neq 31$. The operation Φ is not associative for real numbers.

Example 37

A binary operation Δ is defined on the set of integers as $a \Delta b = (a + b)(a - b)$, for any a and $b \in \mathbb{Z}$. Then

(a) Evaluates 2 Δ 5?

(b) Is ℤ closed under Δ?

(c) Is Δ commutative?

(d) Is Δ associative?

	Question	Explanation
1	What is 2 Δ 5?	Substitute the values of a and b into the right-hand side of the definition, namely $(a + b)(a - b)$. $2 \Delta 5 = (2 + 5)(2 - 5) = -21$
2	Is ℤ closed under Δ	$a + b \in \mathbb{Z}$ because ℤ is closed under addition $a - b \in \mathbb{Z}$ because ℤ is closed under subtraction Therefore, ℤ is closed under Δ because for any a and $b \in \mathbb{Z}$, $a \Delta b = (a + b)(a - b) \in \mathbb{Z}$, since the product of two integers is again an integer.
3	Is Δ commutative?	Is $a \Delta b = b \Delta a$ for all possible values a and b? $a \Delta b = (a + b)(a - b)$ and $b \Delta a = (b + a)(b - a)$ $(a + b)(a - b) = (b + a)(b - a)$? This is not true for all integers. If $a = 2$ and $b = 5$; $a \Delta b = (2 + 5)(2 - 5) = -21$ $b \Delta a = (b + a)(b - a) = (5 + 2)(5 - 2) = 21$ The operation Δ is not commutative for all integers. $-21 \neq 21$
4	Is Δ associative?	Is $a \Delta (b \Delta c) = (a \Delta b) \Delta c$? $a \Delta [(b + c)(b - c)] = a \Delta (b^2 - c^2) = (a + (b^2 - c^2))(a - (b^2 - c^2))$ $= (a + b^2 - c^2)(a - b^2 + c^2)$ $(a \Delta b) \Delta c = ((a + b)(a - b)) \Delta c = ((a^2 - b^2) + c)\ ((a^2 - b^2) - c)$ $= (a^2 - b^2 + c)((a^2 - b^2) - c)$ So $a \Delta (b \Delta c) \neq (a \Delta b) \Delta c$. Δ is not associative If $a = 2$, $b = 5$ and $c = 6$, then $a \Delta (b \Delta c) = 2 \Delta (5 + 6)(5 - 6) = 2 \Delta (-11) = (2 + -11)(2 - (-11))$ $= (-9)(13) = -117$ $(a \Delta b) \Delta c = (2 + 5)(2 - 5) \Delta 6 = -21 \Delta 6 = (-21 + 6)(-21 - 6)$ $= (-15)(-27) = 405 \quad -117 \neq 405$

Exercises

1. If @ is defined on the set of whole numbers as $m @ n = mn + nm$, then which one is false

   (a) The set of whole numbers is closed under @

   (b) @ is commutative

   (c) @ is associative

   (d) 4 @ 5 = 40

   (e) None of the above

2. If © is defined on the set of integers as a © $b = (a + b)^2$, for $a, b \in \mathbb{Z}$, the which one is false?
   (a) The set of integers is closed under ©
   (b) © is commutative
   (c) © is associative
   (d) 4 © 5 = 81
   (e) None of the above
3. If the binary operation ☐ on the set of natural numbers is defined as m ☐ $n = mn$, for $m, n \in \mathbb{N}$, the n which one is true?
   (a) ℕ is closed under ☐
   (b) ☐ is commutative
   (c) © is associative
   (d) 10 ☐ 2 = 1024
4. If Θ is a binary operation defined as $a\ \Theta\ b = a - 2b$, find
   (a) 7 Θ 3 = ______________
   (b) 9 Θ 5 = ______________
   (c) –5 Θ 5 = ______________
   (d) 100 Θ 50 = ______________
   (e) If 3 Θ d = –9, then d = ___
   (f) Is Θ commutative?_______

Key Terms

- Approximation
- Associative property
- Binary operation
- Closure property
- Commutative property
- Denominator
- Fractions
- Integers
- Identity property
- Inverse property
- Irrational numbers
- Numerator
- Rational numbers
- Real numbers
- Standard form

Summary

- To convert a rational number of the form $\frac{a}{b}$ to a decimal divide $\boldsymbol{a}$ by $\boldsymbol{b}$
- When you divide $\boldsymbol{a}$ by $\boldsymbol{b}$ one of the following two happens:
  - The division process ends or terminates when a remainder of zero is obtained. In this case, the decimal is called a terminating decimal.
  - The division process does not terminate as the remainder never becomes zero. Such a decimal is called a repeating decimal

- Both terminating and repeating decimals are rational numbers. They can be converted to the form $\frac{a}{b}$, where ***a*** and ***b*** are integers and $b \neq 0$.
- The sets of Natural numbers, Whole numbers, Integers and rational numbers denoted by $\mathbb{N}$, $\mathbb{W}$, $\mathbb{Z}$ and $\mathbb{Q}$, respectively are described by
- $\mathbb{N} = \{1, 2, 3, \ldots\}$ $\mathbb{W} = \{0, 1, 2, 3, \ldots\}$ $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
  $= \left\{\frac{a}{b} : a \in z, b \in Z, b \neq 0\right\}$.
- Rational numbers are all numbers that can be written as a ratio of two integers where the denominator is not equal to 0. The set of rational numbers is denoted by $\mathbb{Q}$
- All Natural numbers are rational numbers
- All whole numbers are rational numbers
- All integers are rational numbers
- $\mathbb{N} \subseteq W \subseteq \mathbb{Z} \subseteq \mathbb{Q}$
- All rational numbers are either terminating or repeating decimals.
- There are decimal numbers which are neither terminating nor repeating. Such numbers are called irrational numbers. The set of irrational numbers is represented by a symbol $\mathbb{Q}'$.
- All numbers that we know so far are either rational or irrational.
- The union of the sets of rational and irrational numbers gives us a new set of numbers called the set of real numbers. The set of real numbers is represented by $\mathbb{R}$. So $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$.
- Every real number corresponds to a unique point on the line.
- Every point on the line corresponds to a unique real number.
- The set of real numbers have the following basic properties.

For any three real numbers *a*, *b* and *c*	
Closure Property	$a + b \in \mathbb{R}$, $a - b \in \mathbb{R}$, $a \times b \in \mathbb{R}$ and $a \div b \in \mathbb{R}$ $(b \neq 0)$
Commutative Property	$a + b = b + a$ and $a \times b = b \times a$
Associative Property	$a + (b + c) = (a + b) + c$ and $a \times (b \times c) = (a \times b) \times c$

For any three real numbers ***a***, ***b*** and ***c***	
Distributive Property	$a \times (b + c) = (a \times b) + (a \times c)$ Remember! $a + (b \times c) \neq (a + b) \times (a + c)$
Identity property	$a + 0 = 0 + a = a$ and $a \times 1 = 1 \times a = a$ 0 is the additive identity element and 1 is the multiplicative identity element
Inverse property	$a + -a = 0$ and $a \times \frac{1}{a} = 1\ (a \neq 0)$ $-a$ is the additive inverse of $\boldsymbol{a}$ and vice versa and $\frac{1}{a}$ is the multiplicative inverse of $\boldsymbol{a}$ and vice versa
multiplicative property of zero	$0 \times a = 0$ [0 times any number is equal to 0]

- We can approximate a number by rounding it to the nearest 10, 100, 1000 and so on.
- A **standard form** or a scientific notation is a method or form of writing a given number in a form that follows certain rules. The standard form of 271000000 is 2.71×10^8. Similarly the standard form of 0.0000000271 is 2.71×10^{-8}.
- Binary Operation is any operation or sign that combines any two elements of a given set according to some clearly defined rule. A binary operation is simply a rule for combining two values to create a new value.

Exercises

1. Which one is equal to $\frac{2}{5} + \frac{-3}{2} - \frac{3}{4}$?

   (a) $\frac{-37}{20}$ (b) $\frac{-20}{37}$ (c) $\frac{-16}{20}$ (d) $\frac{16}{37}$

2. The simplified form of $\frac{7}{12} \times \frac{30}{11} \times \frac{22}{7} \times \frac{20}{4}$ is

   (a) 1 (b) 2 (c) 3 (d) 4 (e) None

3. Which operations are <u>not</u> associative on the set of rational numbers?

   (a) Addition and multiplication
   (b) Subtraction and multiplication
   (c) Subtraction and division
   (d) Addition and division

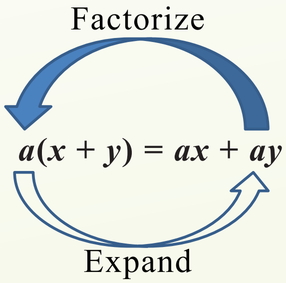
4. For any two rational numbers x and y, $x - y$ is also a rational number. This statement refers to
   (a) Commutative property (c) Associative property
   (b) Closure property (d) Identity property
5. Which of the following statement is false?
   (a) $\mathbb{N} \cap \mathbb{W} \cap \mathbb{Z} = \mathbb{Q}$ (c) $\mathbb{Z} \cap \mathbb{Q} \cap \mathbb{Q}' = \mathbb{Z}$
   (b) $\mathbb{W} \cup \mathbb{Z} \cup \mathbb{Q} = \mathbb{R}$ (d) $\mathbb{Q}' \cap \mathbb{Q} = \mathbb{R}$
   (e) All are false
6. Which rational number is equal to $1.3\overline{25}$?
   (a) $\frac{1213}{990}$ (b) $\frac{1312}{90}$ (c) $\frac{1312}{990}$ (d) $\frac{1213}{90}$
7. What is the result of $3.05 + 1.532 - 0.582$?
   (a) 0.4 (b) 4 (c) 0.14 (d) 0.41
8. When the number 42567 is rounder to the nearest thousands, it is equal to
   (a) 42000 (b) 42500 (c) 42600 (d) 43000
9. The area of the African continent is about 30,000,000 km. How is this number written in standard form?
   (a) 3.0×10^7 (c) 3.0×10^{-7}
   (b) 30.0×10^6 (d) 30.0×10^{-6}
10. A binary operation $\mathbf{\Omega}$ on the set of whole numbers is defined by $p\,\mathbf{\Omega}\,q = (p \times q)\,(p + q)$, where $p, q \in \mathbb{W}$. Then which of the following is true?
   (a) $4\,\mathbf{\Omega}\,5 = 180$ (c) $\mathbb{W}$ is closed under $\mathbf{\Omega}$
   (b) $\mathbf{\Omega}$ is commutative (d) All are true

M10CH03

CHAPTER 3

ALGEBRAIC EXPRESSIONS

Chapter Contents



Perfect Square Trinomial

$$(a + b)^2 = a^2 + 2ab + b^2$$

Square the first term (a)

Multiply the two terms by 2

Square the last term (b)

Chapter Outcomes

Upon completion of this chapter, learners will be able to:

- express Statements in algebraic expression;
- discuss Numerical Statement;
- form Algebraic Expressions;
- evaluate Algebraic Expressions;
- show Relations between two algebraic expressions;
- demonstrate Expansion of algebraic expressions;
- add and Subtract Algebraic expressions;
- solve problems on factorization;
- factorize quadratic expressions.

Introduction

In your discussion of real numbers, you had been using number expressions or numerical expressions together with the four basic operations (addition, subtraction, multiplication and division). In this unit, you shall be introduced with algebraic numbers and some other basic concepts of algebra like algebraic expressions, terms, variables, constants coefficients, and the four fundamental operations on them. You will also learn the techniques of adding and subtracting like terms and the methods of factorizing algebraic expressions.

You are already familiar with real numbers like . . . , -3, -2, -1, 0, 1, 2, 3,...., $\frac{2}{3}$, 0.45, $0.\bar{3}$, $0.\overline{23}$, $\sqrt{2}$, $\sqrt{7}$ etc and operations of addition (+), subtraction (–), multiplication (×) and division (÷) on these numbers.

For example, **2 + 7** is a **numerical expression** formed from numbers 2 and 3 using the operation addition. The following are few examples of **numerical expressions** formed from real numbers and operations on them

14 – 6	7(4 + 2)	8 + 7 ÷ 6
5 × 10	10 – (2 × 6)	

Numerical expressions have numbers in them, and often operations but they don't include any **variables**:

However, sometimes in mathematics, letters called **variables** are also used as symbols to represent numbers. Study the following statement:

The sum of two whole numbers is 10. What are the numbers?

Solution

- If the first whole number is 0, then the second must be 10 because 0 + 10 = 10
- If the first whole number is 1, then the second must be 9 because 1 + 9 = 10
- If the first whole number is 2, then the second must be 8 because 2 + 8 = 10
- If the first whole number is 3, then the second must be 7 because 3 + 7 = 10
- If the first whole number is 4, then the second must be 6 because 4 + 6 = 10

- If the first whole number is m and the second is n, then $m + n = 10$

  Note: In the above explanation, expressions like 0 + 10, 1 + 9, 2 + 8, 3 + 7 and 4 + 6 are called **numerical expressions**. But the expression $m + n$ is called an *expression of algebra* or an **algebraic expression**.

  An **algebraic expression** is a mathematical expression that can contain numbers, variables, and operations (addition, subtraction, multiplication, division.

  Example:

  - $5x – 7$
  - $5x + 4y + 10$
  - $2x^2y – 3xy^2$
  - $(–4b)^2 + 6ab$ are algebraic expressions
- In $5x – 7$, the letter x is called a **variable**. A variable is a letter or symbol that represents an unknown value. Any letter can be used but *x, y, m, n, z, k, p, q* are common.
- The number 7 is called a **constant**. A constant is a number that cannot change its value.
- The number 5 in $5x$ is called a coefficient. A coefficient is a number by which a variable is multiplied
- A **term** in an algebraic expression is a *constant*, a *variable* or a *constant and a variable combined by multiplication or division.*

Example 1

$5x – 7$, $5x$ and $−7$ are two **terms**, but $5x – 7$ is not a term. It is an algebraic expression.

Activity 1

Identify the *variable*, the *constant*, the *coefficient* and the *terms* in the following algebraic expressions and complete the table

Algebraic expression	Variable(s)	Constant	Coefficient	Term(s)
$y + 2$	y	2	1	y and 2
$3m + 6n – 4$	m and n	−4	3 and 6	$3m$, $6n$ and −4
$−9 + 10x$	x	−9	10	$10x$ and −9

Algebraic expression	Variable(s)	Constant	Coefficient	Term(s)
$8r - 2x + 3$	r and x	3	8 and −2	$8r$, $-2x$ and 3
$x^2 + 2$	x	2	1	x^2 and 2
$x \mid 2 \text{ or } \frac{1}{2}x$	x	–		$\frac{1}{2}x$
$2(\sqrt{3x}) - 5x + 1$	x	1	2	$2\sqrt{3}x$, $-3x$ and 1
$3 + 4x - 5wyz$	x, w, y, z	3	4 and 5	3, $4x$ and $-5wyz$

Note: If a variable has no coefficient written in front of it, we assume the coefficient is 1. For example, the expression $y + 2$ can be thought of as $1y + 2$. Similarly $3 - m$ is thought of as $3 - 1m$.

Exercises

Identify the terms, variables, coefficients and constants for the following algebraic expressions and complete the table given below

No	Algebraic expression	Term(s)	Variable(s)	Coefficient	Constant
1	$12m - 6n + 9$				
2	$5x - 7y$				
3	$23 - k$				
4	$4x^3 + 2xy - 1$				
5	$x^2 + y^2 + 3xy$				
6	$-2x + y + \frac{7}{4}$				
7	$\frac{3}{2}x^3 + x^2 + x + 1$				
8	$3x^2 - 2y^2z - z^2x + 5$				

Activity 2

Think of a number n, multiply the number n by 8, divide the result by 2, add 5, and then subtract 4 times the original number n

1. What is your result in terms of n?
2. Which ever number you take why is the final result always equal to 5?

In order to form an algebraic expression we need to translate word or verbal phrases given in to algebraic expressions. Translating verbal or word expressions in to mathematical or algebraic expressions is fundamental in solving many word problems.

Below is a table of common English words or verbal expressions converted into an algebraic expression. You can use this table to assist in translating word or verbal phrases in to algebraic expressions or mathematical expressions.

Operation	Word or verbal expression	Algebraic expression
Addition	A number plus three	$x + 3$
	Three more than a number	
	The sum of a number and three	
	The total of three and a number	
	A number increased by three	
	Three added to a number	
Subtraction	A number minus seven	$n - 7$
	Seven less than a number	
	The difference of a number and seven	
	Seven less a number	
	A number decreased by seven	
	Seven subtracted from a number	
Multiplication	Two times a number	$2n$
	The product of two and a number	
	Twice a number or double a number	
	A number multiplied by negative two	

<table>
<tr><th>Operation</th><th>Word or verbal expression</th><th>Algebraic expression</th></tr>
<tr><td></td><td>Half of a number</td><td rowspan="3">$\frac{n}{2}$</td></tr>
<tr><td></td><td>The ratio of n and 2</td></tr>
<tr><td></td><td>n multiplied by $\frac{1}{2}$</td></tr>
<tr><td rowspan="2">Division</td><td>The quotient of a number and seven</td><td rowspan="2">$\frac{x}{7}$</td></tr>
<tr><td>A number divided by 7</td></tr>
</table>

Exercises

Write an algebraic expression for the following word or verbal phrases

1. Four more than m ________________
2. The product of 7 and c __________________
3. Nine less than x ________________________
4. A number divided by the sum of 4 and 7. _______________
5. Twice the sum of a number plus 4. _______________
6. The sum of ¾ of a number and 7. ________________
7. Ten times a number increased by 150. __________________
8. A number x is increased by 7 ___________________
9. A number y is decreased by 7 ___________________
10. A number a is multiplied by 7 __________________

To evaluate an algebraic expression, just replace the variables with their number values given. Then, find the value of the numerical expression using **order of operations**.

Example 2

Evaluate the following algebraic expressions for the given values of the variables

(a) $10 - 2x$; when $x = 5$

(b) $3x - 4y^2$; when $x = 2$ and $y = -5$

(c) $\frac{3x}{y+5}$; when $x = 3$ and $y = 4$

Solution

(a) $10 - 2x$, when $x = 5$ gives $10 - (2 \times 5) = 10 - 10 = 0$.

(b) $3x - 4y^2$, when $x = 2$ and $y = -5$ gives $(3 \times 2) - 4(-5)^2 = 6 - 4(25)$
$= 6 - 100 = -94$.

(c) $\frac{3x}{y+5}$, when $x = 3$ and $y = 4$ gives $\frac{3 \times 3}{4+5} = \frac{9}{9} = 1$.

Order of Operations

Activity 3

Evaluate **8 ÷ 2 + 2 × 10 – 3**. Three students solved this problem in the following three ways. Which student is correct?

Student A	Student B	Student C
8 ÷ 2 + 2 × 10 – 3	**8 ÷ 2 + 2 × 10 – 3**	**8 ÷ 2 + 2 × 10 – 3**
= 4 + 2 × 10 – 3	= 8 ÷ 4 × 10 – 3	= 4 + 2 × 10 – 3
= 6 × 10 – 3	= 2 × 10 – 3	= 4 + 20 – 3
= 60 – 3	= 20 – 3	= 24 – 3
= 57	= 17	= 21

Mathematics has rules that we all have to follow, so we will all come up with the same (correct) solution. One of the rules or procedures we follow is the "***Order of Operations***." The "Order of Operations" is just that the order in which we perform an operation in a math problem. It is given by the abbreviation **PEDMAS**

P stands for parenthesis [that we have to perform first the expression inside a parenthesis or bracket]

E stands for Exponents [that we have to perform next the expression having exponents or powers]

DM or **MD** stands for division and multiplication [that we have to perform next division or multiplication whichever comes first in order from left to right

AS or **SA** stands for addition and subtraction [that we have to perform next addition or subtraction whichever comes first in order from left to right

Note: Multiplication and division are equal operations—we perform them left to right, in the order they are in the problem. Similarly,

Addition and subtraction are equal operations—we perform them left to right, in the order they are in the problem

Example 3

Simplify the following numeric expressions

(a) $12 - 3^2 + 10 \div 2$

(b) $12 - (3^2 + 10) \div 2$

(c) $8 \div \frac{1}{2} \times 3$

Solution

(a) $12 - 3^2 + 10 \div 2$

$= 12 - 9 + 10 \div 2$ (the power 3^2 is performed first)

$= 12 - 9 + 5$ (Division is performed next)

$= 3 + 5$ (subtraction is performed)

$= 8$ (Addition is performed last)

(b) $12 - (3^2 + 10) \div 2$

$= 12 - (9 + 10) \div 2$ (parenthesis comes first and the power 3^2 inside the parenthesis is performed)

$= 12 - (19 \div 2)$ (parenthesis still comes first)

$= 12 - 9.5$ (Division is performed)

$= 2.5$ (subtraction is performed last)

(c) $8 \div \frac{1}{2} \times 3$

$= 8 \div \frac{1}{2} \times 3$ [Since division and multiplication are equal operations—we perform them left to right, in the order they are in the problem

$= 16 \times 3 = 48.$

Example 4

Evaluate the following algebraic expressions for the given values of the variables using **order of operations**

(a) $m^2 - (n^3 - 4k)$; if $m = 7$, $n = 3$ and $k = 5$

Solution

$$m^2 - (n^3 - 4k) = 7^2 - (3^3 - 4 \times 5)$$
$$= 49 - (27 - 20)$$
$$= 49 - (7) = 42.$$

Exercises

Evaluate the following algebraic expressions for the given values of the variables using **order of operations**

1. $8x - 6$; if $x = 7$
2. $x^2 - x$; if $x = 4$
3. $2x - 4y - 5$; if $x = 6$ and $y = 9$
4. $x(y^3 + 8) \div 12$; if $x = 3$ and $y = 4$
5. $\frac{3a}{2b+1}$; if $a = 11$ and $b = 1$
6. $5x^3 + 3x^2 - 4x - 4$; if $x = 1$
7. $1 - x^5 + 2x^6 + 7x$; if $x = \frac{1}{2}$
8. $x^3 + 3x^2 + 3x + 2$; if $x = -1$

Sometimes algebraic expressions might involve brackets or parenthesis and addition or subtraction signs inside parenthesis just like $2(x + 4) + 3(x - 5)$. Such expressions which involve brackets can be re written in equivalent forms without any brackets by applying the distributive property of multiplication over addition. This process is called '*expanding*' or '*removing*' brackets.

Expanding using the distributive Laws

The distributive law says, for any real numbers ***a***, ***b*** and ***c***

$$\mathbf{2(a + b) = 2a + 2b}$$
$$\mathbf{3(a - b) = 3a - 3b} \text{ and}$$
$$\mathbf{a(b + c) = ab + ac}$$

Note that the left side of the algebraic expressions involve parenthesis and the right sides do not involve parenthesis. However both the left and the right sides are equal. The right side expression is the **expanded** or **distributed form** of the left.

Example 5

Observe the following:

(a) $3(x + 2) = 3x + 6$ [Multiply each term inside the bracket by the term outside the bracket]

(b) $5(x + 2y) = 5x + 10y$

(c) $3x(x + y) = 3x^2 + 3xy$

(d) $-2(a + b) = -2a - 2b$

(e) $-3(a - b) = -3a + 3b$ $[-3 \times -b = 3b]$

(f) $x(y + 2) - 3(y + 2x) = xy + 2x - 3y - 6x$

(g) $a(b + c) + b(a - c) = ab + ac + ba - bc$

Remember! Some addition and subtraction problems look very similar to distributive property problems

- $-3 + (x + 1)$
- $(5 - x) - 8$ and
- $(2x - 4) + 3$
- $2(n \times 4)$
- $(3 \times y)(-2)$

BUT THEY DO NOT REQUIRE DISTRIBUTIVE PROPERTY

Exercises

1. Expand the following algebraic expressions using the distributive property

(a) $6(x + 3)$

(b) $x(x + y)$

(c) $2(m + 4) + 3(m + 6)$

(d) $2(3x - 4)$

(e) $t(3t + 4)$

(f) $4(t - 2) - 3(t + 1)$

(g) $-3(x + y)$

(h) $4(y - 5)$

(i) $7(m - 3) - 2(m - 4)$

(j) $5(m - 4)$

(k) $-x(x + 2)$

(l) $-4(x + 1) + 3(x + 2)$

(m) $y(y^2 - 2)$

(n) $-3(m - n)$

(o) $5(2t + 1) + 3(t + 2)$

2. Find the perimeter of an equilateral triangle if the length of one of the sides is $2x + 3$(Give your answer in terms of x)

Simplification of algebraic expressions

A. Like and unlike terms

Study the algebraic expression $2x + 5y + 7 + 3x + 9 + 2xy$. It has 6 terms. Of these six terms the terms $2x$ and $3x$ have the same variable, that is x. So they are called **like terms**. Similarly 7 and 9 are both constants. They are also called like terms because they do not have a variable.

$5y$ does not have a like term because no other term has the variable y. The term $2xy$ does not have a like term because no other term has the variables x and y.

Like terms

Two or more terms are called like terms if they have EXACTLY the same variables.

Example:

$2x$ and $3x$, $5y$ and $-4y$, $7xy$ and xy, $2xy^2$ and $8xy^2$ are like terms.

Note: Like terms can have different coefficients

But $2x$ and $2y$, $5y$ and $5x$, $7xy$ and $3xy^2$, $2xy^2$ and $8x^2y$ are **unlike** terms.

B. Adding and subtracting algebraic expressions

To add or subtract algebraic expressions we use a similar method of adding and subtracting numbers. However, in the case of algebraic expressions, we need to sort and place the like terms and the unlike terms together which makes it easier to add or subtract.

Like terms can be added or subtracted by adding or subtracting their coefficients and taking the variable as it is.

Example:

$2x$ and $3x$ are like terms. So $2x + 3x = (2 + 3)\,x = 5x$.

$7xy$ and xy are like terms. So $7xy - xy = (7 - 1)xy = 6xy$.

EXAMPLE 6

Simplify the following algebraic expressions:

(a) $9x + 3y + 4x + 2y$

Solution

$9x + 3y + 4x + 2y$ [Identify the like terms; $9x$ and $4x$ are like terms and $3y$ and $2y$ are also like terms]

$= 9x + 4x + 3y + 2y$ [Associative property of addition enables us to combine like terms side by side. When you combine like terms, you MUST take the sign in front of the term with it]

$= (9 + 4)x + (3 + 2)y$ [Add or subtract the coefficients of the like terms]

$= 13x + 5y$

(b) $2(x + 4) + 3(x - 5) - 2y$

If you see an addition or subtraction problem inside a set of parenthesis, like the one given above, remove the parenthesis using the distributive property BEFORE simplifying the expression

Solution

$= 2x + 8 + 3x - 15 - 2y$ [parenthesis removed using distributive property of × over +]

$= 2x + 3x - 2y + 8 - 15$ [combining like terms side by side by taking the sign in front of the term with it]

$= (2 + 3)x - 2y - 7$

$= 5x - 2y - 7$

(c) $x(4 - x) - x(3 - x)$

Solution

$x(4 - x) - x(3 - x)$

$= 4x - x^2 - 3x + x^2$ [parenthesis removed using distributive property of × over +]

$= 4x - 3x - x^2 + x^2$ [combining like terms side by side by taking the sign in front of the term with it]

$= x + 0$

$= x$

(d) $\frac{1}{2}a - \frac{1}{3}b + \frac{3}{4}a + b$

Solution

$\frac{1}{2}a - \frac{1}{3}b + \frac{3}{4}a + b$

$= \frac{1}{2}a + \frac{3}{4}a - \frac{1}{3}b + b$ [combining like terms side by side by taking the sign in front of the term with it]

$= \left(\frac{1}{2} + \frac{3}{4}\right)a + \left(-\frac{1}{3} + 1\right)b$

$= \frac{10}{8}a + \frac{2}{3}b$

$= \frac{5}{4}a + \frac{2}{3}b$

EXERCISES

Simplify the following algebraic expressions:

(a) $2st + 3t - s + 5t + 4s$

(b) $2a - 4b + 3ab - 5a + 2b$

(c) $x(2x + 3y - 4) - x^2 + 4xy - 12$

(d) $4(2x + 1) - 3x$

(e) $4(p - 5) + 3(p + 1)$

(f) $6(p + 3q) - (7 + 4q)$

(g) $4rs - 2s - 3(rs + 1) - 2s$

(h) $-3x + 4 + 2x^2 - 7x - 2$

(i) $9x^2 - 3x - \frac{2}{3} - \left(-4x^2 + 3x + \frac{2}{3}\right)$

Factorizing an algebraic expression means finding all the factors of the expression and re writing it as a product of its factors.

There are many methods of factorizing an algebraic expression. In this topic, we use common terms, regrouping and identities to factorize an algebraic expression.

Factorization by using common terms

We know that $6 = 2 \times 3$. 2 and 3 are the factors of 6. So the term 6 can be factorized and re written as $6 = 2 \times 3$

Similarly, $6x$ if factorized as $6x = 2(3x)$ or $6x = 3(2x)$

EXAMPLE 7

Factorize

(a) $6x + 9$ (b) $-7a^2 + 21a$

Solution

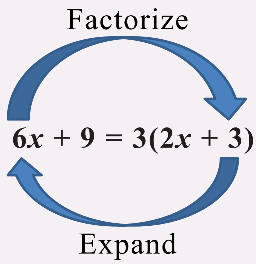
(a) There are two terms $6x$ and 9. Find the Highest Common Factor (HCF) that divides both terms of $6x$ and 9. The common factor is 3.

Write 3 outside a bracket as **3(? + ?)**

Find the missing terms inside the bracket by dividing each term by the common factor.

That means $6x \div 3 = 2x$ and $9 \div 3 = 3$. So $2x$ and 3 are the missing terms. Write these inside the bracket as $\mathbf{3(2x + 3)}$

Check your answer by expanding the bracket and see that the expanded form matches the original equation: $3(2x + 3) = (3 \times 2x) + (3 \times 3) = 6x + 9$

> Factorization is the reverse process of expansion of an algebraic expression
>
> 
>

(b) Here, 7 is a common factor of 7 and 21 and a is a common factor of a^2 and a.
Hence, $-7a^2 + 21a = 7a\,(3 - a)$.

Example 8

Factorize

(a) $12a^2 - 8ac + 4a^2b$ (b) $12x^2y + 3xy^3 - 6x3y^2$

Solution

(a) There are three terms $\mathbf{12a^2}$, $\mathbf{-8ac}$ and $\mathbf{4a^2b}$. Find the HCF that divides all the three terms. The HCF of the three is $4a$.

Write $\mathbf{4a}$ outside a bracket as $\mathbf{4a(? + ? + ?\ )}$

Find the missing terms inside the bracket by dividing each term by the common factor

$12a^2 \div 4a = 3a$

$-8ac \div 4a = -2c$

$4a^2b \div 4a = ab$

Write these inside the bracket as $4a(3a - 2c + ab)$

Therefore $12a^2 - 8ac + 4a^2b$ is factorized as $4a\,(3a - 2c + ab)$

Check: $4a\,(3x - 2c + ab) = (4a \times 3a) + (4a \times -2c) + (4a \times ab)$

$= 12a^2 + (-8ac) + 4a^2b$

$= 12a^2 - 8ac + 4a^2b.$

(b) Clearly, $3xy$ is a common factor of the expression $12x^2y + 3xy^3 - 6x^3y^2$

Therefore $12x^2y + 3xy^3 - 6x3y^2 = 3xy\,(4x + y^2 - 2x^2y)$

Factorization by regrouping the terms

Example 9

Factorize

(a) $6 - 10xy + 3x - 15x^2y$ (b) $z^3 - z^2 - 9z + 9$

Solution

(a)

(i) Regroup the terms as $(6 + 3x) - (10xy + 15x2y)$

(ii) Factor each group $3\,(2 + x) - 5xy\,(2 + x)$

(iii) Take out the common factor of each group $(2 + x)\,(3 - 5xy)$

Therefore, $6 - 10xy + 3x - 15x^2y = (2 + x)\,(3 - 5xy)$

(b) The expression has four terms. Since there is no a common factor that divides all the four, try to factorize the first two terms together and the last two terms together

$z^3 - z^2 = z^2\,(z - 1)$

$- 9z + 9 = -9(z - 1)$

So $z^3 - z^2 - 9z + 9 = z^2\,(z - 1) - 9(z - 1)$

Note that $z - 1$ is common to both. So take it out as a common factor $(z - 1)\,(z^2 - 9)$

Therefore $z^3 - z^2 - 9z + 9 = (z - 1)\,(z^2 - 9)$

$= (z - 1)\,(z - 3)(\,z + 3\,)$

[Note: $z^2 - 9 = (z - 3)(\,z + 3\,)$]

Exercises

Factorize the following algebraic expressions

(a)	$3y + 3z$	(d)	$4y^2 + y$	(g)	$12x^2y - 8xy^2$
(b)	$3a+ 6y$	(e)	$x^2 - x$	(h)	$2xy + 2y + 3x + 3$
(c)	$3mn - 6n^2$	(f)	$6a^2b^3 + 12a^4b$	(i)	$x^3 + x - 3x^2 - 3$

Algebraic Fractions, as the name denotes, are simply fractions with algebraic expressions in their numerator or denominator or both. The following are examples of algebraic fraction:

$$\frac{3}{x}\ ,\ \frac{2}{x-3}\ ,\ \frac{y+5}{y+3}\ ,\ \frac{2x+10}{x^2+5x}$$

You know that you cannot divide a number by 0. Because division by 0 is impossible, variables in the denominator have certain restrictions. The denominator can never be equal 0.

For example,

In $\frac{3}{x}$, x cannot be 0, i.e., $(x \neq 0)$

In $\frac{2}{x-3}$, $x - 3$ cannot be 0, i.e., $(x \neq 3)$

In $\frac{y+5}{y+3}$, $y + 3$ cannot be 0, i.e., $(y \neq -3)$

In $\frac{2x+10}{x^2+5x}$, $x^2 + 5x$ cannot be 0, i.e., $(x \neq 0, x \neq -5)$

$x^2 + 5x = 0$
$x(x + 5) = 0$
$x = 0$ or $x = -5$

Simplifying algebraic fractions

Simplifying algebraic fractions means reducing them to their lowest term by finding common factors and canceling out the common factors

Example 10

Simplify $\frac{3x^2}{x}$.

Solution

$\frac{3x^2}{x} = \frac{x(3x)}{x} = \frac{(3x^2)}{1} = 3x^2$ [x in the denominator cancels out one $\boldsymbol{x}$ in the numerator]

Or $= \frac{3x^3}{x} = \frac{3 \times x \times x \times x}{x} = 3 \times x \times x = 3x^2$ [one x in the denominator cancels out one x in the numerator]

Example 11

Simplify $\frac{15x^3}{35x^4}$ $(x \neq 0)$.

Solution

$$\frac{15x^3}{35x^4} = \frac{3 \times \cancel{5}\ \cancel{x^3}}{7 \times \cancel{5} \times x \times \cancel{x^3}} = \frac{3}{7x}$$

Example 12

Simplify $\frac{2x^2-6x}{10x}(x\neq 0)$.

Solution

$$\frac{2x^2-6x}{10x}=\frac{\cancel{2x}(x-3)}{\cancel{2x}(5)}=\frac{x-3}{5}$$ [$2x$ is a common factor to both]

Example 13

Simplify $\frac{2-x}{4x-8}$ $(x\neq 2)$.

Solution

$$\frac{2-x}{4x-8}=\frac{2-x}{4x-8}=\frac{-1(\cancel{x-2})}{4(\cancel{x-2})}=\frac{-1}{4}=-\frac{1}{4}$$ [$x-2$ is common factor to both]

Example 14

Simplify $\frac{12a+12b}{3a+3b}(a\neq -b)$.

Solution

$$\frac{12a+12b}{3a+3b}=\frac{12(\cancel{a+b})}{3(\cancel{a+b})}=\frac{12}{3}=4$$

Example 15

Simplify $\frac{\frac{x+1}{4y}}{\frac{x+4}{8y}}$ $(y\neq 0, x\neq -4)$.

Solution

$$\frac{\frac{x+1}{4y}}{\frac{x+4}{8y}}=\frac{x+1}{4y}\div\frac{x+4}{8y}$$ [Use reciprocal multiplication]

$$=\frac{x+1}{\cancel{4y}_{1}}\times\frac{\cancel{8y}^{2}}{x+4}=\frac{2(x+1)}{x+4}$$

EXERCISES

Simplify the following algebraic fractions

(a) $\dfrac{4x}{12x}$

(b) $\dfrac{27y^2}{36y}$

(c) $\dfrac{5x^2}{25x^4}$

(d) $\dfrac{5a^2-10a}{5a^2}$

(e) $\dfrac{12ab-3b^2}{3ab}$

(f) $\dfrac{18b^2+30b}{9b^3}$

(g) $\dfrac{2ax+2bx}{6x^2}$

(h) $\dfrac{16-a^2}{2a-8}$

(i) $\dfrac{-32a^3b^3}{48a^3b^3}$

(j) $\dfrac{\frac{4}{xy}}{\frac{x+2}{6}}$

Addition and subtraction of algebraic fractions

If you remember how to add or subtract fractions with the same or different denominators, adding and subtracting fractional expressions would be easy.

Same denominators

If $\dfrac{a}{b}$ and $\dfrac{c}{b}$ are two fractions having the same denominator b ($b \neq 0$), then

(i) $\dfrac{a}{b}+\dfrac{c}{b}=\dfrac{a+c}{b}$ (ii) $\dfrac{a}{b}-\dfrac{c}{b}=\dfrac{a-c}{b}$, similarly

$$\frac{4x}{x+2}+\frac{3}{x+2}=\frac{4x+3}{x+2} \text{ and } \frac{4x}{x+2}-\frac{3}{x+2}=\frac{4x-3}{x+2}$$

To add or subtract algebraic fractions having the same denominators, simply add or subtract the numerators and then take one of the denominators

EXAMPLE 16

Add or subtract the following algebraic fractions having same denominators

(a) $\dfrac{5}{4x}+\dfrac{-9}{4x}$ (b) $\dfrac{4x+7}{6x}-\dfrac{2x-4}{6x}$ (c) $\dfrac{x-4}{x+1}+\dfrac{3}{x+1}$

Solution

(a) $\dfrac{5}{4x}+\dfrac{-9}{4x}=\dfrac{5+(-9)}{4x}=\dfrac{-4}{4x}=\dfrac{-1}{x}=-\dfrac{1}{x}$ $(x \neq 0)$

(b) $\dfrac{4x+7}{6x}-\dfrac{2x-4}{6x}=\dfrac{(4x+7)-(2x-4)}{6x}=\dfrac{4x+7-2x+4}{6x}=\dfrac{2x+11}{6x}$ $(x \neq 0)$

(c) $\dfrac{x-4}{x+1}+\dfrac{3}{x+1}=\dfrac{x-4+3}{x+1}=\dfrac{x-1}{x+1}$ $(x\neq -1)$

Adding or subtracting algebraic fractions having different denominators

To add or subtract fractions with different denominators, we must first learn how to find the **Least Common Multiple (LCM**) of two or more expressions.

The Least Common Multiple (LCM) of two or more algebraic expressions is the smallest expression that can be divided by each of the given terms. It is found by using the following steps:

Steps

1. Factorize each of the given expressions.
2. From the common factors, take the one with the highest power. (If common factors have equal exponent, take any one of them)
3. Take all the factors that are not common
4. Multiply all the factors that you have considered in the above steps. The product is the LCM

EXAMPLE 17

Find the LCM of y^2, xyz, x^3

Solution

(i) Factorize each:

$y^2 = y \times y$

$xyz = x \times y \times z$

$x^3 = x \times x \times x$

(ii) Take the common factors with the highest power

- y is a common factor to y^2 and xyz, so we take y^2 since it has 2 as exponent compared to the other y with one exponent
- x is a common factor to xyz and x^3, so we take x^3 since it has 3 as exponent compared to x with one exponent

(iii) Though z is not a common factor we will take it.

(iv) Multiply the factors considered in the above steps, that is, $y^2 \times x^3 \times z$ *or* $x^3 y^2 z$ is the LCM of the three terms y^2, xyz, x^3.

Example 18

Find the LCM of ab, bc and cd

Solution

Factorize each:

$ab = a \times b$

$bc = b \times c$

$cd = c \times d$

- b is a common factor to ab and bc, so we take any one b since both have equal exponent
- c is a common factor to bc and cd, so we take any one c since both have equal exponent
- Though a is not a common factor we will take it.
- Though d is not a common factor we will take it.
- Multiply the factors considered in the above steps: So $abcd$ is the LCM of ab, bc and cd

Example 19

Find the LCM of $3x^2yz$ and $4x^3 y^3$

Solution

$3x^2yz = 3 \times x^2 \times y \times z$

$4x^3 y^3 = 4 \times x^3 \times y^3$

Of the common factors x^2 and x^3 we take x^3

Of the common factors y and y^3 we take y^3

Take all other non-common factors, that is, 3, 4 and z

Multiplying all we get $3 \times 4 \times x^3 \times y^3 \times z = 12\, x^3 y^3 z$

Therefore LCM $(3x^2yz, 4x^3 y^3) = 12\, x^3 y^3 z$

- LCM of x^2, x, x^3 is x^3 because x^3 is the least product that contains all x, x^2 and x^3
- LCM of ab, bc and cd is $abcd$ because it is $abcd$ which is the least product that contains all the three

Example 20

Find the LCM of $4x^2y^2$, $6xy$ and $10xy^2$

Solution

$$4x^2y^2 = 2^2 \times x^2 \times y^2$$
$$6xy = 2 \times 3 \times x \times y$$
$$10xy2 = 2 \times 5 \times x \times y^2$$

LCM $(4x^2y^2, 6xy, 10xy^2) = 2^2 \times 3 \times 5 \times x^2 \times y^2 = 60\, x^2\, y^2$

Example 21

Find the LCM of $2x^2$, $x^2 + x$ and $x^3 + 2x$

Solution

$$2x^2 = 2 \times x^2$$
$$x^2 + x = x(x + 1)$$
$$x^3 + 2x = x(x^2 + 2)$$

LCM $(2x^2, x^2 + x, x^3 + 2x) = 2 \times x^2 \times (x + 1) \times (x^2 + 2)$

LCM $= 2x^2(x + 1)\,(x^2 + 2\,)$

Once we saw how to find the LCM of two or more expressions, next let us see how to add or subtract algebraic fractions with different denominators

Example 22

Add or subtract the following algebraic fractions having different denominators

(a) $\dfrac{2x+5}{3}-\dfrac{x-2}{4}$

Solution

Method 1: Make the denominators the same by finding the LCM of the denominators.

The LCM of 3 and 4 is 12. So convert each expression to another equivalent expression having denominator 12.

$$\frac{2x+5}{3}\times\frac{4}{4}=\frac{4(2x+5)}{12} \text{ and } \frac{x-2}{4}\times\frac{3}{3}=\frac{3(x-2)}{12}$$

$$\frac{2x+5}{3}-\frac{x-2}{4}=\frac{2x+5}{3}\times\frac{4}{4}-\frac{x-2}{4}\times\frac{3}{3}$$

$$=\frac{4(2x+5)}{12}-\frac{3(x-2)}{12}$$ [Now they have the same denominators]

$$=\frac{8x+20}{12}-\frac{3x-6}{12}=\frac{8x+20-(3x-6)}{12}$$ [since they have the same denominators, subtract only the numerators)

$$=\frac{8x+20-3x+6}{12}=\frac{5x+26}{12}$$

Method 2: Cross multiply the numerators and denominators as follows:

Solution

$$\frac{2x+5}{3}-\frac{x-2}{4}=\frac{4(2x+5)-3(x-2)}{3\times 4}$$

$$=\frac{8x+20-3x+6}{12}=\frac{5x+26}{12}$$

Remember

If $\frac{a}{b}$ and $\frac{c}{d}$ are any two fractions with $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b}\pm\frac{c}{d}=\frac{(a\times d)\pm(b\times c)}{b\times d}$$

(b) $\frac{6x}{x^2-4}-\frac{3}{x-2}$ $(x \neq -2 \text{ and } x \neq 2)$

Solution

Make the denominators the same by finding the LCM of the denominators.

LCM $(x^2-4, x-2)=(x-2)(x+2)$

$$\frac{6x}{x^2-4}=\frac{6x}{(x-2)(x+2)},\quad \frac{3}{x-2}=\frac{3}{x-2}\left(\frac{x+2}{x+2}\right)=\frac{3(x+2)}{(x-2)(x+2)}$$ [Now both have the same denominators]

$$\frac{6x}{x^2-4}-\frac{3}{x-2}=\frac{6x}{(x-2)(x+2)}-\frac{3}{x-2}\left(\frac{x+2}{x+2}\right)$$

$$=\frac{6x}{(x-2)(x+2)}-\frac{3(x+2)}{(x-2)(x+2)}$$

$$=\frac{6x-(3x+6)}{(x-2)(x+2)}=\frac{3(x-2)}{(x-2)(x+2)}=\frac{3(x-2)}{(x-2)(x+2)}=\frac{3}{x+2}$$

(c) $\frac{x+2}{3x}+\frac{x-3}{6x}$ $(x \neq 0)$

Solution

Method 1: Make the denominators the same by finding the LCM of the denominators.

LCM (3x, 6x) = 6x

$$\frac{x+2}{3x}\times\frac{2}{2}=\frac{2(x+2)}{6x}$$

$$\frac{x+2}{3x}+\frac{x-3}{6x}=\left[\frac{x+2}{3x}\times\frac{2}{2}\right]+\frac{x-3}{6x}$$

$$=\frac{2(x+2)}{6x}+\frac{x-3}{6x}=\frac{2x+4+x-3}{6x}=\frac{3x+1}{6x}$$

Method 2: Cross multiply the numerators and denominators as follows:

Solution

$$=\frac{x+2}{3x}+\frac{x-3}{6x}=\frac{6x(x+2)+3x(x-3)}{6x(3x)}=\frac{6x^2+12x+3x^2-9x}{6x(3x)}=\frac{9x^2+3x}{6x(3x)}=\frac{3x(3x+1)}{6x(3x)}$$

$$=\frac{3x+1}{6x}$$

EXERCISES

Add or subtract the following algebraic fractions

(a) $\frac{a+b}{2}+\frac{a-b}{2}$

(b) $\frac{2x-3}{x-2}+\frac{x-4}{x-2}$

(c) $\frac{10x-5}{2x+1}+\frac{8-4x}{2x+1}$

(d) $\frac{3}{b-1}-\frac{4}{b-2}$

(e) $\frac{4(x+1)}{3}-\frac{5(x-2)}{2}$

(f) $\frac{1}{3x}+\frac{1}{6x}-\frac{1}{x^2}$

(g) $\frac{x}{3}+\frac{x}{4}$

(h) $\frac{3x-5}{x^2-25}-\frac{2}{x+5}$

DEFINITION

A binomial is an algebraic expression having exactly two terms

Example:

$3x + 5$

$x^2 - y^2$

$3x + 3y$

$x^4 - \frac{7}{9}y$ ... are binomials

A monomial is an algebraic expression having exactly one term

Example:

3, x, xy, $3x^2$, $-5xyz$ etc are monomials

A trinomial is an algebraic expression having exactly three terms

Example:

$2x + 3y + 1$, $x^2 - xy + y^3$, $5mn - 7m - 10$ are trinomials

How do we multiply two binomials?

Suppose we want to find the product of $(x + 5)(x + 10)$

We imagine that the term $(x + 5)$ is a single quantity and use it to multiply both the x and the 10 in the second pair of brackets:

$(x + 5)(x + 10)$

$= (x + 5)x + (x + 5)10$

$= x^2 + 5x + 10x + 50$

$= x^2 + 15x + 50$

We must ensure that each term in the first bracket multiplies each term in the second

$(x + 5)(x + 10) = x^2 + 10x + 5x + 50 = x^2 + 15x + 50$

EXAMPLE 23

Find the product of $(x – 5)(2x + 3)$

Solution

$(x – 5)(2x + 3)$

$= x(2x + 3) – 5(2x + 3)$

$= 2x^2 + 3x – 10x – 15$

$= 2x^2 – 7x – 15$

1. Expand the brackets by multiplying $(2x + 3)$ by x and $(2x + 3)$ by $–5$
2. Simplify by collecting like terms: $3x – 10x = –7x$

EXAMPLE 24

Find the product of $(x^2 – 2x^3 + 8)(x + 2)$

Solution

$(x^2 – 2x^3 + 8)(x + 2)$

$x^2(x + 2) – 2x^3(x + 2) + 8(x + 2)$

$= x^2(x)+ x^2(2) – 2x^3(x) + (–2x^3)(2) + 8(x) + 8(2)$

$= x^3 + 2x^2 – 2x^4 – 4x^3 + 8x + 16$

$= x^3 – 4x^3 + 2x^2 – 2x^4 + 8x + 16$

$= (1 – 4)x^3 + 2x^2 – 2x^4 + 8x + 16$

$= –3x^3 + 2x^2 – 2x^4 + 8x + 16$

$= –2x^4 – 3x^3 + 2x^2 + 8x + 16$

1. Expand the brackets by multiplying $(x + 2)$ by x^2, $(x + 2)$ by $–2x^3$ and $(x + 2)$ by 8
2. Simplify by collecting like terms

EXERCISES

Find the product of the following algebraic expressions

(a) $(x + 2)(x + 3)$

(b) $(2p + 3q)(5p – 2q)$

(c) $(a + b)(c + 3)$

(d) $(x + 2)(2x^2 – x – 1)$

(e) $(y-3)(y+2)$
(f) $(4p+3)(2p-q-5)$
(g) $(2x+1)(3x-2)$
(h) $(2z+3)(2z+3)$
(i) $(3x-1)(3x+1)$
(j) $(x^2-2x+1)(x^2+4x+3)$
(k) $(5x-1)(x-5)$
(l) $(3x^2-2x+1)(x^2-4x-5)$

Do you remember what perfect squares are? When a number is multiplied by itself, the product will be a perfect square

Example 25

$0 \times 0 = 0$	$3 \times 3 = 9$	$7 \times 2 = 49$
$1 \times 1 = 1$	$5 \times 5 = 25$	$8 \times 8 = 64$ etc.
$2 \times 2 = 4$	$6 \times 6 = 36$	

So numbers 0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100 etc. are called perfect squares

Similarly you can multiply an algebraic expression by itself to get a perfect square expression

Example 26

Find the product of $(a+b)(a+b)$ or $(a+b)^2$

Solution

$(a+b)(a+b) = a(a+b) + b(a+b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2$

Therefore $(a+b)^2 = a^2 + 2ab + b^2$

Note that $a^2 + 2ab + b^2$ is a perfect square trinomial if is the square of $(a+b)$

Example 27

Find the product of $(a-b)(a-b)$ or $(a-b)^2$

Solution

$(a-b)(a-b) = a(a-b) - b(a-b) = a^2 - ab - ba + b^2 = a^2 - 2ab + b^2$

Therefore $(a-b)^2 = a^2 - 2ab + b^2$

Note that $a^2 - 2ab + b^2$ is a perfect square trinomial

Square of a binomial

$(a + b)^2 = a^2 + 2ab + b^2$

$(a - b)^2 = a^2 - 2ab + b^2$

When a binomial is multiplied by itself it gives a trinomial which is a perfect square

Perfect Square Trinomial

$$(a + b)^2 = a^2 + 2ab + b^2$$

Square the first term (a) — Multiply the two terms by 2 — Square the last term (b)

NOTE: $(a \pm b)^2 \neq a^2 \pm b^2$

EXAMPLE 28

Expand $(3x + y)^2$ using the property of the square of a binomial.

Solution

$(3x + y)^2 =$

- Square the first term $(3x)^2 = 9x^2$
- Square the second term y^2
- Multiply the two terms by 2, $2(3x)(y) = 6xy$
- Add each result $9x^2 + 6xy + y^2$

Therefore, $(3x + y)^2 = (3x)^2 + 2(3x)(y) + y^2 = 9x^2 + 6xy + y^2$

Note that $\mathbf{9x^2 + 6xy + y^2}$ **is a trinomial which is a perfect square**

EXAMPLE 29

Expand $(2x^2 - 5y)^2$ using the property of the square of a binomial.

- Square the first term $(2x^2)^2 = 4x^4$
- Square the second term $(-5y)^2 = 25y^2$
- Multiply the two terms by 2 , $2(2x^2)(-5y) = -20x^2y$
- Add each result $4x^4 - 20x^2y + 25y^2$

Therefore, $(2x^2 - 5y)^2 = (2x^2)^2 + 2(2x^2)(-5y) + (-5y)^2 = 4x^4 - 20x^2y + 25y^2$

EXERCISES

Expand the following binomial squares using the property $\mathbf{(a \pm b)^2 = a^2 \pm 2ab + b^2}$

(a) $(x - z)^2$

(b) $(m + n)^2$

(c) $(2a - b)^2$

(d) $(3m + 2b)^2$

(e) $(6x - 3y)^2$

(f) $(m^2 + n^2)^2$

(g) $(2a^2 - b^2)^2$

(h) $[(3m^2 + 2b)^2]^2$

If two terms in a binomial are perfect squares separated by subtraction, then you can easily factor them.

For example, a^2 **is a perfect square** b^2 **is also a perfect square**

The difference of the two perfect squares, that is, $a^2 - b^2$ or $b^2 - a^2$ is called difference of two squares and can be factorized as follows

Difference of two squares

- $a^2 - b^2 = (a - b)(a + b)$
- $x^2 - y^2 = (x - y)(x + y)$
- $m^2 - n^2 = (m - n)(m + n)$

Example 30

Factorize $x^2 - 4$ using the property of difference of two squares

Solution

(i) Find the square roots of x^2 and 4. The square roots are x and 2 respectively.

(ii) Write the factorization as the sum and difference of the square roots. The sum of the roots is $x + 2$ and the difference between the roots is $x - 2$.

So, $x^2 - 4 = (x)^2 - 2^2 = (x - 2)(x + 2)$

Example 31

Factorize $9x^2 - 16$ using the property of difference of two squares

Solution

(i) Find the square roots of $9x^2$ and 16. The square roots are $3x$ and 4 respectively.

(ii) Write the factorization as the sum and difference of the square roots. The sum of the roots is $3x + 4$ and the difference between the roots is $3x - 4$.

So, $9x^2 - 16 = (3x)^2 - 4^2 = (3x - 4)(3x + 4)$

Example 32

Factorize $36x^2 - 1$ using the property of difference of two squares

Solution

$36x^2 - 1 = (6x)^2 - 1^2 = (6x - 1)(6x + 1)$

Example 33

Factorize $4x^2 - 49y^2$ using the property of difference of two squares

Solution

$$4x^2 - 49y^2 = (2x)^2 - (7y)^2 = (2x - 7y)(2x + 7y)$$

Example 34

Factorize $x^2 - 3$ using the property of difference of two squares

Solution

$$x^2 - 3 = x^2 - (\sqrt{3})^2 \quad [\text{ Note that } (\sqrt{3})^2 = \sqrt{3} \times \sqrt{3} = 3 \,]$$
$$= (x - \sqrt{3})(x + \sqrt{3})$$

Example 35

Factorize $(2a + 3b)^2 - (3b + 4b)^2$ using the property of difference of two squares

Solution

$$(2a + 3b)^2 - (3b + 4b)^2 = [(2a + 3b) - (3b + 4b)][(2a + 3b) + (3b + 4b)]$$
$$= (2a + 3b - 3b - 4b)(2a + 3b + 3b + 4b)$$
$$= (2a - 4b)(2a + 10b)$$

Example 36

Factorize $x^4 - y^4$ using the property of difference of two squares

Solution

$$x^4 - y^4 = (x^2)^2 - (y^2)^2 = (x^2 - y^2)(x^2 + y^2)$$
$$= (x - y)(x + y)(x^2 + y^2)$$

Exercises

Factorize each of the following using the property of difference of two squares

(a) $x^2 - 25$

(b) $x^2 - 100$

(c) $4x^2 - 9y$

(d) $4x^2 - 49$

(e) $\frac{1}{9}x^2 - 81$

(f) $64x^2 - \frac{1}{4}$

(g) $a^2 - 25b^2$

(h) $a^2b^2 - 16$

(i) $a^4 - b^4$

(j) $(3a + 4b)^2 - (4b + 5b)^2$

Quadratic expressions are of the form $ax^2 + bx + c$, where a, b, c are real numbers. You can see that the quadratic algebraic expression of the form $ax^2 + bx + c$ has three terms and the highest exponent of the variable is 2.

Factorizing the quadratic expression of the form $ax^2 + bx + c$ means re writing it as a product of its two linear factors of the form a $(x + p)(x + q)$

Case i. When $a = 1$

$ax^2 + bx + c = x^2 + bx + c$

Rewrite $x^2 + bx + c$ as a product of its linear factors of the form $(x + p)(x + q)$

$x^2 + bx + c = (x + p)(x + q)$

$x^2 + bx + c = x(x + q) + p(x + q)$

$x^2 + bx + c = x^2 + qx + px + pq$

$x^2 + bx + c = x^2 + (p + q)x + pq$

The left and right sides will be equal if $b = p + q$ and $c = pq$. This means, if we can get two number p and q such that their sum is equal to b and their product equal to c, we can easily factorize $x^2 + bx + c$ as $(x + p)(x + q)$

Example 37

Factorize $x^2 + 5x + 6$

Solution

Think of two numbers whose sum is 5 and product 6. These numbers are 2 and 3

$2 + 3 = 5 = b$

$2 \times 3 = 6 = c$ So, $x^2 + 5x + 6 = (x + 2)(x + 3)$ [To easily find the unknown numbers, first find the factors of c and think of the other number that goes with the factors of c to satisfy the sum and product requirement]

Example 38

Factorize $x^2 + 7x + 10$

Solution

Think of two numbers whose sum is 7 and product 10.

The factors of 10 are $\pm 1, \pm 2, \pm 5, \pm 10$

The required numbers are 2 and 5 because $2 + 5 = 7 = b$ and $2 \times 5 = 10 = c$

So, $x^2 + 7x + 10 = (x + 2)(x + 5)$

Example 39

Factorize $y^2 - 12y + 11$

Solution

Find two numbers whose sum is –12 and product 11.

The factors of 11 are ± 1, ± 11.

The required numbers are –1 and –11 because $-1 + -11 = -12 = b$ and $-1 \times -11 = 11 = c$.

So, $y^2 - 12y + 11 = (y - 1)(y - 11)$.

Exercises

Factorize the following quadratic expressions

(a)	$x^2 + 8x + 12$	(d)	$x^2 + 2x - 24$	(g)	$x^2 - 21x + 110$
(b)	$x^2 + x - 12$	(e)	$x^2 - 12x + 35$	(h)	$x^2 - 14x + 40$
(c)	$x^2 + 5x + 6$	(f)	$x^2 - 3x - 18$		

Case ii. When $a \neq 1$

How can we factorize $ax^2 + bx + c$ when a is any number different from 1

When a is any number other than 1, we find two numbers p and q whose sum is b and product $a \times c$. Then we substitute $px + qx$ in place of the middle term bx and factorize the first two terms together and the last two terms together.

Example 40

Factorize $2x^2 + 7x + 3$

Solution

Think of two numbers whose sum is 7 and product $2 \times 3 = 6$.

The required numbers are 1 and 6, because $1 + 6 = 7 = b$ and $1 \times 6 = 6 = ac$

$2x^2 + 7x + 3 = 2x^2 + x + 6x + 3$ [Note that the middle term $7x$ is substituted by $x + 6x$]

Find a common factor of the first two terms and another common factor for the last two terms and factorize

$2x^2 + 7x + 3 = 2x^2 + x + 6x + 3$

$2x^2 + 7x + 3 = x(2x + 1) + 3(2x + 1)$

$2x^2 + 7x + 3 = (2x + 1)(x + 3)$ [taking out $2x + 1$ as a common factor]

Example 41

Factorize $6x^2 + 5x - 6$

Solution

Think of two numbers whose sum is 5 and product $6 \times -6 = -36$.

The required numbers are 9 and −4 because $9 + -4 = 5 = b$ and $9 \times -4 = -36 = ac$

$6x^2 + 5x - 6 = 6x^2 + 9x - 4x - 6$ [Note that the middle term $5x$ is substituted by $9x - 4x$]

Factorize the first two terms and the last two terms together

$6x^2 + 5x - 6 = 6x^2 + 9x - 4x - 6$

$6x^2 + 5x - 6 = 3x(2x + 3) - 2(2x + 3)$

$6x^2 + 5x - 6 = (2x + 3)(3x - 2)$ [taking out $2x + 3$ as a common factor]

Example 42

Factorize $2x^2 - 5x + 2$

Solution

Think of two numbers whose sum is −5 and product $2 \times 2 = 4$.

The required numbers are −4 and −1 because $-4 + -1 = -5 = b$ and $-4 \times -1 = 4 = ac$

$2x^2 - 5x + 2 = 2x^2 - 4x - x + 2$ [Note that the middle term $-5x$ is substituted by $-4x - x$]

Factorize the first two terms and the last two terms together

$2x^2 - 5x + 2 = 2x^2 - 4x - x + 2$

$2x^2 - 5x + 2 = 2x(x - 2) - 1(x - 2)$

$2x^2 - 5x + 2 = (x - 2)(2x - 1)$ [taking out $x - 2$ as a common factor]

Exercises

Factorize the following quadratic expressions

(a) $3x^2 - 8x - 3$

(b) $2x^2 - 14x + 20$

(c) $7x^2 + 18x + 11$

(d) $2x^2 - 7x + 6$

(e) $25x^2 - 10x + 1$

(f) $9x^2 + 6x + 1$

(g) $4x^2 - 1$

(h) $6x^2 - 7x + 2$

The product $(a - b)(a^2 + ab + b^2) = a^3 - b^3$ and $(a + b)(a^2 - ab + b^2) = a^3 + b^3$

Hence, the factors of $a^3 - b^3$ are $a - b$ and $a^2 + ab + b^2$ and the factors of $a^3 + b^3$ are $a + b$ and $a^2 - ab + b^2$.

Therefore we have

$$a^3 - b^3 = (a - b)\,(a^2 + ab + b^2)$$
$$a^3 + b^3 = (a + b)\,(a^2 - ab + b^2)$$

Example 43

Factorize the expressions

(a) $x^3 - 8$ (b) $64x^3 - 125$ (c) $3x^3 - 81y^6$

Solution

(a) $x^3 - 8 = x^3 - 2^3$ (substitute a by x and b by 2 in the difference of cubes $a^3 - b^3$)

$= (x - 2)\,(x^2 + 2x + 4)$

(b) $64x^3 - 125 = (4x)^3 - 5^3$ (substitute a by $4x$ and b by 5)

$= (4x - 5)\,(16x^2 + 20x + 25)$

(c) $3x^3 - 81y^6 = 3\,(x^3 - 27y^6) = 3(x^3 - (3y^2)^3)$

$= 3\,(x - 3y^2)\,(x^2 + 3xy^2 + 9y^4)$

Example 44

Factorize the expressions

(a) $x^3 + 1$ (b) $1000a^3 + \frac{27}{8}b^3$ (c) $25p^3 + 1.6q^3$

Solution

(a) $x^3 + 1 = x^3 + 1^3$ (substitute a by x and b by 1 in the sum of cubes $a^3 + b^3$)

$= (x + 1)\,(x^2 - x + 1)$

(b) $1000a^3 + \frac{27}{8}b^3 = (10a)^3 + \left(\frac{3}{2}b\right)^3$ (substitute a by $10a$ and b by $\frac{3}{2}b$)

$= (10a + \frac{3}{2}b\ \ (100a^2 + 15ab + \frac{9}{4}b^2$

(c) $25p^3 + 1.6q^3 = 25p^3 + \frac{8}{5}q^3$

$= \frac{1}{5}\left(125p^3 + 8q^3\right)$

$= \frac{1}{5}\left((5p)^3 + (2q)^3\right)$ (substitute a by $5p$ and b by $2q$)

$= \frac{1}{5}(5p + 2q)\,\left(25p^2 + 10pq + 4q^2\right)$

Exercises

Factorize the following expressions

1. $x^3 - 1$
2. $a^6 - b^6$
3. $729x^3 - \frac{1}{8}y^3$
4. $x^3 + 8y^3$
5. $6x^3 + 48y^3$
6. $\frac{1}{2}a^3 + 4b^3$

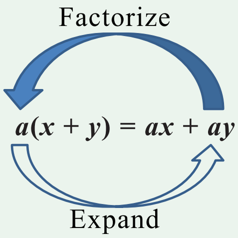
Key Terms

- Algebraic expression
- Algebraic Statements
- Binomial
- Constant
- Coefficient
- Difference of two squares
- Factorization
- Like terms
- Monomial
- Numerical Statement
- Perfect square
- Quadratic expressions
- Term
- Unlike terms
- Variable

Summary

- Mathematical expressions can be numerical or algebraic.
- If a mathematical expression involves only numbers and operations, then it is called a numerical expression. Example: $8 \div 2 + 2 \times 10 - 3$
- If a mathematical expression involves numbers, variables and operations, then it is called an algebraic expression.
  Example: $2x^2 + 3xy - 2$
- The letter or symbol in an algebraic expression is called a variable.
- The number that the variable is being multiplied by is called the **coefficient**
- Any number not joined to a variable is called a constant.
- The parts of an algebraic expression that are separated by addition or subtraction signs are called **terms.**
- Two or more terms that have EXACTLY the same variables are called like terms.

- Like terms can be added or subtracted by adding or subtracting their coefficients
- Algebraic expressions can be evaluated by substituting the values of the variables
- An algebraic expression of the form $a(x + y)$ can be expanded or distributed as $ax + ay$ using the distributive laws.
- Factorization of an algebraic expression is just the reverse operation of expanding an expression. $ax + ay$ is factorized by taking the common factor of the two terms as $a(x + y)$. Factorization is the reverse process of expansion of an algebraic expression.



- An algebraic expression having only one term is called a **monomial**.
- An algebraic expression having only two terms is called a **binomial.**
- An algebraic expression having tree terms is called a trinomial.
- Any two binomials can be multiplied as:
  $(x + y)(a – b) = x(a – b) + y(a – b)$
- The square of a binomial is a product obtained by multiplying the binomial with itself.
  For example, $(x + y)\,(x + y) = (x + y)^2 = x^2 + 2xy + y^2$
  $(x – y)\,(x – y) = (x – y)^2 = x^2 – 2xy + y^2$
- The square of a binomial gives a trinomial which is a perfect square
- The difference of two perfect squares like $m^2 – n^2$ is called difference of two squares and is expanded as $m^2 – n^2 = (m – n)\,(m + n)$
- Quadratic expressions of the form $ax^2 + bx + c$ can be factorized and written as a product of two linear factors. To factorize quadratic expressions we usually find two numbers whose sum is the coefficient of the middle term b and product $a \times c = ac$.

Exercises

1. Which terms are like terms in the expressions $3x\,(3 – 2y)$ and $2(xy + x^2)$?
   (a) $9x$ and $2x^2$
   (b) $–6xy$ and $2xy$
   (c) $9x$ and $2xy$
   (d) $–6xy$ and $2x^2$

2. The coefficient of xy in $3x^2zy + 7xy - 2z^2x$ is

(a) $3z$ (b) -2 (c) $7yz$ (d) 7

3. The value of $a^3 + a^2b + ab^2 + b^3$ at $a = 1$ and $b = -2$ is

(a) -5 (b) 5 (c) 1 (d) 0

4. How many scarfs of half meter length can be made from a total of y meter cloth?

(a) $y + 2$ (b) $2y$ (c) $y+\frac{1}{2}$ (d) $\frac{y}{2}$

5. The length of a side of a square is given as $2x + 3$. Which expression represents the perimeter of the square?

(a) $2x + 16$ (b) $6x + 9$ (c) $8x + 3$ (d) $8x + 12$

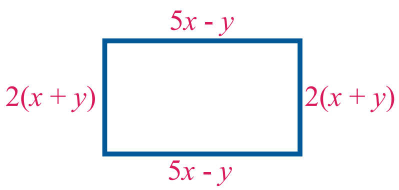
6. If $A = 3x^2 - 4x + 1$, $B = 5x^2 + 3x - 8$, $C = 4x^2 - 7x + 3$, then $A + B - C$ is equal to

(a) $4x^2 + 6x - 10$

(b) $4x^2 - 6x - 10$

(c) $-4x^2 + 6x - 10$

(d) $4x^2 + 6x + 10$



7. Which one is the perimeter of the rectangle with length $7x$ and width y

(a) $14x$ (c) $14x + 2y$

(b) $14x - 2y$ (d) $6x + 2y$

8. Which one is equal to $(5b + 1)^2$?

(a) $25b^2 + 5b + 1$ (c) $25b^2 + 10b + 1$

(b) $25b^2 - 5b + 1$ (d) $25b^2 - 10b + 1$

9. Which one is equal to $x^2 - 4y^2$?

(a) $(x - 4y)(x + 4y)$ (c) $(2x - y)(2x + y)$

(b) $(x - 2y)(x + 2y)$ (d) $(x - 16y)(x + 16y)$

10. The simplified form of $\frac{x^2+5x}{x^2+7x+10}$ for ($x \neq -2$ and $x \neq -5$) is:

(a) $\frac{x+2}{x}$ (b) $\frac{x-2}{x}$ (c) $\frac{x}{x-2}$ (d) $\frac{x}{x+2}$

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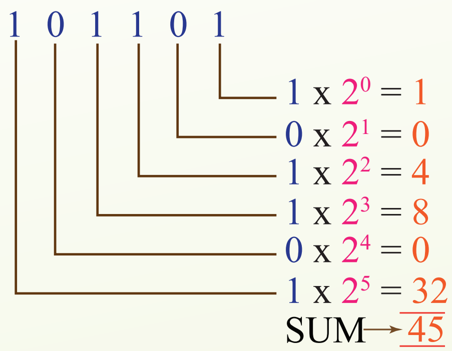
CHAPTER 4

NUMBER BASE

Chapter Contents



Chapter Outcomes

Upon completion of this chapter, learners should be able to:

- explain the relationship between different number base systems;
- convert a given base ten system to other base and vice versa;
- add and subtract in bases five and eight;
- operate with different base systems;
- solve simple equations on bases.

Introduction

We know that the number system that we are using is composed of ten symbols. These symbols are **0, 1, 2, 3, 4, 5, 6, 7, 8,** and **9**. They are called the **Hindu - Arabic numerals** because they were originated in India, adopted by Arabs and then took their current form in Europe. Our number system is called the **base ten system** because its counting system is based only on the ten symbols zero to nine.

Apart from our commonly used base ten number system; other base number systems, different from ten, are also used for different purposes. In this unit therefore, you will also be introduced to other base number systems like base two, base five, base eight and base sixteen number systems.

Activity 1

1. What is the largest number that you can write using the basic whole numbers **0**, **1**, **2**, **3**, **4**, **5**, **6**, **7**, **8**, and **9** without repeating any of these?
2. Consider the decimal number 265. 314. Write the value of each digit.

People are using different number systems with different bases for different purposes. A **base**, (sometimes also called a radix) in math, is defined as the total count of digits used to express numbers in a number system. The most commonly and widely used bases are **binary** (**Base**-**2**) number system which uses two symbols **0** and **1**, **octal** (**Base**-**8**) number system which uses eight symbols **0, 1, 2, 3, 4, 5, 6, 7** , decimal (**Base**-**10**) number system which uses ten symbols **0, 1, 2, 3, 4, 5, 6, 7, 8, 9** , and hexadecimal (**Base-16**) number system which uses sixteen symbols **0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F**). Apart from these, any numerical system can assume any base greater than or equal to two, like **base-3** which uses three symbols **0, 1 and 2), base-4** which uses four symbols **0, 1, 2** and **3), base-5** which uses five symbols **0, 1, 2, 3, 4 and 5).**

Our number system

The number system that we are using having ten symbols **0, 1, 2, 3, 4, 5, 6, 7, 8,** and **9** is called the base ten system. It is called a **base ten system** because its counting system is based on the ten symbols zero to nine. Our number system is also called the decimal number system because any real number can be represented in a decimal form. Why ancient people used the ten symbols 0 to 9 is not clearly known;

but there is the assumption that people might have used these ten symbols as their base of counting just because humans have ten fingers.

The number system that we are using is also called **positional number system.** This is because the position of a symbol or digit determines the value of that symbol carries in that given position. For example the position of the symbol 2 in the number 325,497, gives it a value 20,000 which is much greater than the value 90 of the symbol 9 in that same number. That is why our number system is called a **positional number system** because position matters and determines the value.

Consider the number **5,247,056. 931**. When you put this number in a place value table, it looks like the following:

Place Value table

Millions 10^6	Hundred thousands 10^5	Ten thousands 10^4	Thousands 10^3	Hundreds 10^2	Tens 10^1	Ones 10^0	Decimal point	Tenths 10^{-1} =	Hundredths 10^{-2} =	Thousandths 10^{-3} =
5	2	4	7	0	5	6	•	9	3	1

As seen on the table above, each digit has a unique place value. Using that place value of each digit we can expand **5,247,056. 931** as

$(5 \times 10^6) + (2 \times 10^5) + (4 \times 10^4) + (7 \times 10^3) + (0 \times 10^2) + (5 \times 10^1) + (6 \times 10^0) + (9 \times 10^{-1}) + (3 \times 10^{-2}) + (1 \times 10^{-3})$

Observe that the base is always 10. That means each place is a multiple of 10. That is why our number system is also called a base 10 system. As you move from right to left the exponents increase by 1 and as you move from left to right the exponents decrease by 1.

Recall that any number to the power of zero is 1 ($n^0 = 1, n \neq 0$), and any number to the power of one is the number itself ($n^1 = n$).

Therefore, the place values of each digit is indicated in the table given below

Given is the number 5,247,056. 931 **Expanded form**	**Value of each digit**
The value of 5 is $5 \times 1{,}000{,}000 = 5 \times 10^6$	= 5,000,000
The value of 2 is $2 \times 100{,}000 = 2 \times 10^5$	= 200,000
The value of 4 is $4 \times 10{,}000 = 4 \times 10^4$	= 40,000
The value of 7 is $7 \times 1{,}000 = 7 \times 10^3$	= 7,000
The value of 0 is $0 \times 100 = 0 \times 10^2$	= 0
The value of 5 is $5 \times 10 = 5 \times 10^1$	= 50
The value of 6 is $6 \times 1 = 6 \times 10^0$	= 6
The value of 9 is $9 \times \ = 9 \times 10^{-1}$	= 0.9
The value of 3 is $3 \times \ = 3 \times 10^{-2}$	= 0.03
The value of 1 is $1 \times \ = 1 \times 10^{-3}$	= 0.001
Adding all the values gives the number	**5,247,056. 931**
Five million, two hundred forty seven thousand, fifty six, point nine three one	

Hence, in the number **5,247,056. 931**, there are

5 one millions,

2 hundred thousands,

4 ten thousand,

7 one thousands,

0 hundreds,

5 tens

6 ones 9 tenths, 3 hundredths and 1 thousandth

Base 10 is only just one of the many systems used. People are using different base number systems for different purposes. The following are among the widely used number systems in the area of science and computer science.

Binary or **Base-2** number system; It uses only two symbols **0** and **1**

Quinary or **Base-5** number system; It uses five symbols **0, 1, 2, 3** and **4**

Octal or **Base-8** number system; It uses eight symbols **0, 1, 2, 3, 4, 5, 6** and **7**

Decimal or **Base-10** number system; It uses ten symbols **0, 1, 2, 3, 4, 5, 6, 7, 8** and **9**

Hexadecimal or **Base-16** number system; It uses sixteen symbols **0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E** and **F**)

Converting from other bases to base ten and vice versa

A Binary (base-2) system

Binary is the most commonly used non-base10 system. It is used for coding in computers. The binary or base-2 number system has only two symbols 0 and 1. It is only by repeatedly writing these two symbols at different positions that big numbers are created. The following are numbers zero to nine in the binary counting system.

Base 10	0	1	2	3	4	5	6	7	8	9	10
Base 2	0	1	10	11	100	101	110	111	1000	1001	1010

We don't read binary numbers like 10 as "ten" and 11 as "eleven" and 100 as "one hundred." This is wrong because "Ten" and "eleven" and "one hundred" are names of numbers in the decimal number system, and they do not apply to binary. We should read binary 10 as one-zero, 11 as one-one, and 100 as one-zero-zero.

Remember! $\mathbf{(2)_{10} = (10)_2}$

Positions in the Binary Number System

Remember that the positions in the base ten system are powers of 10; but positions in the binary number system are powers of 2. Here is a table that shows the values of each digit in a binary counting system:

ACTIVITY 2

Think of a group of society or an ethnic group of people living in an Island somewhere in the world. They know only number symbols 0 and 1. They don't have any additional number symbol to represent the number 2 and other bigger numbers. However, they managed to represent bigger numbers grater or equal to two using only the number symbols 0 and 1.

Can you guess the way they represented the number two, three, four and five only using the symbols 0 and 1? How do you think is the number ten represented by this group of people using only 0 and 1?

Place value table (Base 2)						
2^6 (64)	2^5 (32)	2^4 (16)	2^3 (8)	2^2 (4)	2^1 (2)	2^0 (1)
1	1	1	1	0	1	1

The lowest place value (in any base system) is 1. Each place value is equal to the previous place value times the base. In base 2, the place values from right to left are 1, 2, 4, 8, 16, 32, 64 and so on. That means, in the binary number **1101111** for example, there is 1 sixty four, 1 thirty two, 1 sixteen, 1 eight, 0 four(no four), 1 two and 1 one.

Now let us try to assign the values of each digit for the binary number **1101111** as it is indicated on the table above.

The binary number **1101111** is expanded using the values of each digit seen on the table above as $(\mathbf{1} \times 2^6) + (\mathbf{1} \times 2^5) + (\mathbf{1} \times 2^4) + (\mathbf{1} \times 2^3) + (\mathbf{0} \times 2^2) + (\mathbf{1} \times 2^1) + (\mathbf{1} \times 2^0)$
$= 64 + 32 + 16 + 8 + 0 + 2 + 1 = 123$.

As can be seen above, each digit is multiplied by the power of 2 based on its position (the position starts from right to left) and the products are added. Then the first digit has a weight of 1 (2^0), the second digit has a weight of 2 (2^1), the third a weight of 4 (2^2), the fourth a weight of 8 (2^3) and so on.

Remember 123(One hundred twenty three) is the number in the decimal or base 10 system. That means **1101111** in binary or base 2 system is equal to **123** in base 10 system. We usually write this as $\mathbf{(1101111)_2 = (123)_{10}}$

The base of any number is written beside the number as a subscript. For example,

- $(1011)_2$ is read as 1011 base 2
- $(324)_5$ is read as 324 base 5
- $(17)_8$ is read as 17 base 8
- $(324)_{10}$ is read as 324 base 10 (in base ten system the subscript is omitted, not usually written)

1. Converting binary numbers to base ten system

To convert any binary or base-2 number in to our base ten system, we use the following steps:

Step 1: Expand by multiply each digit of the binary number by the corresponding power of two:

Step 2: Solve the powers:

Step 3: Add up the numbers written above: So, the sum is the decimal equivalent of the given binary number

Example 1

Convert the following binary numbers in to decimal or base 10.

(a) $(101101)_2$

Solution

$(101101)_2 = (1 \times 2^5) + (0 \times 2^4) + (1 \times 2^3) + (1 \times 2^2) + (0 \times 2^1) + (1 \times 2^0)$

$= 32 + 0 + 8 + 4 + 0 + 1 = (45)_{10}$

Therefore, $(101101)_2 = (45)_{10}$

(b) $(101100101)_2$

1 0 1 1 0 1

$1 \times 2^0 = 1$
$0 \times 2^1 = 0$
$1 \times 2^2 = 4$
$1 \times 2^3 = 8$
$0 \times 2^4 = 0$
$1 \times 2^5 = 32$
SUM → 45

(c) $(101100101)_2$

Solution

$$(101100101)_2 = (1 \times 2^8) + (0 \times 2^7) + (1 \times 2^6)\ (1 \times 2^5) + (0 \times 2^4) + (0 \times 2^3) + (1 \times 2^2) + (0 \times 2^1) + (1 \times 2^0)$$

$$= 256 + 0 + 64 + 32 + 0 + 0 + 4 + 0 + 1 = (357)_{10}$$

Therefore, $(101100101)_2 = (357)_{10}$

In general, if $d_n\, d_{n-1} \ldots d_3\, d_2\, d_1$ *is* a binary number with n digits:

The decimal number is equal to the sum of all binary digits times their power of 2

That is,

Base 10 $= (d_n \times 2^{n-1}) + (d_{n-1} \times 2^{n-2}) + \ldots + (d_4 \times 2^3) + (d_3 \times 2^2) + (d_2 \times 2^1) + (d_1 \times 2^0)$

EXERCISES

Convert the following binary numbers in to decimal or base 10.

(a) $(111001)_2$ (c) $(11101111)_2$

(b) $(110010)_2$ (d) $(11111111)_2$

2. Converting base ten system to binary

To convert any decimal or base 10 number to base 2, we follow the following steps:

Step1: Divide the given number by 2 and write down the remainder, which must be 0 or 1.

Step2: Divide the quotient by 2 and write down the remainder to the left of the previous remainder

Step 3: Repeat this process until the result of the division is 0.

EXAMPLE 2

Convert the following base 10 numbers to binary:

a) $(19)_{10}$

Division	Quotient	remainder
19 ÷ 2	9	1
9 ÷ 2	4	1
4 ÷ 2	2	0
2 ÷ 2	1	0
1 ÷ 2	0	1

The result of the division has become 0. So we stop here and write the remainders starting from the bottom to top to get the required number in base 2.

So $(19)_{10}$ = $(\mathbf{10011})_2$

b) $(172)_{10}$

Division	Quotient	remainder
172 ÷ 2	86	0
86 ÷ 2	43	0
43 ÷ 2	21	1
21 ÷ 2	10	1
10 ÷ 2	5	0
5 ÷ 2	2	1
2 ÷ 2	1	0
1 ÷ 2	0	1

The result of the division has become 0. So we stop here and write the remainders starting from the bottom to top to get the required number in base 2.

So $(172)_{10}$ = $(\mathbf{10101100})_2$

EXERCISES

Convert the following base 10 numbers to binary:

(a) $(18)_{10}$ (c) $(190)_{10}$ (e) $(651)_{10}$

(b) $(254)_{10}$ (d) $(344)_{10}$

Quinary (base-5) system

Just like the binary (base 2) number system, the base 5 number system is another system of counting which is different from our base 10 or decimal system. The base-5 number system (which is also called **quinary** number system) uses only five symbols **0**, **1**, **2**, **3**, and **4**. It is only by repeatedly writing these five symbols at different positions that big numbers are created. The following are numbers zero to fifteen in the base 5 counting system.

Base 10	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Base 5	0	1	2	3	4	10	11	12	13	14	20	21	22	23	24	30

ACTIVITY 3

Suppose you are given number symbols **0**, **1**, **2**, **3**, and **4** only. To write the number five and more other numbers you have to repeatedly use these five symbols at different positions exactly the way base ten and base two systems have used. So, how do you write the numbers five, six, seven, eight, nine, ten using only **0**, **1**, **2**, **3**, and **4?**

Remember that the positions in the base five system are powers of 5; Here is a table that shows the base 5 number **1423** and the values of each digit in a base 5 counting system:

Place value table (Base 5)						
5^6 (15625)	5^5 (3125)	5^4 (625)	5^3 (125)	5^2 (25)	5^1 (5)	5^0 (1)
			1	4	2	3

The lowest place value (in any base system) is 1. Each place value is equal to the previous place value times the base. In base 5, the place values from right to left are 1, 5, 25, 125, 625, 3125, 15625 and so on. That means the first digit of any base 5 number has a weight of 1 (5^0), the second digit has a weight of 5 (5^1), the third a weight of 25 (5^2), the fourth a weight of 125 (5^3), the fifth a weight of 625 (5^4) and so on.

Hence in the number $(\mathbf{1423})_5$, there are 3 ones, 2 fives, 4 twenty fives and 1 one hundred twenty five. That means $(\mathbf{1423})_5 = (\mathbf{1} \times 5^3) + (\mathbf{4} \times 5^2) + (\mathbf{2} \times 5^1) + (\mathbf{3} \times 5^0)$

$= 125 + 100 + 10 + 3 = 238.$

Therefore, $(\mathbf{1423})_5 = (238)_{10}$

A. Converting base 5 numbers to base ten system

To convert any base 5 number in to our base ten system, we use the same method that we have used in converting binary to decimal.

Step 1: Expand by multiply each digit of the base 5 number by the corresponding power of five:

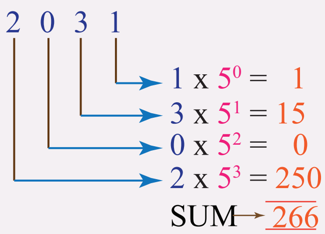
Step 2: Solve the powers:

Step 3: Add up the numbers written above: So, the sum is the decimal equivalent of the given base 5 number

Example 3

Convert the following base 5 numbers to base 10

(a) $(2031)_5$ (b) $(124234)_5$



Solution

(a) $(2031)_5 = (\mathbf{2} \times 5^3) + (\mathbf{0} \times 5^2) + (\mathbf{3} \times 5^1) + (\mathbf{1} \times 5^0)$

$= 250 + 0 + 15 + 1 = (266)_{10}$

Therefore, $(2031)_5 = (266)_{10}$

(b) $(124234)_5 = (1 \times 5^5) + (2 \times 5^4) + (4 \times 5^3) + (2 \times 5^2) + (3 \times 5^1) + (4 \times 5^0)$

$= (1 \times 3125) + (2 \times 625) + (4 \times 125) + (2 \times 25) + (3 \times 5) + (4 \times 1)$

$= 3125 + 1250 + 500 + 50 + 15 + 4 = (4944)_{10}$

Therefore, $(124234)_5 = (4944)_{10}$

EXERCISES

Convert the following base 5 numbers to base 10

(a) $(44)_5$ (c) $(2431)_5$ (e) $(120342)_5$

(b) $(444)_5$ (d) $(10243)_5$

B. Converting base ten system to base 5

To convert any decimal or base 10 number to base 5, follow the following steps:

Step1: Divide the given number by 5 and write down the remainder, which must be 0, 1, 2, 3 or 4.

Step2: Divide the quotient by 5 and write down the remainder to the left of the previous remainder

Step 3: Repeat this process until the result of the division is 0.

EXAMPLE 4

Convert the following base 10 numbers to base 5:

(a) $(266)_{10}$

Division	Quotient	remainder
266 ÷ 5	53	1
53 ÷ 5	10	3
10 ÷ 5	2	0
2 ÷ 5	0	2

The result of the division has become 0. So we stop here and write the remainders starting from the bottom to top to get the required number in base 5.

So $(266)_{10}$ = $(\mathbf{2031})_5$

(b) $(4944)_{10}$

Division	Quotient	remainder
4944 ÷ 5	988	4
988 ÷ 5	197	3
197 ÷ 5	39	2
39 ÷ 5	7	4
7 ÷ 5	1	2
1 ÷ 5	0	1

The result of the division has become 0. So we stop here and write the remainders starting from the bottom to top to get the required number in base 5.

So $(4944)_{10}$ = $(\mathbf{124234})_5$

EXERCISES

Convert the following base 10 numbers to base 5:

(a) $(98)_{10}$ (c) $(2626)_{10}$ (e) $(12560)_{10}$

(b) $(321)_{10}$ (d) $(3748)_{10}$

Octal (base-8) number system

The octal number system uses only number symbols from 0 to 7, that is, **0, 1, 2, 3, 4, 5, 6** and **7.**

Any number greater than 7 is written by repeatedly writing these symbols in different positions. The octal number system has many applications and importance in computers and digital numbering systems. The following table shows numbers up to 16 in an octal counting system

Base 10	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Base 8	0	1	2	3	4	5	6	7	10	11	12	13	14	15	16	17	20

The positions in an octal system are powers of 8. That means the first digit of any base 8 number has a weight of 1 (8^0), the second digit has a weight of 8(8^1), the third a weight of 64 (8^2), the fourth a weight of 512 (8^3), the fifth a weight of 4096(8^4) and so on.

Place value table (Base 8)						
8^6 (262144)	8^5 (32768)	8^4 (4096)	8^3 (512)	8^2 (64)	8^1 (8)	8^0 (1)
			1	3	7	5

As we can see in the table above, in the number $(\mathbf{1375})_8$, there are 5 ones, 7 eights, 3 sixty fours and 1 five hundred twelve. That means $(\mathbf{1375})_8 = (\mathbf{1} \times 8^3) + (\mathbf{3} \times 8^2) + (\mathbf{7} \times 8^1) + (\mathbf{5} \times 8^0) = 512 + 192 + 56 + 5 = 765$.

Therefore, $(\mathbf{1375})_8 = (765)_{10}$.

A. Converting base 8 numbers to base ten system

Converting any base 8 number in to our base ten system follows the same procedure used in binary and base 5 systems. You just expand by multiplying each digit of the base 8 number by the corresponding power of eight and add.

Example 5

Convert the following base 8 or octal numbers to base 10

(a) $(1456)_8$ (b) $(21365)_8$

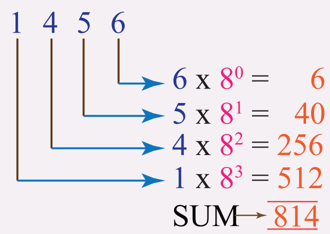
Solution

(a) $(1456)_8$

$= (\mathbf{1} \times 8^3) + (\mathbf{4} \times 8^2) + (\mathbf{5} \times 8^1) + (\mathbf{6} \times 8^0)$

$= 512 + 256 + 40 + 6 = (814)_{10}$

Therefore, $(1456)_5 = (814)_{10}$



(b) $\mathbf{(21365)_8}$

$= (2 \times 8^4) + (1 \times 8^3) + (3 \times 8^2) + (6 \times 8^1) + (5 \times 8^0)$

$= (2 \times 4096) + (1 \times 512) + (3 \times 64) + (6 \times 8) + (5 \times 1)$

$= 8192 + 512 + 192 + 48 + 5 = (8949)_{\mathbf{10}}$

Therefore, $(21365)_5 = (8949)_{\mathbf{10}}$

Exercises

Convert the following base 8 or octal numbers to base 10

(a) $(67)_8$ (c) $(347)_8$ (e) $(10276)_8$

(b) $(456)_8$ (d) $(2054)_8$

B. Converting base ten system to base 8

To convert any decimal or base 10 number to base 8, follow the following steps:

Step1: Divide the given number by 8 and write down the remainder, which must be 0, 1, 2, 3, 4, 5, 6, or 7.

Step2: Divide the quotient by 8 and write down the remainder to the left of the previous remainder.

Step 3: Repeat this process until the result of the division is 0.

Example 6

Convert the following base 10 numbers to base 8:

(a) $(814)_{10}$

Division	Quotient	remainder
814 ÷ 8	101	6
101 ÷ 8	12	5
12 ÷ 8	1	4
1 ÷ 8	0	1

The result of the division has become 0. So we stop here and write the remainders starting from the bottom to top to get the required number in base 8.

So $(814)_{10} = \mathbf{(1456)_8}$

(b) $(8949)_{10}$

Division	Quotient	remainder
8949 ÷ 8	1118	5
1118 ÷ 8	139	6
139 ÷ 8	17	3
17 ÷ 8	2	1
2 ÷ 8	0	2

The result of the division has become 0. So we stop here and write the remainders starting from the bottom to top to get the required number in base 5.

So $(8949)_{10} = \mathbf{(21365)_8}$

Exercises

Convert the following base 10 numbers to base 8:

(a) $(85)_{10}$ (c) $(1234)_{10}$ (e) $(8000)_{10}$

(b) $(141)_{10}$ (d) $(2189)_{10}$

Having a good knowledge of converting numbers from one base to another base is very important in mathematics.

A given number in base x can be converted to any other base y using the following steps:

Step 1: Convert the number from base x to base 10 using expansion method.

Step 2: Convert the number from base 10 to base y using division method.

Example 7

Convert the number $(110101)_2$ to base 5 and 8.

Solution

To convert $(110101)_2$ to base 5 and 8, convert it first to base 10

That is $(110101)_2 =)$ $(\mathbf{1} \times 2^5) + (\mathbf{1} \times 2^4) + (\mathbf{0} \times 2^3) + (\mathbf{1} \times 2^2) + (\mathbf{0} \times 2^1) + (\mathbf{1} \times 2^0)$

$= 32 + 16 + 0 + 4 + 0 + 1 = (53)_{\mathbf{10}}$

Now convert $(53)_{10}$ to base 5 and base 8 by dividing and collecting the remainders.

(i) To convert $(53)_{\mathbf{10}}$ to base 5 divide it by 5 and collect the remainders each time

$$\frac{53}{5} = 10 + \text{remainder } 3$$

$$\frac{10}{5} = 2 + \text{remainder } 0$$

$$\frac{2}{5} = 0 + \text{remainder } 2$$

Writing the remainders starting from the bottom to top we get the required number in base 5.

So $(110101)_2 = (53)_{10} = (203)_5$

(ii) To convert $(53)_{\mathbf{10}}$ to base 8 divide it by 8 and collect the remainders

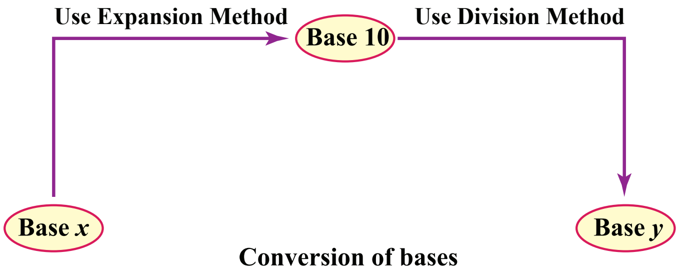
$$\frac{53}{8} = 6 + \text{remainder } 5$$

$$\frac{6}{8} = 0 + \text{remainder } 6$$

Writing the remainders starting from the bottom to top we get the required number in base 8.

So $(110101)_2 = (53)_{10} = (65)_8$

In general, to convert any base to another base use the following method



Conversion of bases

Exercises

1. Express each of the following numbers as binary
   (a) $(34)_5$ (c) $(1024)_5$
   (b) $(2524)_8$ (d) $(332210)_8$
2. Express each of the following numbers to base 5
   (a) $(1111)_2$ (c) $(1011101)_2$
   (b) $(2524)_8$ (d) $(332210)_8$
3. Express each of the following numbers to octal (base 8)
   (a) $(1111)_2$ (c) $(1011101)_2$
   (b) $(2043)_5$ (d) $(10324)_5$

Addition in base five

When we add numbers in base 5, we use a similar method of adding numbers in base 10. We add digits right-to-left one column at a time. If the sum is less than 5, we simply record it as a digit of the sum and move to the next column. If the sum of a column is 5 or greater, then in base 5, it is a two digit number of the form 1k. The k goes at the bottom of the column and becomes a digit of the sum. The 1 gets carried to the next column to the left. Continue this way until every column has been added.

Remember! $(5)_{10} = (10)_5$

Example 8

(a) 3 + 2 = 5 in base 10, but 3 + 2 = 10 in base 5

(b) 3 + 3 = 6 in base 10, but 3 + 3 = 11 in base 5,

$[(6)_{10} = (5 + 1)_{10} = (10 + 1)_5 = (11)_5]$

(c) 3 + 4 = 7 in base 10, but 3 + 2 = 12 in base 5

$[(7)_{10} = (5 + 2)_{10} = (10 + 2)_5 = (12)_5]$

(d) 4 + 4 = 8 in base 10, but 4 + 4 = 13 in base 5

$[(8)_{10} = (5 + 3)_{10} = (10 + 3)_5 = (13)_5]$

(e) 4 + 5 = 9 in base 10; $[(9)_{10} = (5 + 4)_{10} = (10 + 4)_5 = (14)_5]$

(f) 5 + 5 = 10 in base 10; $[(10)_{10} = (5 + 5)_{10} = (10 + 10)_5 = (20)_5]$

(g) $(15)_{10} = (5 + 5 + 5)_{10} = (10 + 10 + 10)_5 = (30)_5$

(h) $(23)_{10} = (5 + 5 + 5 + 5 + 3)_{10} = (10 + 10 + 10 + 10 + 3)_5 = (43)_5$

Activity 4

Add and complete the following base 5 addition table In base five addition table, if sum is 5 and more than 5, then add 5 to it and compare your answer with the one given to the right.

Example: 2 + 6 = 8, add 5 to it and make it 13.

+	0	1	2	3	4
0					
1					
2					
3					
4					

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

Example 9

Add $(303)_5 + (222)_5$

Therefore, $(303)_5 + (222)_5 = (1030)_5$

$$\begin{array}{r} 1\ \ \ \\ 303 \\ +\ 222 \\ \hline 1030 \end{array}$$

→ 3 + 2 = 10 (Write down 0 and carry 1 to the 5th place)

→ 1 + 0 + 2 = 3 (Write down 3 in the 5th place)

→ 3 + 2 = 10 (Write down 0 under the third column and 1 under the next fourth column)

EXAMPLE 10

Add $(3314)_5 + (1332)_5$

$$\begin{array}{r} 111 \\ 3314 \\ +\ 1332 \\ \hline 10201 \end{array}$$

→ 4 + 2 = 11 (Write down 1 and carry 1 to the 2nd column)

→ 1 + 1 + 3 = 10 (Write down 0 and carry 1 to the 3rd column)

→ 1 + 3 + 3 = 12 (Write down 2 and carry 1 to the 4th column)

→ 1 + 3 + 1 = 10 (Write down 0 under column four and write 1 under the 5th column)

Therefore, $(3014)_5 + (1332)_5 = (10201)_5$

EXERCISES

Add the following base 5 numbers

(a) $(203)_5 + (132)_5 =$ ______

(b) $(314)_5 + (141)_5 =$ ______

(c) $(444)_5 + (444)_5 =$ ______

(d) $(1342)_5 + (2211)_5 =$ ______

(e) $(1241)_5 + (3014)_5 =$ ______

(f) $(4321)_5 + (1234)_5 =$ ______

Addition in base 8

When we add numbers in base 8, we use a similar method of adding numbers in base 10. We add digits right-to-left one column at a time. If the sum is less than 8, we simply record it as a digit of the sum and move to the next column. If the sum of a column is 8 or greater, then in base 8, it is a two digit number of the form 1*k*. The *k* goes at the bottom of the column and becomes a digit of the sum. The 1 gets carried to the next column to the left. Continue this way until every column has been added.

Remember! $(8)_{10} = (10)_8$

EXAMPLE 11

(a) 3 + 5 = 8 in base 10, but 3 + 5 = 10 in base 8.

(b) 3 + 6 = 9 in base 10, but 3 + 6 = 11 in base 8.
$[(9)_{10} = (8 + 1)_{10} = (10 + 1)_8 = (11)_8]$

(c) 3 + 7 = 10 in base 10, but 3 + 7 = 12 in base 8.
$[(10)_{10} = (8 + 2)_{10} = (10 + 2)_8 = (12)_8]$

(d) 4 + 7 = 11 in base 10, but 4 + 7 = 13 in base 8.
$[(11)_{10} = (8 + 3)_{10} = (10 + 3)_8 = (13)_8\ ]$

(e) 5 + 7 = 12 in base 10, but 5 + 7 = 14 in base 8.
$[(12)_{10} = (8 + 4)_{10} = (10 + 4)_8 = (14)_8\ ]$

(f) 6 + 7 = 13 in base 10, but 6 + 7 = 15 in base 8.
$[(13)_{10} = (8 + 5)_{10} = (10 + 5)_8 = (15)_8\ ]$

(g) 7 + 7 = 14 in base 10, but 7 + 7 = 16 in base 8.
$[(14)_{10} = (8 + 6)_{10} = (10 + 6)_8 = (16)_8\ ]$

(h) 8 + 7 = 15 in base 10;
$[(15)_{10} = (8 + 7)_{10} = (10 + 7)_8 = (17)_8]$

(i) 8 + 8 = 16 in base 10;
$[(16)_{10} = (8 + 8)_{10} = (10 + 10)_8 = (20)_8]$

(j) $[(16)_{10} = (8 + 8)_{10} = (10 + 10)_8 = (20)_8]$

(k) $[(25)_{10} = (8 + 8 + 8 + 1)_{10} = (10 + 10 + 10 + 1)_8 = (31)_8]$

Example 12

Add $(167)_8 + (765)_8$

```
   111
   167
 + 765
  1154
```

→ 7 + 5 = 14 (Write down 4 and carry 1 to the 2nd column)
→ 1 + 6 + 6 = 15 (Write down 5 and carry 1 to the 3rd column)
→ 1 + 1 + 7 = 11 (Write down 1 and carry 1 to the 4th column)
→ 1 + 0 + 0 = 1 (Write 1 under the 5th column)

Therefore, $(167)_8 + (765)_8 = (1154)_8$

Example 13

Add $(7675)_8 + (6574)_8$

```
   1111
   7675
 + 6574
  16471
```

→ 5 + 4 = 11 (Write down 1 and carry 1 to column 2)
→ 1 + 7 + 7 = 17 (Write down 7 and carry 1 to the 3rd column)
→ 1 + 6 + 5 = 14 (Write down 4 and carry 1 to the 4th column)
→ 1 + 7 + 6 = 16 (Write down 6 and carry 1 to the 5th column)
→ 1 + 0 + 0 = 1 (Write down 1 under the 5th column)

Therefore, $(7675)_8 + (6574)_8 = (16471)_8$.

Exercises

Add the following base 8 numbers

(a) $(1554)_8 + (7364)_8 =$ ______

(b) $(7512)_8 + (4544)_8 =$ ______

(c) $(7722)_8 + (7672)_8 =$ ______

(d) $(7273)_8 + (4565)_8 =$ ______

(e) $(5141)_8 + (7651)_8 =$ ______

(f) $(3043)_8 + (5632)_8 =$ ______

Subtraction in base 5

Subtraction in base five is similar to subtraction in base 10. Remember; in base 10 system we borrow 1 ten from the tens place, 1 one hundred (10 tens) from the hundreds place or 1 one thousand (10 hundreds) from the thousands place. However in base 5, we borrow 1 five from the fives place (column 2 from right to left) or 1 twenty five (5 fives) from the twenty fives place (column 3 from right to left) or 1 one hundred twenty five (5 twenty fives) from column 4

Let us try to subtract the following base 5 numbers:

Example 14

$(234)_5 - (120)_5 =$ ______

Solution

$$\begin{array}{r} 234 \\ -\underline{120} \\ \underline{\underline{114}} \end{array}$$

This subtraction doesn't need borrowing from other place values. So subtract it just as you did it in base 10 system.

Therefore, $(234)_5 - (120)_5 = (114)_5$

Example 15

$(231)_5 - (114)_5 =$ ______

Solution

$$\begin{array}{r} 231 \\ -114 \\ \hline \end{array} \qquad \begin{array}{r} 2\overset{2}{\cancel{3}}\overset{6}{\cancel{1}} \\ -114 \\ \hline \underline{\underline{112}} \end{array}$$

Since 1 is smaller than 4, this subtraction needs borrowing. We have to borrow 1five from the 3 fives in the fives place. When we borrow 1 five, the ones place will be 6 ones. Now rewrite it as follows and subtract.

Therefore, $(231)_5 - (114)_5 = (112)_5$

Example 16

$(2213)_5 - (1342)_5 =$ ______

Solution

$$\begin{array}{r} 2213 \\ -1342 \\ \hline \end{array}$$

The ones place doesn't need borrowing; so subtract 2 from 3. Column 2 (the fives place) needs borrowing. So borrow 1 twenty five (5 fives) from column 3; this will make column two 6 fives and column three will be reduced to 1 (1 twenty five).

$$\begin{array}{r} 2\,\overset{1}{\cancel{2}}\,\overset{6}{\cancel{1}}\,3 \\ -1342 \\ \hline 21 \end{array} \qquad \begin{array}{r} \overset{1}{\cancel{2}}\,\overset{\overset{4}{\cancel{1}}}{\cancel{2}}\,\overset{6}{\cancel{1}}\,3 \\ -1342 \\ \hline 0121 \end{array}$$

To subtract numbers under column three, borrow 1one hundred twenty five (3 twenty fives); this will make column three 4 twenty fives and column four will be reduced to 1.

[Therefore, $(2213)_5 - (1342)_5 = (121)_5$]

Exercises

Find the difference of the following base 5 numbers

(a) $(203)_5 - (132)_5 =$ ________

(b) $(314)_5 - (141)_5 =$ ________

(c) $(401)_5 - (144)_5 =$ ________

(d) $(4314)_5 - (1402)_5 =$ ________

(e) $(3102)_5 - (1103)_5 =$ ________

(f) $(4321)_5 - (1234)_5 =$ ________

Subtraction in base 8

Subtraction in base 8 is the same as subtraction in base 10 or base 5. The only difference here is that we borrow groups of 8. That means, we borrow 1 eight from the eighth place (column 2 from right to left) or 1 sixty four (8 eights) from the sixty fourth place (column 3 from right to left) or 1 five hundred twelve (8 sixty fours) from column 4.

Example 17

$(357)_8 - (126)_8 =$ __________

Solution

$$\begin{array}{r} 357 \\ -126 \\ \hline 131 \end{array}$$

This subtraction doesn't need borrowing. Subtract as usual.

So $(357)_8 - (126)_8 = (131)_8$

Example 18

$(456)_8 - (173)_8 =$ __________

Solution

$$\begin{array}{r} \overset{3}{\cancel{4}}\,\overset{13}{\cancel{5}}\,6 \\ -173 \\ \hline 263 \end{array}$$

This subtraction needs borrowing only in column two. Borrow 1 sixty four (8 eights) from column three; this makes number 5 of column two 13 eights and column three will be reduced to 3.

So $(345)_8 - (146)_8 = (177)_8$

Example 19

$(4102)_8 - (2643)_8 =$ ________

Solution

$$\begin{array}{r} 4102 \\ -2643 \\ \hline \end{array}$$

Borrow from column four (from right to left) 1 five hundred twelves (8 sixty fours).

[8 sixty fours = 7 sixty fours + 7 eighths + 8 ones]

$$\begin{array}{r} {}^{3}\not{4}\,{}^{8}\not{1}\,{}^{7}\not{0}\,{}^{10}\not{2} \\ -2643 \\ \hline 1237 \\ \hline\hline \end{array}$$

Now, give 7 sixty fours to column three, 7 eighths to column two and 8 ones to column one. Column four will be reduced to 3.

So $(4102)_8 - (2643)_8 = (1237)_8$

Exercises

Find the difference of the following base 8 numbers

(a) $(367)_8 - (154)_8 =$ ________
(b) $(374)_8 - (165)_8 =$ ________
(c) $(500)_8 - (247)_8 =$ ________
(d) $(5353)_8 - (4074)_8 =$ ________
(e) $(7145)_8 - (1361)_8 =$ ________
(f) $(4715)_8 - (1237)_8 =$ ________

Multiplication with Base 5

To multiply numbers in base 5, just multiply them in base 10 and convert to base 5.

Example 20

Find the following product

(a) $2 \times 3 = 6$ in base 10, but $(2)_5 \times (3)_5 = (11)_5$
(b) $2 \times 4 = 8$ in base 10, but $(2)_5 \times (4)_5 = (13)_5$
(c) $3 \times 3 = 9$ in base 10, but $(3)_5 \times (3)_5 = (14)_5$
(d) $3 \times 4 = 12$ in base 10, but $(3)_5 \times (4)_5 = (22)_5$
(e) $4 \times 4 = 16$ in base 10, but $(4)_5 \times (4)_5 = (31)_5$

Activity 5

Complete the following bases multiplication table and compare your answer the table given below.

×	0	1	2	3	4
0					
1					
2					
3					
4					

Now we can complete the base 5 multiplication table as follows

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	11	13
3	0	3	11	14	22
4	0	4	13	22	31

Example 21

Find the product of the following base five numbers

(a) $(32)_5 \times (3)_5 =$ _____

Solution

$$\begin{array}{r} {}^{1} \\ 32 \\ \times\ 3 \\ \hline \underline{\underline{201}} \end{array}$$

$3 \times 2 = 11$ (write 1 in one place and carry 1 five to the 2nd column

$3 \times 3 + 1 = 20$ (write 0 in the fives place under column two and write 2 under the 3rd column

Therefore, $(32)_5 \times (3)_5 = (201)_5$

(b) $(343)_5 \times (24)_5 =$ _____

Solution

$$\begin{array}{r} {}^{3\ 2} \\ 343 \\ \times 24 \\ \hline 3032 \\ 1241 \\ \hline \underline{\underline{20442}} \end{array}$$

Multiply 343 by 4 and then by 2. Always convert products which are 5 and above to base 5. Note that:

- $(4)_5 \times (3)_5 = (22)_5$
- $4 \times 4 + 2 = (18)_{10} = (33)_5$
- $4 \times 3 + 3 = (15)_{10} = (30)_5$

Therefore, $(343)_5 \times (24)_5 = \mathbf{(20442)_5}$.

(c) $(441)_5 \times (323)_5 =$ _____

Solution

Note the following

- $(3)_5 \times (4)_5 = (22)_5$
- $3 \times 4 + 2 = (14)_{10} = (24)_5$
- $(2)_5 \times (4)_5 = (13)_5$

Therefore, $(441)_5 \times (323)_5 =$ **$(320043)_5$**.

$$\begin{array}{r} 441 \\ \times 323 \\ \hline 2423 \\ 1432 \\ 2423 \\ \hline 320043 \\ \hline\hline \end{array}$$

EXERCISES

Find the product of the following base five numbers

(a) $(203)_5 \times (2)_5 =$ _________

(b) $(401)_5 \times (3)_5 =$ _________

(c) $(314)_5 \times (4)_5 =$ _________

(d) $(24)_5 \times (41)_5 =$ _________

(e) $(40)_5 \times (23)_5 =$ _________

(f) $(104)_5 \times (14)_5 =$ _________

(g) $(123)_5 \times (32)_5 =$ _________

(h) $(304)_5 \times (123)_5 =$ _________

(i) $(4104)_5 \times (134)_5 =$ _________

Multiplication with Base 8

To multiply numbers in base 8, just multiply them in base 10 and convert to base 8

EXAMPLE 22

Find the following product

(a) $4 \times 2 = 8$ in base 10 but $(4)_8 \times (2)_8 = (10)_8$

(b) $3 \times 3 = 9$ in base 10 but $(3)_8 \times (3)_8 = (11)_8$

(c) $3 \times 4 = 12$ in base 10 but $(3)_8 \times (4)_8 = (14)_8$

(d) $2 \times 7 = 14$ in base 10 but $(2)_8 \times (7)_8 = (16)_8$

(e) $3 \times 5 = 15$ in base 10 but $(3)_8 \times (5)_8 = (17)_8$

(f) $4 \times 4 = 16$ in base 10 but $(4)_8 \times (4)_8 = (20)_8$

(g) $7 \times 7 = 49$ in base 10 but $(7)_8 \times (7)_8 = (61)_8$ [$49 = 6 \times 8 + 1 = 6 \times 10 + 1 = 61$]

Activity 6

Complete the following bases 8 multiplication table and compare your answer the one given below.

×	0	1	2	3	4	5	6	7
0								
1								
2								
3								
4								
5								
6								
7								

Now we can complete the base 8 multiplication table as follows

×	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	10	12	14	16
3	0	3	6	11	14	17	22	25
4	0	4	10	14	20	24	30	34
5	0	5	12	17	24	31	36	43
6	0	6	14	22	30	36	44	52
7	0	7	16	25	34	43	52	61

Example 23

Find the following product of the following base 8 numbers

$(466)_8 \times (2)_8 =$ ________

Solution

$$\begin{array}{r} {}^{1\,1} \\ 466 \\ \times 2 \\ \hline 1154 \\ \hline\hline \end{array}$$

Multiply 466 by 2

$2 \times 6 = (12)_{10} = (14)_{8.}$ Write 4 and carry 1

$2 \times 6 + 1 = (13)_{10} = (15)_8$. Write 5 and carry 1

$2 \times 4 + 1 = (9)_{10} = (11)_8$. Write 1 under column3 and another 1 under column 4

Therefore, $(466)_8 \times (2)_8 = \mathbf{(1154)_8}$.

Example 24

$(3636)_8 \times (741)_8 =$ ________

Solution

$\overset{3\,1\,3}{3636}$	$4 \times 6 = (24)_{10} = (30)_8$. Write 0 and carry 3
$\times 741$	$4 \times 3 + 3 = (15)_{10} = (17)_8$. Write 7 and carry 1
3636	$4 \times 6 + 21 = (25)_{10} = (31)_8$. Write 1 and carry 3
17170	$4 \times 3 + 3 = (15)_{10} = (17)_8$.
32522	$7 \times 6 = (42)_{10} = (52)_8$. Write 2 and carry 5
	$7 \times 3 = (21)_{10} = (25)_8$
3447736	Therefore, $(3636)_8 \times (741)_8 = (3447736)_8$.

Exercises

Find the product of the following base eight numbers

(a) $(657)_8 \times (5)_8 =$ ________

(b) $(716)_8 \times (3)_8 =$ ________

(c) $(504)_8 \times (4)_8 =$ ________

(d) $(25)_8 \times (16)_8 =$ ________

(e) $(47)_8 \times (23)_8 =$ ________

(f) $(323)_8 \times (53)_8 =$ ________

(g) $(666)_8 \times (45)_8 =$ ________

(h) $(647)_8 \times (201)_8 =$ ________

(i) $(3250)_8 \times (142)_8 =$ ________

Dear student! So far we have seen four different number systems. These are our common decimal or base 10 system, base 2 or binary system, base 5 or quinary system and base 8 or octal system. However, this doesn't mean that these are the only base systems that exist. In principle, you can have any number system having any integer base which is greater or equal to two. That means we can count numbers in base 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, etc. What is common to any base n number system is that, the number symbols that base system uses begin by 0 and end by $n - 1$. Study the following table

Number base	Number symbols it uses	Counting in that base
Base 3	0, 1, 2	0, 1, 2, 10, 11, 12, 20, 21 22,100, 101
Base 4	0, 1, 2, 3	0, 1, 2, 3, 10, 11, 12, 13, 20, 21, 22
Base 6	0, 1, 2, 3, 4, 5	0, 1, 2, 3, 4, 5, 10, 11, 12, 13,14, 15, 20
Base 7	0, 1, 2, 3, 4, 5, 6	0, 1, 2, 3, 4, 5, 6, 10, 11, 12, 13,14,15, 16, 20
Base 9	0, 1, 2, 3, 4, 5, 6, 7, 8	0, 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13,14,15, 16, 17, 18, 20
Base10	0, 1, 2, 3, 4, 5, 6, 7, 8, 9	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13,14,15, 16, 17, 18, 19, 20

Addition and subtraction with different bases

When we add or subtract numbers in any base $\boldsymbol{n}$, we use a similar method of adding or subtracting numbers in base 10. We add or subtract digits right-to-left one column at a time. If the sum is less than $\boldsymbol{n}$, we simply record it as a digit of the sum and move to the next column. If the sum of a column is $\boldsymbol{n}$ or greater, then in base $\boldsymbol{n}$, it is a two digit number of the form 1k. The k goes at the bottom of the column and becomes a digit of the sum. The 1 gets carried to the next column to the left. Continue this way until every column has been added.

Similarly during subtraction, in base n, we borrow one n from the n^{th} place (column 2 from right to left) or one n^2, from n^2 place (column 3 from right to left) or one n^3 from column 4 etc.

Example 25

Find the sum and difference of the following numbers in the given base

(a) $(21)_3 + (12)_3$

Solution

$$\begin{array}{r} {\scriptstyle 1} \\ 321 \\ +203 \\ \hline 1130 \\ \hline \end{array}$$

$1 + 2 = (3)_{10} = (10)_3$

$1 + 2 + 1 = (4)_{10} = (11)_3$

Therefore, $(21)_3 + (12)_3 = (110)_3$

(b) $(321)_4 + (203)_4$

Solution

$$\begin{array}{r} {\scriptstyle 1} \\ 21 \\ +12 \\ \hline 110 \\ \hline \end{array}$$

$1 + 3 = (4)_{10} = (10)_4$

$1 + 2 + 0 = (3)_{10} = (3)_4$

$3 + 2 = (5)_{10} = (11)_4$

Therefore, $(321)_4 + (203)_4 = (1130)_4$

(c) $(153)_7 + (216)_7$

Solution

$$\begin{array}{r} {\scriptstyle 1\,1} \\ 153 \\ +216 \\ \hline 402 \\ \hline \end{array}$$

$3 + 6 = (9)_{10} = (12)_7$

$1 + 5 + 1 = (7)_{10} = (10)_7$

$1 + 1 + 2 = (5)_{10} = (11)_4$

Therefore, $(321)_4 + (203)_4 = (1130)_4$

(d) $(121)_3 - (22)_3$

Solution

$$\begin{array}{r} {\scriptstyle 0\ \ \overset{4}{\not 1}\ \ 4} \\ \not 1\,\not 2\,\not 1 \\ -22 \\ \hline 22 \\ \hline \end{array}$$

This subtraction needs borrowing. Borrow 1 three (3 ones) from column two; this makes number 1 of column one 4 and column two will be reduced to 1. To subtract numbers from column 2, borrow 1 nine (3 threes) from column 3

So $(121)_3 - (22)_3 = (22)_3$

Exercises

Find the sum and difference of the following numbers in the given base

(a) $(2020)_3 + (1111)_3$

(b) $(3333)_4 + (2222)_4$

(c) $(4056)_7 + (4302)_7$

(d) $(201)_3 - (122)_3$

(e) $(203)_4 - (130)_4$

(f) $(605)_7 - (336)_7$

In this sub topic, you will be introduced to the techniques of solving some basic equations involving different bases.

Activity 7

If 4 + 6 = 20, what is the base of this counting system?

Example 26

If 6 + 7 = 15, what is the base?

Solution

Let the base be x

$(6)_x + (7)_x = (15)_x$

$(6 \times x^0) + (7 \times x^0) = (1 \times x^1) + (5 \times x^0)$ [Changing all to base 10]

$6 + 7 = 5 + x$

$13 = 5 + x$

$8 = x$ so the base is octal or 8.

Example 27

If $(24)_x + (25)_x = (52)_x$, then find the base x.

Solution

Change them all to base 10 as follows:

$(24)_x + (25)_x = (52)_x$

$(2 \times x^1) + (4 \times x^0) + (2 \times x^1) + (5 \times x^0) = (5 \times x^1) + (2 \times x^0)$

$2x + 4 + 2x + 5 = 5x + 2$

$4x + 9 = 5x + 2$

$4x - 5x = 2 - 9$

$-x = -7$

$x = 7.$

Example 28

If $(4321)_7 + (1234)_7 + (x)_7 = (12341)_7$, find x.

Solution

$(4321)_7 + (1234)_7 + (x)_7 = (12341)_7$

$(5555)_7 + (x)_7 = (12341)_7$ [Note: $(4321)_7 + (1234)_7 = (5555)_7$]

$(x)_7 = (12341)_7 - (5555)_7$ [Note: $(12341)_7 - (5555)_7 = (3453)_7$]

$(x)_7 = (3453)_7$

$x = 3453$.

Example 29

If $(231)_n - (143)_n = (44)_n$. Find the base n.

Solution

Change them all to base 10

$\mathbf{(231)_n - (143)_n = (44)_n}$

$(2 \times n^2) + (3 \times n^1) + (1 \times n^0) - (1 \times n^2) + (4 \times n^1) + (3 \times n^0) = (4 \times n^1) + (4 \times n^0)$

$2n^2 + 3n + 1 - (n^2 + 4n + 3) = 4n + 4$

$2n^2 - n^2 + 3n - 4n + 1 - 3 = 4n + 4$

$n^2 - n - 2 = 4n + 4$

$n^2 - 5n - 6 = 0$

$(n - 6)(n + 1) = 0$

$n = 6$ or $n = -1$ [But number base is not negative]

Therefore $n = 6$.

Exercises

1. Find the base y if $(11)_y = 6$
2. If $(32k)_4 = 56$. Find k
3. If $(103)_x = 67$, find x
4. If $(211)_x = (320)_4$. Find x
5. If $(123)_x = (102)_6$, find x
6. If $(35)_7 + (43)_x = (61)_{10}$, find x.
7. If $(133)x = (57)_8$, find x.
8. If $(123)_y - (83)_{10} = y$, find y.
9. If $(543)_n = (244)_n + (255)_n$. Find n.

Key Terms

- Base
- Binary
- Decimal
- Octal
- Quinary
- Radix

SUMMARY

- Starting from ancient times people were counting with different base systems. Our number system that uses number symbols 0 to 9 is called the decimal or base 10 system. There are also other systems which use different bases for counting. The most common ones are base 2(binary), base 8(octal) and base 10 (decimal) systems. They are broadly used in computer science to represent data in a computer system.
- We can always convert numbers from one base to another either by expansion or by division method. For example
- To convert any base $\boldsymbol{n}$ to base 10, we expand by multiply each digit of the base $\boldsymbol{n}$ number by the corresponding power of $\boldsymbol{n}$ and find the sum.
- To convert a base 10 number to any base $\boldsymbol{n}$, divide the given number by $\boldsymbol{n}$ and write down the remainder. Continue dividing the quotient by $\boldsymbol{n}$ until the result of the division is 0 and collect all remainders from bottom to top.
- To convert any base x to base y, first convert the base x number to base 10 by expansion method and then change the base 10 number to base y by division method
- We can also add, subtract or multiply numbers of different bases paying attention to the counting rules of that particular base system.

EXERCISES

1. Base of a number system is the total number of digits used in that number system.

   (a) True (b) False

2. Number system with base 'n' has its digits in the range between 0 and $n - 1$ [0 and $n - 1$ included]

   (a) True (b) False

3. To convert a binary number to a decimal number, we multiply every digit in the binary number by a power of –

   (a) 2 (c) 5

   (b) 3 (d) 8

4. Which number is not an octal number?

   (a) 5 (c) 7

   (b) 6 (d) 8

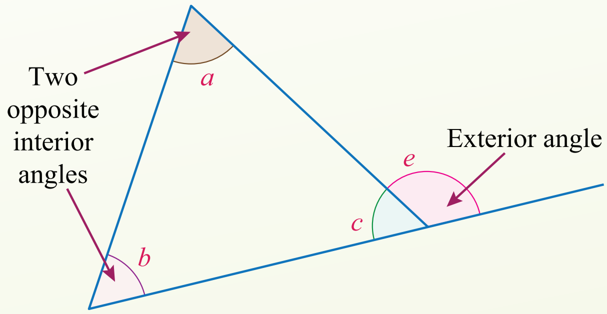
5. When the binary number $(11001011)_2$ is converted to decimal (base 10), it is equal to
   (a) 302 (c) 320
   (b) 203 (d) 230
6. When (1000)10 is converted to base 7, it is equal to ______
   (a) 1616 (c) 3626
   (b) 2626 (d) 4646
7. What is the sum of 7 + 7 in a base 8 system?
   (a) 16 (c) 14
   (b) 15 (d) 13
8. $(32)_5 + (43)_5 =$ ______
   (a) $(75)_5$ (c) $(130)_5$
   (b) $(80)_5$ (d) $(125)_5$
9. $(456)_8 - (173)_8 =$ ______
   (a) $(33)_8$ (c) $(263)_8$
   (b) $(133)_8$ (d) $(333)_8$
10. $(45)_7 \times (63)_7 =$ ______
    (a) $(2421)_7$ (c) $(1221)_7$
    (b) $(4221)_7$ (d) $(4121)_7$
11. If $(103)x = 67$, find x
    (a) 6 (c) 8
    (b) 7 (d) 9
12. If $(123)_x = (102)_6$, find x
    (a) 6 (c) 4
    (b) 5 (d) 3

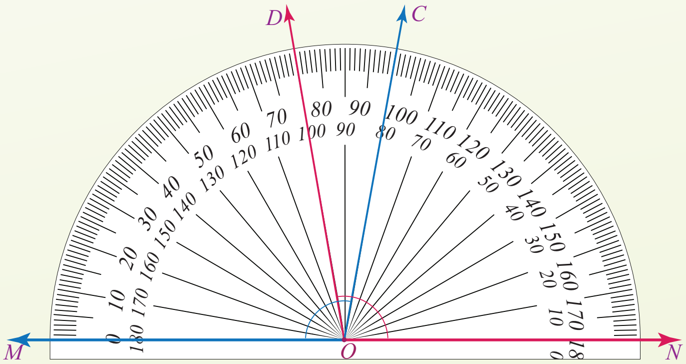
CHAPTER 5

M10CH05

PLANE GEOMETRY

Chapter Contents





Chapter Outcomes

Upon completion of this chapter, you will be able to:

- draw and measure angles;
- explain angle properties of parallel lines;
- draw and name Triangles;
- describe angle properties of triangles;
- define and apply the Pythagoras theorem;
- discuss the properties of polygons;
- describe quadrilaterals like trapezium, Parallelograms, Rectangles, Squares, Rhombuses and calculate their angles.

Introduction

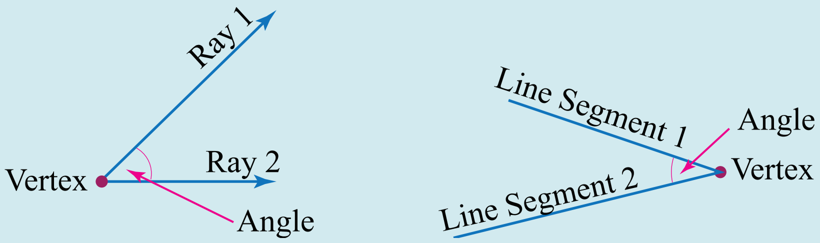
Geometry is a branch of mathematics that deals with the measurement, properties, and relationships of points, lines, angles, plane and solid figures. The part of geometry that studies about points, lines, angles, flat(plane) surfaces & their properties is called **plane geometry**. The other part which studies rigid three-dimensional objects and their properties is called **solid geometry**.

In this unit, you will be introduced with the basic concepts of plane geometry like angles, parallel lines, triangles, polygons, quadrilaterals and their properties and measurement.

Measuring an angle

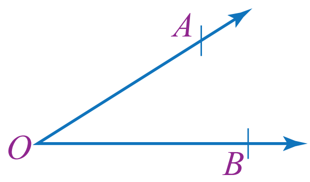
DEFINITION

An **angle** is the union of two rays or line segments with a common endpoint



The common end point for the two rays is called the **vertex** of the angle.

Naming an angle



The angle formed by the union of two rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is named as angle *AOB* or angle *BOA*

Angle *AOB* is symbolically represented as any one of the following: $\angle AOB$ or $\angle BOA$ or $\widehat{AOB}$ or $\widehat{BOA}$

Note that always the middle letter is the vertex of the angle.

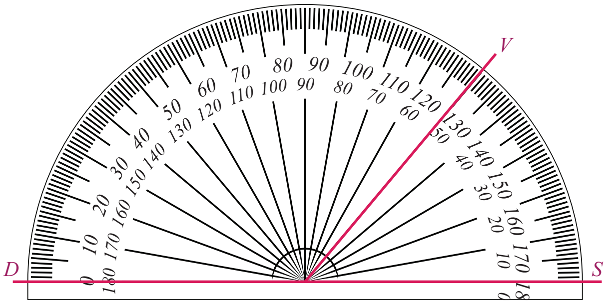
Sometimes, an angle is named simply by stating only the letter of the vertex, as $\angle O$ or $\hat{O}$. [We use this method if the naming doesn't confuse with other adjacent angles]

Measurement of Angles

To measure an angle means to measure the size of the space or opening created between the two rays. This is measured using a special tool called a protractor.

- Angles are measured using a unit called degree.
- A degree is denoted by the symbol "o" just like in 90^{o}, 180^{o} or 360^{o}.
- A protractor is one of the most important geometric tools as it helps in measuring angles in degrees.

When we look at a protractor, we can see measurements from 0 to 180 from left to right at the outer edge and 0 to 180 from right to left on the inner edge. The measurements in both the edges add up to 180°. For example the protractor on the right reads $m\angle STV = 50^0$ and $m\angle DTV = 130^0$ [Note that $50^0 + 130^0 = 180^0$]



The steps to measure an angle are:

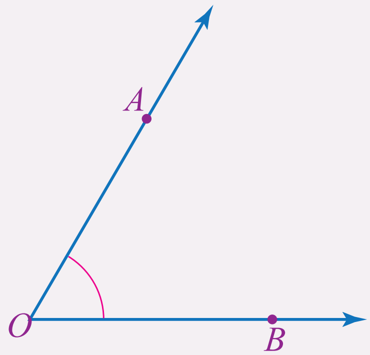
Step 1: Place the center of the protractor on the vertex of the angle.

Step 2: we rotate the protractor so that the 0^o mark lines up with one side of the angle.

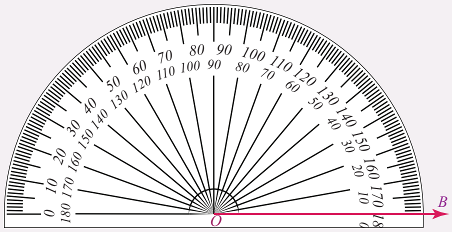
Step 3: The angle is equal to the number of degrees crossed by the other side on the protractor.

Example 1

Measure $\angle AOB$ using a protractor



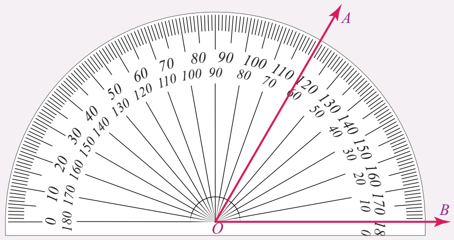
Place the center of the protractor on the vertex O and align the protractor with the ray OB as shown on the right.



The number on the protractor that coincides with the second ray is the measure of the angle.

Do we start reading from the inner 0° on the right or from the outer 0° on the left?

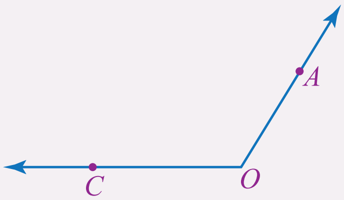
We always start reading from the 0° mark where the first ray (OB) coincides. Since the second ray reads at 60°, $m\angle AOB$ = **60°**



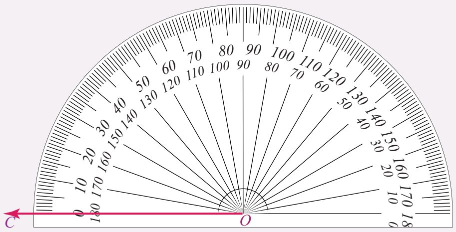
$m\angle AOB$ read as measure of the angle AOB, refers to the real number r that corresponds to the measure of angle AOB.

Example 2

Measure $\angle AOC$ using a protractor

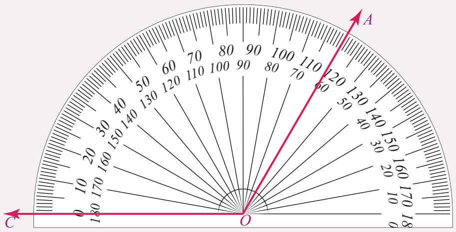


Place the center of the protractor on the vertex O and align the protractor with the ray OC as shown on the right.



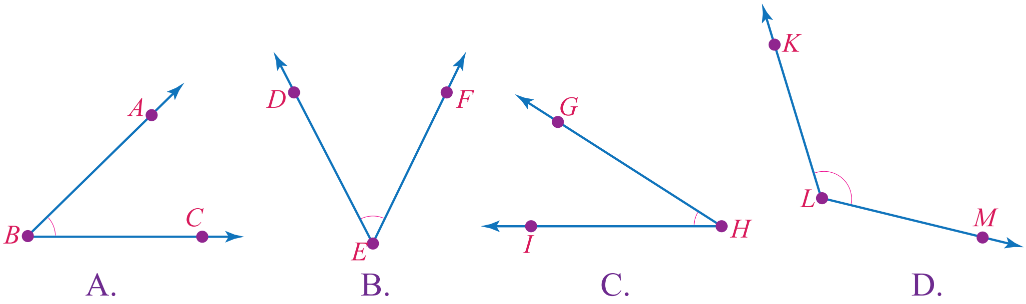
The number on the protractor that coincides with the second ray is the measure of the angle.

Remember! We start reading from the outer 00 on the left because OC coincides to it. The second ray reads 1200, therefore, $m\angle AOC = 120°$



Exercises

1. Find the measures of the following angles using a protractor



2. Write the measures of the angles shown on the protractor given below

(a) $m\angle AON =$ ________

(b) $m\angle DON =$ ________

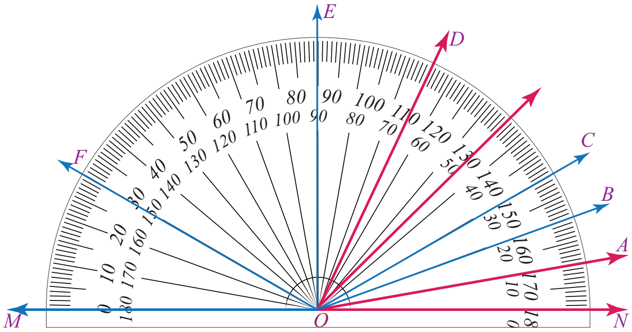
(c) $m\angle EON =$ ________

(d) $m\angle EOM =$ ________

(e) $m\angle FOM =$ ________

(f) $m\angle COM =$ ________

(g) $m\angle BOM =$ ________



Drawing angles

A protractor is used not only for measuring but also for constructing or drawing angles. The following examples illustrate how you can draw an angle if the measure of the angle is given.

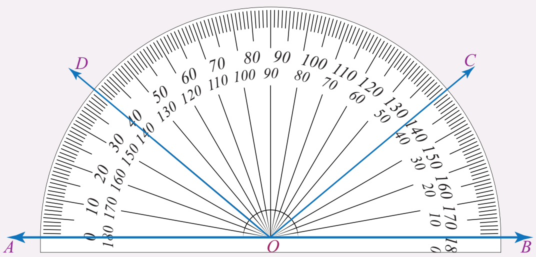
Example 3

Draw an angle of 40° using a protractor

Solution

Steps:

- Draw a baseline *AB*.
- Mark the point *O* and place the center of the protractor at *O*.
- Align the baseline of the protractor with the line *OB*.



- In the inner readings, look for 40º and mark it as point *C*.
- Now using a ruler, join *O* and *C*.
- $m\angle COB = 40°$

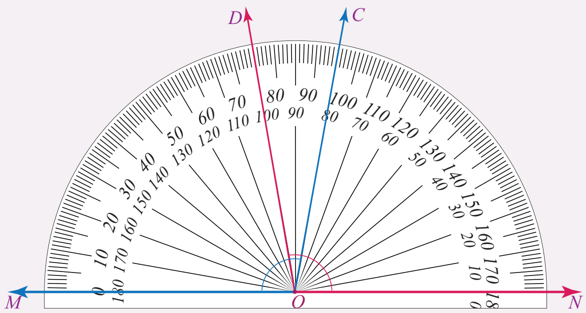
Remember, if you take ray *OA* as your base line and align the protractor with it, you start reading from the outer 0^0 on the left and mark *D* at 40^0. Joining *O* and *D*, you will have $m\angle DOA = 40°$.

Example 4

Draw an angle of 100° using a protractor

Solution

- Draw a baseline *MN*.
- Mark the point *O* and place the center of the protractor at *O*.
- Align the baseline of the protractor with the line *OM*.
- In the OUTER readings, look for 100º and mark it as point *C*.
- Now using a ruler, join *O* and *C*.
- $m\angle MOC = 100°$



Remember! If you align the baseline of the protractor with the line *ON* instead of *OM*, you have to start reading from the INNER 0^0 on the right side. In this case the angle will be $m\angle NOD = 100°$.

Exercises

Draw the angles with the measures given below using a protractor

(a) 45^0 (b) 90^0 (c) 135^0 (d) 180^0

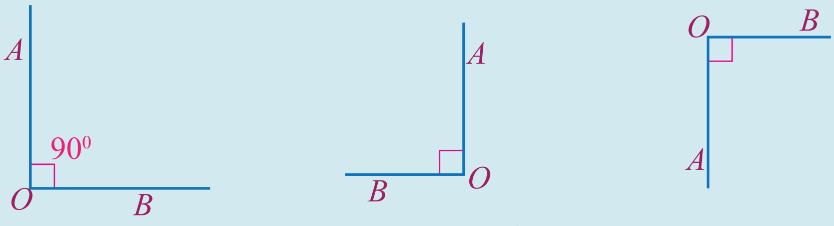
Types of angles

When you draw and measure angles, you may have come across with angles having different sizes and measurements. Some angles have a measure which is exactly equal to 90^0, some may have a measure which is greater than 90^0 and still some others may have a measure less than 90^0.

Angles may be named or classified according to their measures in degrees.

Definition

An angle whose measure is 90^0 is called a right angle

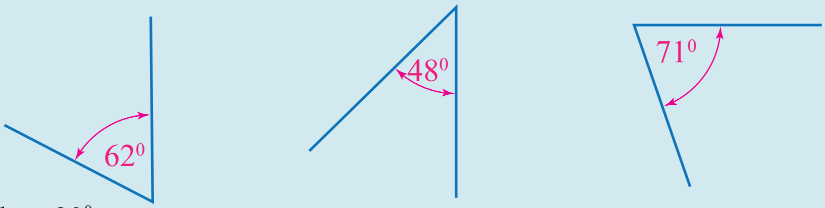


$m\angle AOB = 90^0$

Right angle = 90^0

Definition

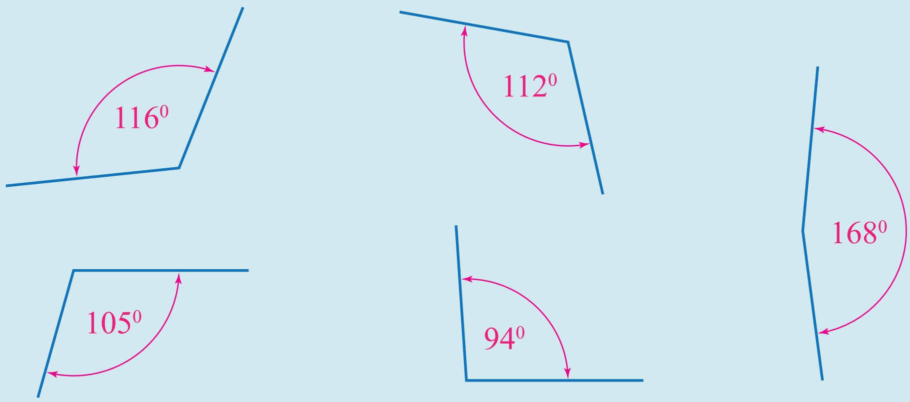
An angle whose measure is less than 90^0 is called an **acute** angle



Acute angle < 90^0

Definition

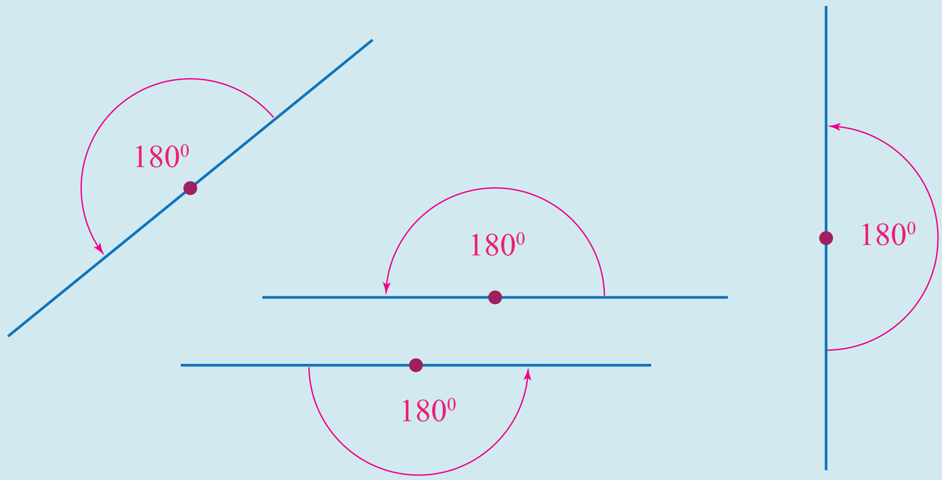
An angle whose measure is greater than 90^0 but less than 180^0 is called an **obtuse** angle**.**



All the above angles are obtuse because their measures are greater than 90^0 but less than 180^0 **90^0 < obtuse angle < 180^0**

Definition

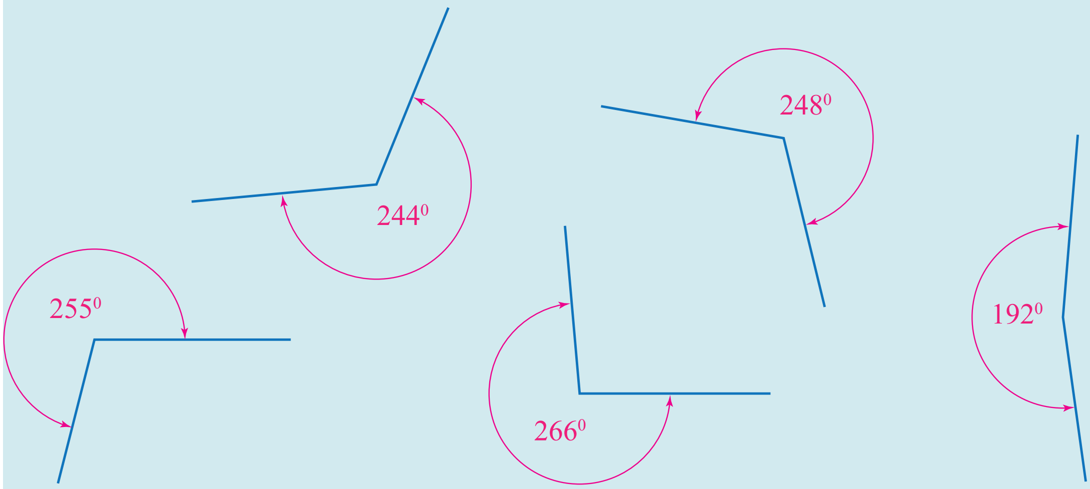
A **straight angle** is an angle whose measure is exactly equal to 180^0.



A straight line measures 180^0

Definition

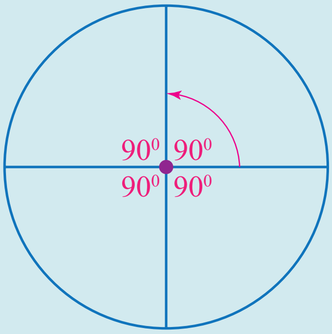
A **reflex** angle is an angle whose measure is greater than 180^0 but less than 360^0

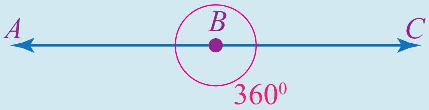


$180^0 <$ reflex angle $< 360^0$

Definition

Full rotation: A complete circle or one full rotation measures 360^0





A Circle measures 360^0

A half circle which is a straight line measures 180^0

Quarter of a circle measures 90^0, it is a right angle

EXERCISES

1. Name the following angles as right, acute, obtuse, straight, and reflex just by looking at them without using a protractor.

   Answers:

   (e) ________________

   (f) ________________

   (g) ________________

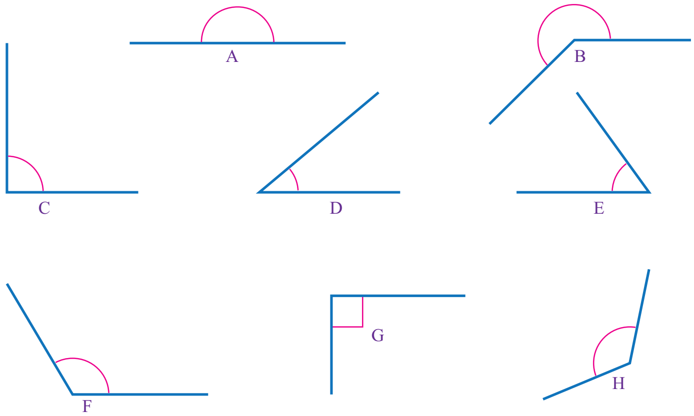
   (h) ________________

   (i) ________________

   (j) ________________

   (k) ________________

   (l) ________________



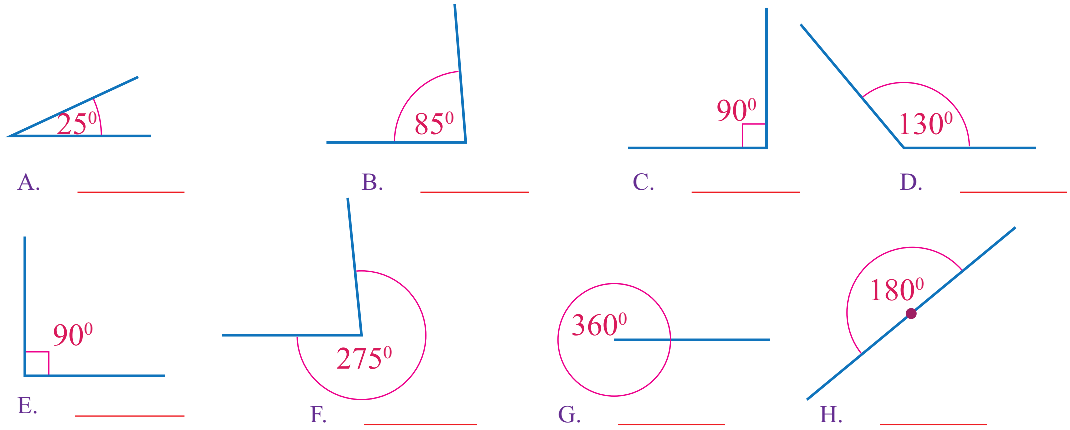
2. The name of the angle formed by the chair shown below is

   (a) Acute angle

   (b) Right angle

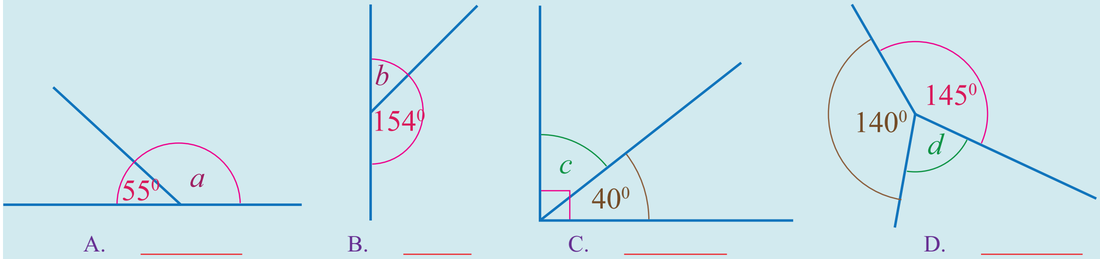
   (c) Obtuse angle

3. Name the following angles as right, acute, obtuse, straight, and reflex



Activity 1

Find the measures of the missing angles indicated by letters *a*, *b*, *c* and *d*?

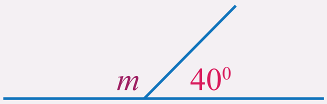


In order to calculate the missing angles, you need to remember the basic facts that

1. A right angle measures 90^0
2. Angle of a straight line measures 180^0
3. A circle or full rotation measures 360^0

Example 5

Find the measures of the missing angles in the following figures?

(a) 

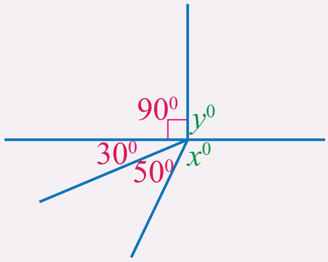
Solution

$m + 40^0 = 180^0$ [Angle of a straight line measures 180^0]

$m + 40^0 - 40^0 = 180^0 - 40^0$

$m = 140^0$

(b)



Solution

$x + 50^0 + 30^0 = 180^0$ **[Angle of a straight line measures 180^0]**

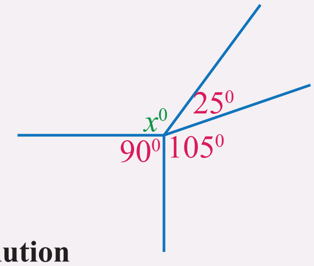
$x + 80^0 = 180^0$

$x = 100^0$

$y + 90^0 = 180^0$ **[Angle of a straight line measures 180^0]**

$y = 90^0$

(c)



Solution

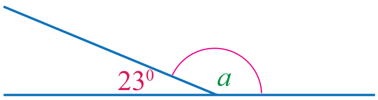
$x + 20^0 + 105^0 + 90^0 = 360^0$ **[A circle measures 360^0]**

$x + 215^0 = 360^0$

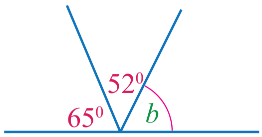
$x = 145^0$

Exercises

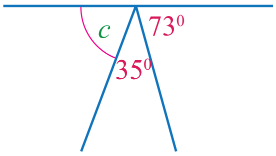
Find the measures of the missing angles in the following figures?



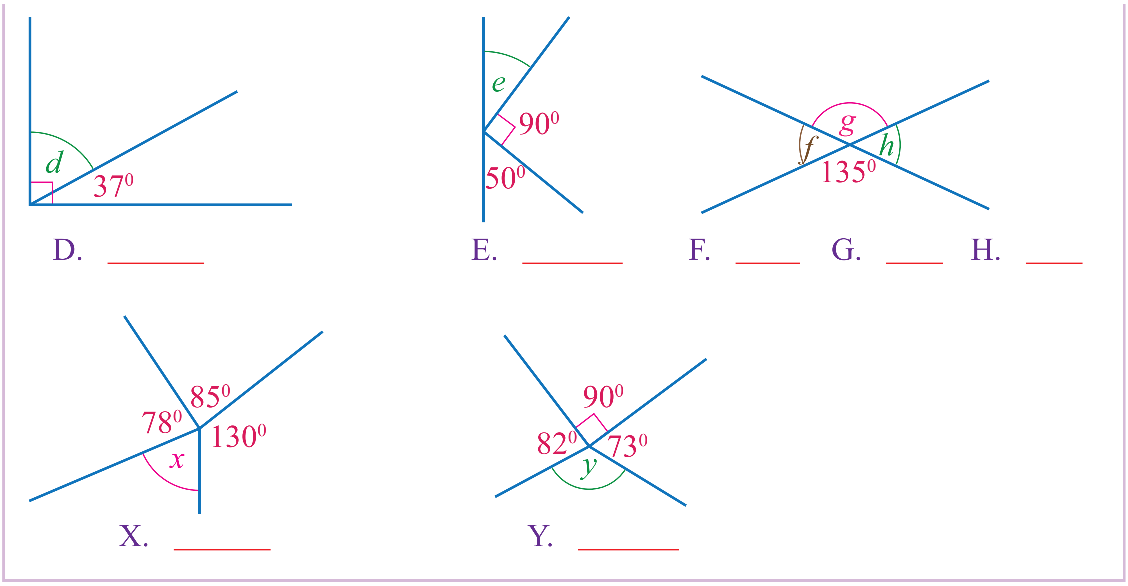
A. ____________



B. ____________



C. ____________



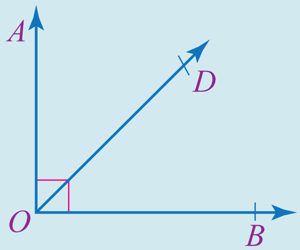
Complementary, supplementary and vertically opposite angles

DEFINITION

Complementary angles

If the sum of the measure of two angles is 90^0, then the two angles are called **complementary**, and each of them is called a complement of the other.

If $m\angle AOB = 90^0$, then $\angle AOD$ & $\angle BOD$ are complementary angles



Example:

A 20^0 angle and a 70^0 angle are complementary.

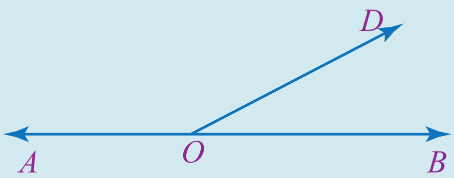
Similarly a 30^0 and a 60^0 angle are complementary.

DEFINITION

Supplementary angles

If the sum of the measures of two angles is equal to 180^0 (two right angles) then the angles are called supplementary, and each of them is called a supplement of the other.

Since $m\angle AOD + m\angle BOD = 180^0$, the angles AOD and BOD are called supplementary angles



Example:

A 30^0 and 150^0 angles are supplementary. Similarity a 60^0 and 120^0 angle are also supplementary.

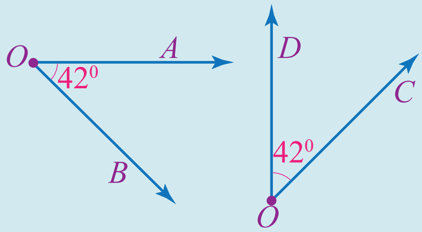
DEFINITION

Congruent angles

Two angles that have the same degree measures are called **congruent**

$\angle AOD \cong \angle DOC$

" $\cong$ " is the symbol for congruence



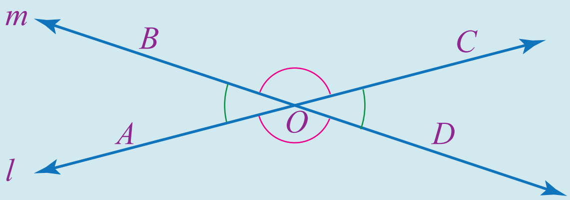
DEFINITION

Vertically opposite angles

Two angles are *vertically opposite* if their sides form two pairs of opposite rays.

$\angle AOB$ **and** $\angle DOC$ are vertically opposite

$\angle AOD$ **and** $\angle BOC$ are vertically opposite



EXAMPLE 6

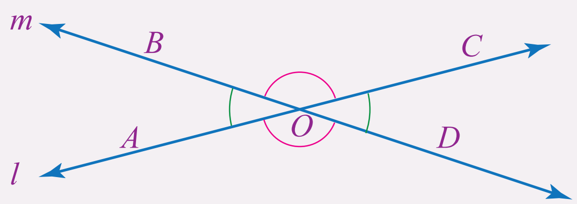
Prove the following theorem

Theorem 1: If two lines intersect, the vertically opposite angles are congruent.

Solution

In the figure below; $\angle AOB$ **and** $\angle COD$ are vertically opposite

$\angle BOC$ **and** $\angle DOA$ are vertically opposite



We want to show that $\angle AOB \cong \angle COD$ and $\angle BOC \cong \angle DOA$

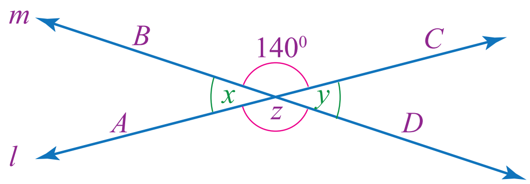
Proof:

Statement	Reason
1. $\angle AOB$ is supplementary to $\angle BOC$	1. Since their Sum = 180°.
2. $\angle COD$ is supplementary to $\angle BOC$	2. Since their Sum = 180°.
3. $\angle AOB \cong \angle COD$ Similarly it can be proved that $\angle BOC \cong \angle DOA$	3. Transitivity of congruence of angles and steps 1 and 2.

Exercises

1. Which of the following pairs of angles are complementary?
   (a) 70°, 20° (b) 20°, 160° (c) 50°, 130°
2. Which of the following pairs of angles are supplementary?
   (a) 105°, 75° (c) 10°, 170°
   (b) 125°, 55° (d) All
3. The complement of 45° is
   (a) 55° (b) 135° (c) 45° (d) 145°
4. The supplement of 179^0 is
   (a) 79° (c) 19°
   (b) 17° (d) 1°
5. Find the measure of the missing angles and give your reason

   z = ________ Reason: ________
   x = ________ Reason: ________
   y = ________ Reason: ________



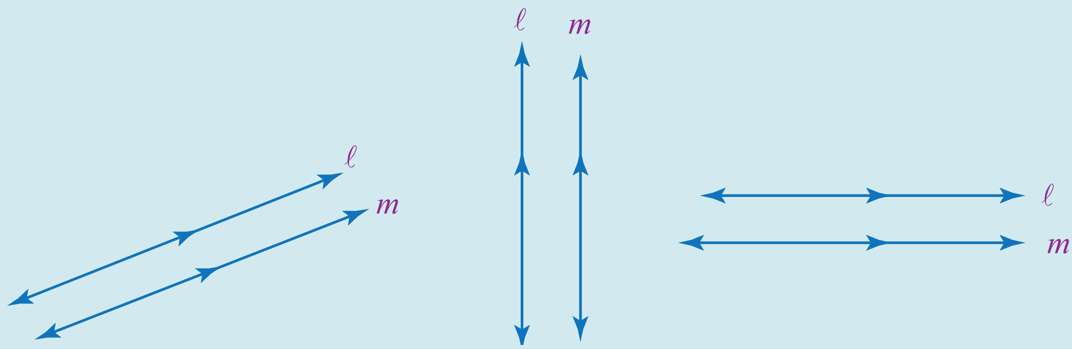
Angles and parallel lines

ACTIVITY 2

What kinds of lines are called parallel lines?
What are perpendicular lines?

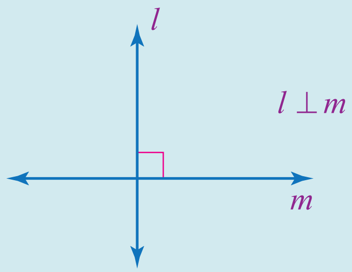
DEFINITION

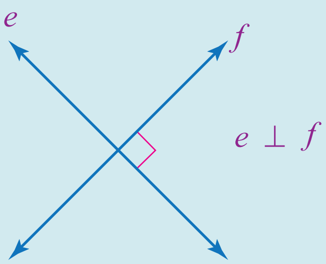
Two straight lines ℓ and m on the same plane are said to be **parallel,** written $\ell \perp m$, if they do not intersect.



DEFINITION

Two intersecting lines ℓ and m are said to be **perpendicular**, written $\ell \perp m$, if the two lines form 90°.





DEFINITION

A transversal is a line that intersects or cuts two other lines in two different points.

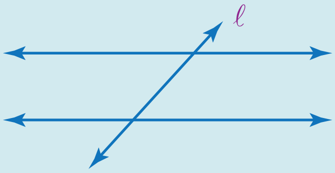


Figure (1)

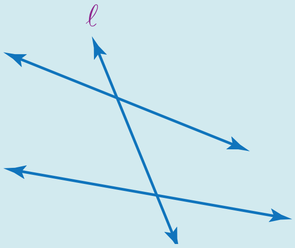


Figure (2)

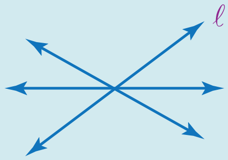
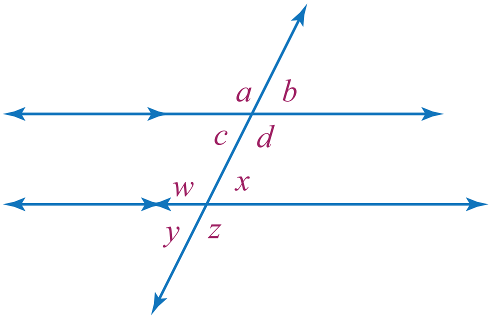


Figure (3)

- Line ℓ in fig(1) and fig (2) is a transversal line
- But ℓ in fig (3) is not a transversal line, since it doesn't cut the two lines at two different points

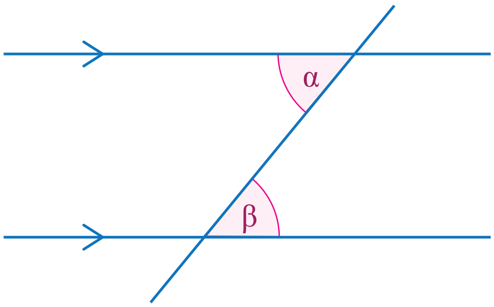
When two parallel lines are cut by a transversal, the following angles are formed.



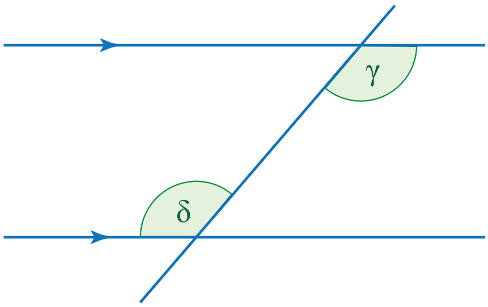
- **Alternate interior angles:** c and x; d and w
- **Corresponding angles:** a and w; b and x; c and y; d and z

Properties of angles and parallel lines

Property 1: If two parallel lines are cut by a transversal, then the alternate interior angles formed are congruent.

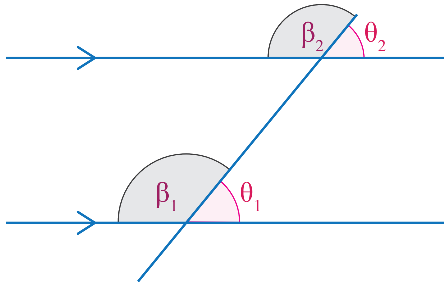


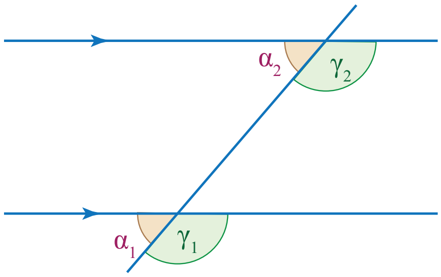
α and β are alternate interior angles
$\alpha = \beta$



δ and γ are alternate interior angles
$\delta = \gamma$

Property 2: If two parallel lines are cut by a transversal, then the corresponding angles formed are congruent.





θ_1 and θ_2 are corresponding angles; $\theta_1 = \theta_2$

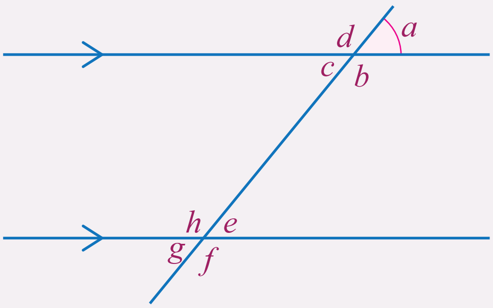
β_1 and β_2 are corresponding angles; β_1 and β_2

α_1 and α_2 are corresponding angles; $\alpha_1 = \alpha_2$

γ_1 and γ_2 are corresponding angles; $\gamma_1 = \gamma_2$

Example 7

If lines m and n are parallel and $a = 65^0$, then find the remaining angles. Give reasons for each of your answer.



Solution

If $a = 65^0$, then $b = 115^0$ [*since* $a + b = 180^0$]

If $b = 115^0$, then $c = 65^0$ [*since* $c + b = 180^0$]

If $b = 115^0$, then $d = 115^0$ [*because b and d are vertically opposite angles 180⁰*]

If $c = 65^0$, then $e = 65^0$ [*since c and e are alternate interior angles*]

If $b = 115^0$, then $h = 115^0$ [*since b and h are alternate interior angles*]

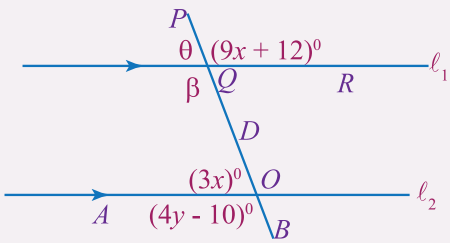
If $b = 115^0$, then $f = 115^0$ [*since b and f are corresponding angles*]

If $c = 65^0$, then $g = 65^0$ [*since c & g are corresponding angles*]

Example 8

Find the measures of the following angles:

1. $m\angle PQR =$ ______
2. $m\angle AOD =$ ______
3. $m\angle AOB =$ ______



Solution

1. As indicated in the figure ℓ_1 and ℓ_2 are parallel lines cut by a transversal

$\theta = (3x)^0$ [corresponding angles]

$\theta + (9x + 12)^0 = 180^0$ [Angle of a straight line]

$(3x)^0 + (9x + 12)^0 = 180^0$ [From the above two steps]

$3x + 9x = 180 - 12$

$12x = 168$

$x = 14$

2. If $x = 14$, then $(9x + 12)^0 = (9 \times 14 + 12)^0 = 138^0$. So, $m\angle PQR = 138^0$
3. Similarly, $(3x)^0 = (3 \times 14)^0 = 42^0$. So, $m\angle AOD = 42^0$
4. $\beta = (9x + 12)^0$ [Vertically opposite angles]
5. $\beta = (4y - 10)^0$ [corresponding angles]

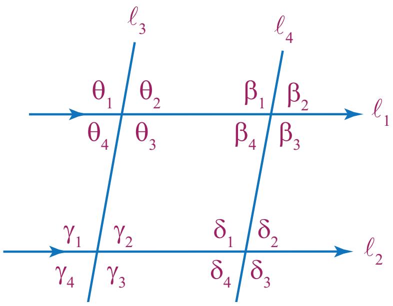
$(9x + 12)^0 = (4y - 10)^0$ [From the above two steps]

$138^0 = (4y - 10)^0$ [since $(9x + 12)^0 = 138^0$]

Therefore, $(4y - 10)^0 = m\angle AOB = 138^0$

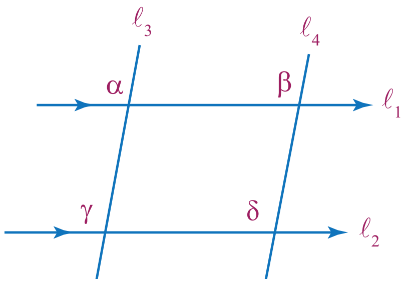
Exercises

1. In the figure shown below $\ell_1 // \ell_2$ and $\ell_3 // \ell_4$ and $\theta_3 = 102^0$. Then find



(i) γ_1
(ii) γ_2
(iii) β_3
(iv) γ_3
(v) δ_2
(vi) δ_3

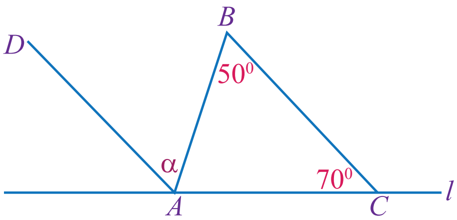
2. In the figure shown below $\ell_1 // \ell_2$ and $\ell_3 // \ell_4$. If $\alpha = (3x + 15)^0$, $\beta = (4x - 5)^0$, $\delta = (5y)^0$, then find the values of x and y?



3. In the figure below, if $\overline{BC} \parallel \overline{AD}$ and ℓ is a line, $m\angle B = 50^\circ$ and $m\angle C = 70^\circ$, then

(i) $\alpha =$ ________

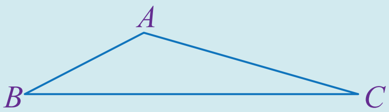
(ii) $\beta =$ ________



DEFINITION

A **triangle** is a simple closed curve composed of *three line segments* that are joined end to end.

A **triangle** is the union of three non- collinear points in a plane and the three segments connecting them



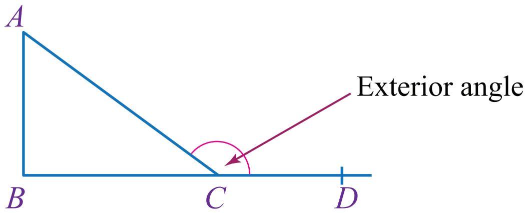
The points A, B and C are called the **vertices** (singular **vertex**) of the triangle.

The segments $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ are called **sides** of the triangle.

A triangle is named by stating the three vertices. The above figure shows triangle ABC, written ΔABC.

The three angles, i.e, $\angle ABC$, $\angle BCA$ and $\angle BAC$ are called **interior angles** of ΔABC.

Exterior angle: When a side of a triangle is produced or extended, the angle so formed with *the adjacent side* is called an **exterior angle.** Side $\overline{BC}$ is produced to D. $\overline{AC}$ is adjacent to $\overline{BC}$. $\angle ACD$ is called an **exterior angle**.



When you extend or produce each side in two directions, you will get six exterior angles. Can you name them?

Types of triangles

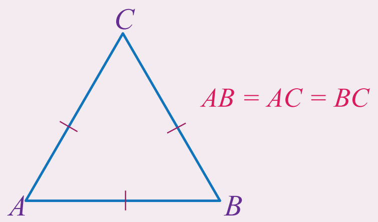
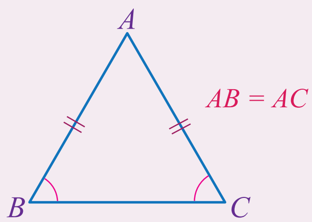
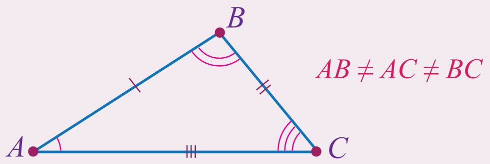
Triangles are classified depending on

(i) the length of their **sides**

(ii) the size of their **angles**

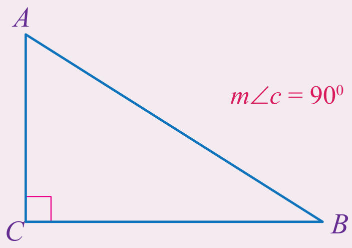
A. Classification of triangles according to the length of their sides.

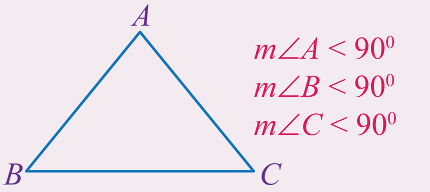
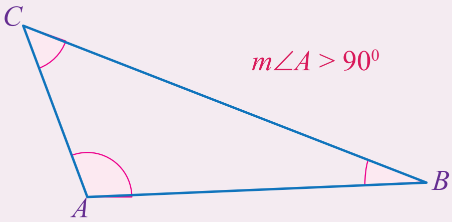
By studying the lengths of the three sides of a triangle, we can group triangles in to the following three classifications:

Equilateral triangle	A triangle which has <u>three equal sides</u>	
Isosceles triangle	A triangle which has <u>two equal sides</u>	
Scalene triangle	*A triangle* which has <u>all three sides different in length (un equal)</u>	

B. Classification of triangles according to the size of their angles.

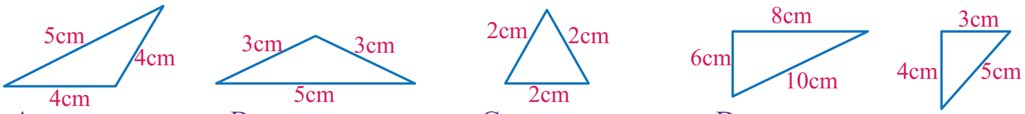
Triangles can also be classified by considering the measures of the three interior angles.

Right triangle	A triangle having one of its three angles a right angle (measures 90^0)	 The side *AB* opposite the right angle is called the **hypotenuse** and Sides *AB* & *BC* are called **legs** of the right triangle

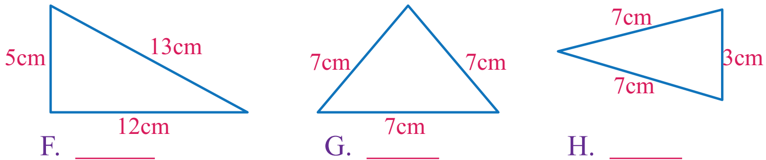
Acute triangle	A triangle having all of its three angles a measure of less than 90^0	
Obtuse triangle	A triangle having one of its three angles a measure of greater than 90^0	

EXERCISES

1. Name the following triangles as equilateral, isosceles or scalene triangle:

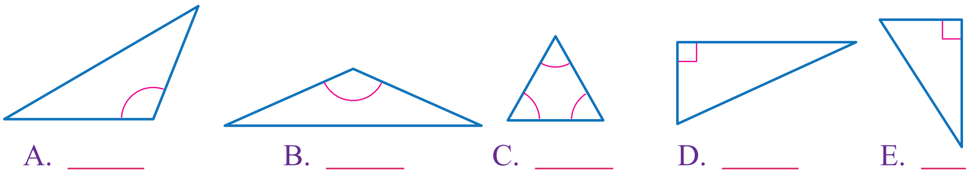


A. ______ B. ______ C. ______ D. ______ E. ______

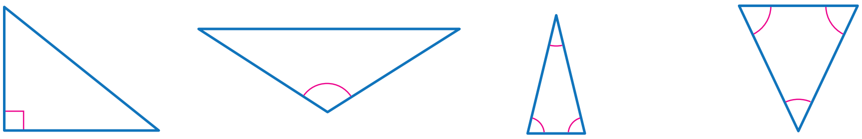


F. ______ G. ______ H. ______

2. Name the following triangles as right, acute or obtuse angle triangle

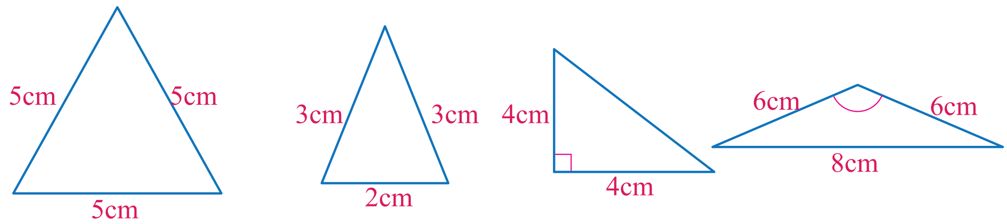


A. ______ B. ______ C. ______ D. ______ E. ______

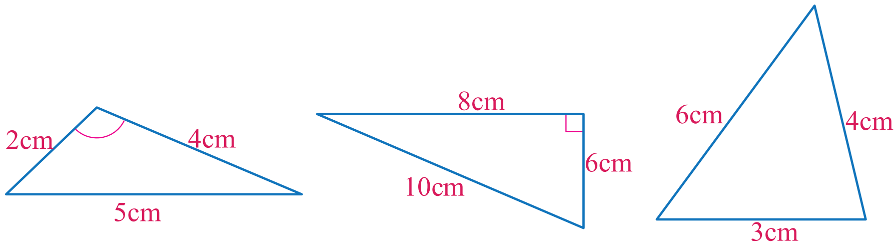


F. ______ G. ______ H. ______ I. ______

3. Classify the following triangles using both their side length and angles and complete the blank spaces



A. Equilateral and acute B. ________ C. ________ D. ________



E. Scalene and obtuse F. ________ G. ________

4. Complete the following table by putting a "✓" mark if a triangle satisfying both the left and right side attributes exists and a "×" mark if such triangle doesn't exist.

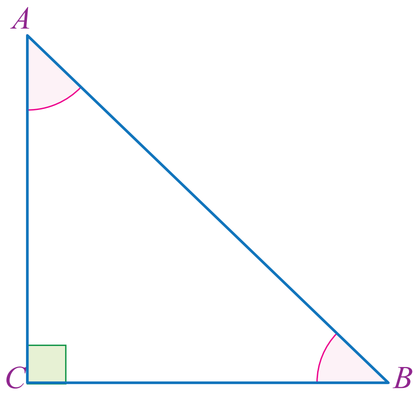
	Right	Acute	Obtuse
Equilateral	×	✓	
Isosceles			
Scalene			

Activity 3

1. Draw any triangle *ABC*. Measure the interior angles at *A*, *B* and *C* using your protractor carefully. What will be the sum of the three interior angles?

2. Draw any triangle ABC. Extend or produce side BC to create an exterior angle at C. Measure carefully the exterior angle using a protractor. Measure the two interior angles at A and B which are opposite to the exterior angle and find their sum. How do you compare the sum of the two with the measure of the exterior angle?

1. Angle sum property



The sum of the measures of the three interior angles of any triangle is 180

For any triangle ABC,

$m\angle A + m\angle B + m\angle C = 180^0$

Example 9

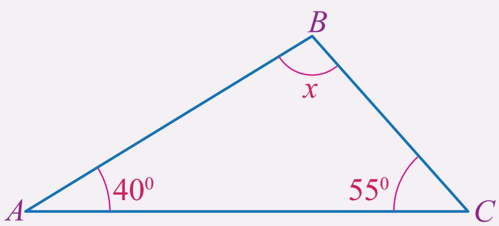
In triangle ABC, if $m\angle A = 40^0$, $m\angle C = 55^0$, then $m\angle B =$ __________

Solution

$m\angle A + m\angle B + m\angle C = 180^0$ [Angle sum property]

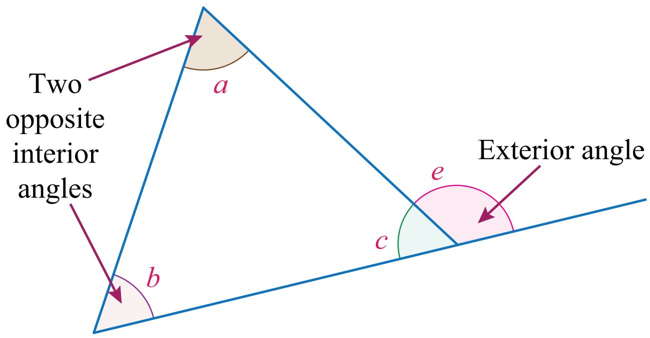
$40^0 + 55^0 + x = 180^0$

$x = 85^0$



2. Exterior angle property

The exterior angle of a triangle is equal to the sum of the measure of the two interior angles opposite to it.



EXAMPLE 10

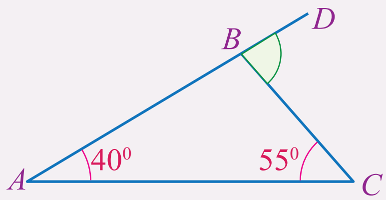
In triangle *ABC*, if side *AB* is produced to *D* and $m\angle A = 40^0$, $m\angle C = 55^0$, then $m\angle DBC =$ ________

Solution

$m\angle DBC = m\angle A + m\angle C$ *[Exterior angle property]*

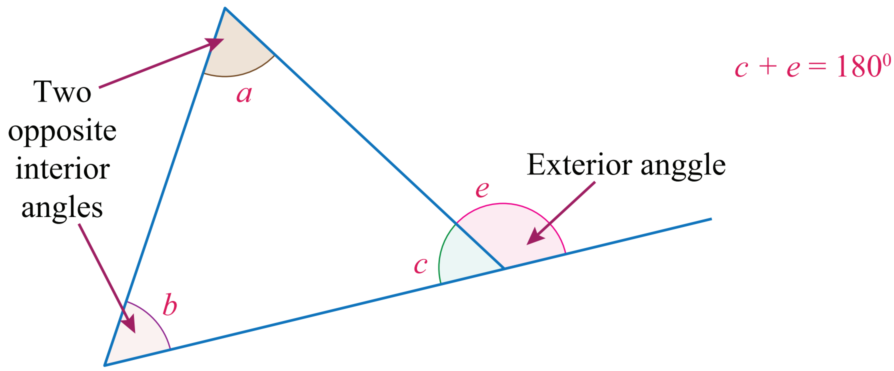
$m\angle DBC = 40^0 + 55^0$

$m\angle DBC = 95^0$



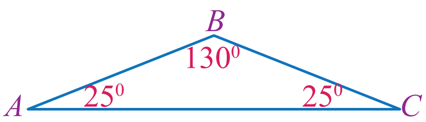
3. Interior angle and adjacent Exterior angle property

The sum of an interior angle and its adjacent exterior angle is 180°



4. Largest angle longest side property

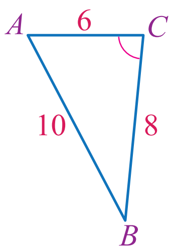
The side opposite to the largest angle of a triangle is the longest side.



$\angle A$ is the largest angle and the side opposite to it,

$\overline{BC}$ is the longest side of the triangle.

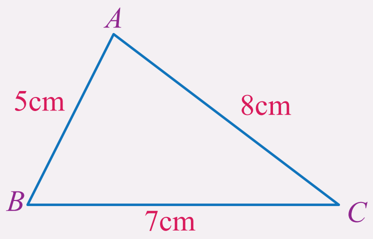
5. Longest side largest angle property



The angle opposite to the longest side of a triangle is the largest angle.

$\overline{AB}$ is the longest side and the angle opposite to it, $\angle ACB$ is the largest angle of the triangle.

Example 11

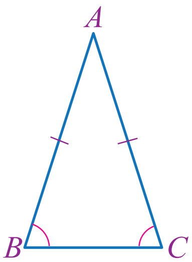


In the triangle ABC given below, which angle is the largest?

Solution

Since AC is the longest side, the angle opposite to it, that is, $\angle B$ must be the largest

6. Base angles of an Isosceles triangle property



The base angles (angles opposite to the two equal sides) of any isosceles triangle are equal in measure

In ΔABC, if $AB = AC$, then $m\angle B = m\angle C$

Example 12

In ΔABC given below, if $AB = AC$ and $m\angle A = 40^0$, then $m\angle B =$ ——; $m\angle C =$ ——

Solution

If $AB = AC$, then $m\angle B = m\angle C$ [Base angles of an Isosceles triangle property]

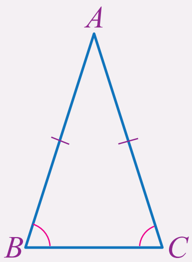
$m\angle B + m\angle C + m\angle A = 180^0$ [Angle sum property]

$x + x + 40 = 180$

$2x + 40 = 180$

$2x = 140$

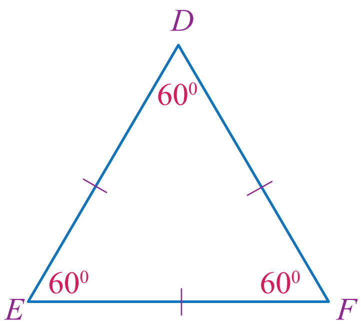
$x = 70^0$ Hence $m\angle B = m\angle C = 70^0$



7. Angles of Equilateral Triangle property

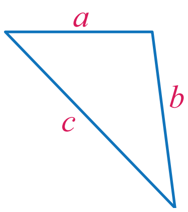
An equilateral triangle is also equiangular (has three equal angles each measuring 60^0)

If ΔDEF is equilateral, then $m\angle D = m\angle E = m\angle F = 60^0$



8. Triangle inequality property

The sum of the length of any two sides of a triangle is greater than the length of the third side



$a + b > c$

$a + c > b$

$b + c > a$

Example 13

Is it possible to have a triangle whose sides are 5 cm, 6 cm and 4 cm?

Solution

By triangle inequality property;

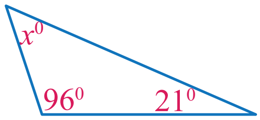
$5 + 6 > 4$ (True)

$5 + 4 > 6$ (True)

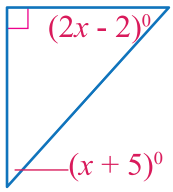
$4 + 6 > 5$ (True), Therefore, it is possible to have a triangle with side lengths 5 cm, 6 cm and 4 cm

Exercises

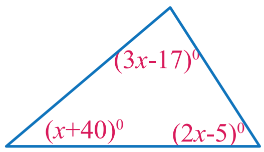
1. Find the value of x in each of the following figures



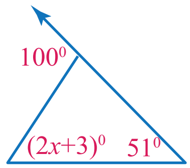
A. $x =$ ______



B. $x =$ ______

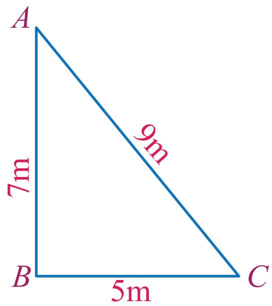


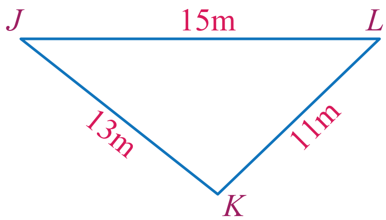
C. $x =$ ______



D. $x =$ ______

2. Identify the largest and smallest angles in the figures given below





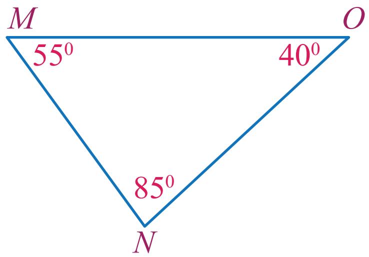
(a) The Largest angle is __________

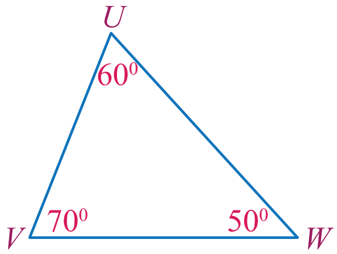
The smallest angle is __________

(b) The Largest angle is __________

The smallest angle is __________

3. Identify the longest and shortest sides in the figures given below





(a) The longest side is ________

The shortest side is ______

(b) The longest side is _________

The shortest side is ________

4. Which of the following triples of numbers can be the measures of the three sides of a triangle?

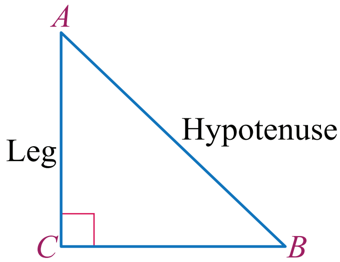
(a) 3, 4, 5

(b) 6, 9, 3

(c) 18, 12, 33

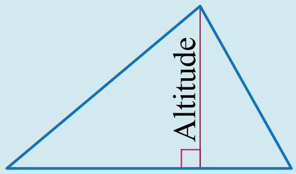
(d) 10, 10, 20

If one of the angles of a triangle is a right angle (measures 90^0), then the triangle is called a right-angled triangle or simply, a right triangle.



DEFINITION

The **altitude** of a **triangle** is the perpendicular drawn from the vertex of the **triangle** to the opposite side. Also, known as the height of the **triangle.**



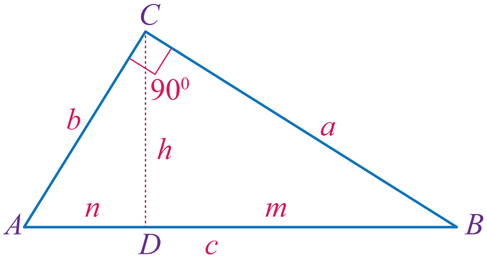
Theorems on Right Angled Triangles

A. The Altitude Theorem

The square of the altitude drawn to the hypotenuse of a right angled triangle is equal to the product of the measures of the segments of the hypotenuse.

ΔABC is a right triangle; h is the altitude drawn from C to the hypotenuse; n and m are segments of the hypo tenuous. Hence,

$$h^2 = n \times m \text{ or } h = \sqrt{nm}.$$



EXAMPLE 14

Suppose ΔABC is a right triangle at C & DC is its altitude. If $AD = 3$ cm & $BD = 12$ cm, then find h.

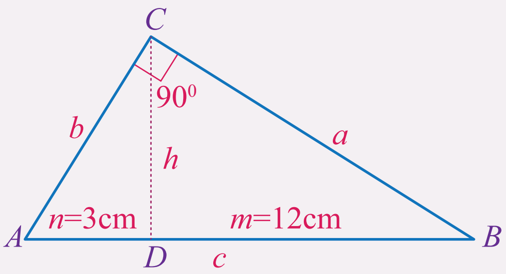
Solution

By the altitude theorem, $h^2 = n \times m$

$h^2 = 3\text{cm} \times 12\text{cm}$

$h^2 = 36 \text{ cm}^2$

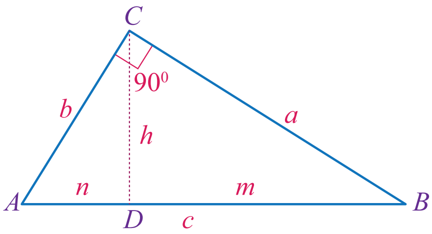
$h = \sqrt{36 \text{ cm}^2} = \mathbf{6 \text{ cm}}$.



B. Euclid's Theorem

Euclid (c. 325 BC – 265 BC) – **Greek Mathematician considered the "Father of Geometry"**. His textbook *'Elements'* remained a highly influential mathematics teaching book until the late 19th Century and is one of the most widely published books in the world.

In aright triangle with an altitude drawn to the hypotenuse, the square of the length of each leg is equal to the product of the hypotenuse and the length of the adjacent segment in to which the altitude divides the hypotenuse.

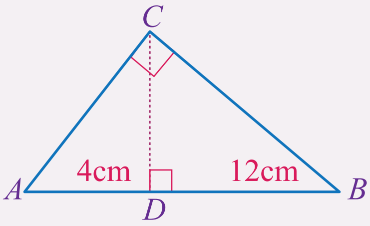


$(AC)^2 = AB \times AD \Rightarrow b^2 = c \times n$

$(BC)^2 = AB \times BD \Rightarrow a^2 = c \times m$

Example 15

In the figure bellow ΔACB is a right triangle with $CD \perp AB$. Find the lengths of $\overline{AC}$ and $\overline{BC}$, if $AD = 4$ and $BD = 12$ cm.



Solution

By Euclid's theorem

$(AC)^2 = AB \times AD$

$(AC)^2 = 16 \times 4$

$(AC)^2 = 64 \Rightarrow \underline{AC = 8 \text{ cm}}$

$(BC)^2 = AB \times BD$

$(BC)^2 = 16 \times 12$

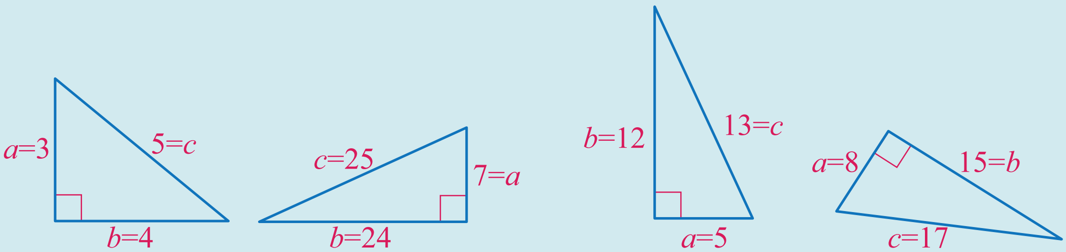
$(BC)^2 = 192$

$\underline{BC = 13.86 \text{ cm}}$

Pythagoras theorem

Activity 4

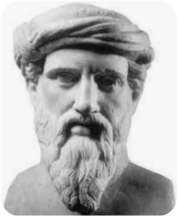
Use the lengths of the right triangles given below to complete the following table:



(i) (ii) (iii) (iv)

Figure	a	b	c	a^2	b^2	c^2	$a^2 + b^2$
(i)							
(ii)							
(iii)							
(iv)							

How do you compare the results of the last two columns, that is, c^2 and $a^2 + b^2$? What can you conclude from the results of the above activity?



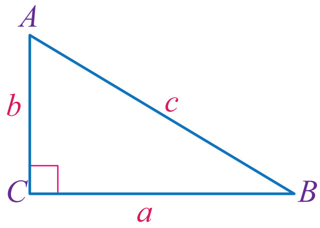
Pythagoras was born in 570 century B.C in the island of Samos; Greece. He made contribution to the field of science and mathematics. He is recognized for presenting the well-known Pythagorean Theorem

In a right angled triangle the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the two legs.

If ΔABC is aright triangle, then $c^2 = a^2 + b^2$

c is the longest side

a and b are the legs



Example 16

Prove that If ΔABC is aright triangle with side lengths as shown in the above figure, then $\boldsymbol{c^2 = a^2 + b^2}$

Solution

Given: ΔACB is aright triangle and $CD \perp AB$

Required: To prove that $\boldsymbol{c^2 = a^2 + b^2}$.

Strategy: Use *Euclid's theorem*.

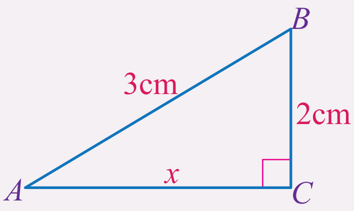
Proof

Statement	Reason
1. $a^2 = c \times m$	1. Euclid's theorem.
2. $b^2 = c \times n$	2. Euclid's theorem
3. $a^2 + b^2 = c \times m + c \times n$	3. Adding steps (1) & (2).
4. $a^2 + b_2 = c\,(n + m)$	4. From step 3
5. $a^2 + b^2 = c \times c$	5. Since $n + m = c$
6. $a^2 + b^2 = c^2$	6. From step 5

Example 17

If the length of the hypotenuse of a right triangle is 3 cm and one leg is 2 cm, then what is the length of the other leg?

Solution



Let x be the length of the unknown leg.

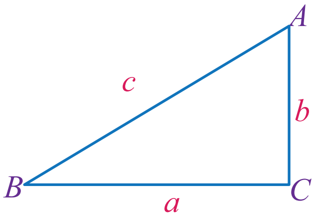
By Pythagoras Theorem $x^2 + 22 = 32$

$\Rightarrow x^2 = 9 - 4$

$\Rightarrow x^2 = 5$

$\Rightarrow \underline{\underline{x = \sqrt{5}}}$ cm

C. *Converse of the Pythagorean Theorem*



If in a triangle, the square of the lengths of the longest side is equal to the sum of the squares of the lengths of the other two sides, then the triangle is a right angled triangle.

If $a^2 + b^2 = c^2$, then ΔACB is a right triangle.

Example 18

The lengths of the sides of a triangle are 8, 15 and 17. Show that this triangle is a right triangle.

Solution

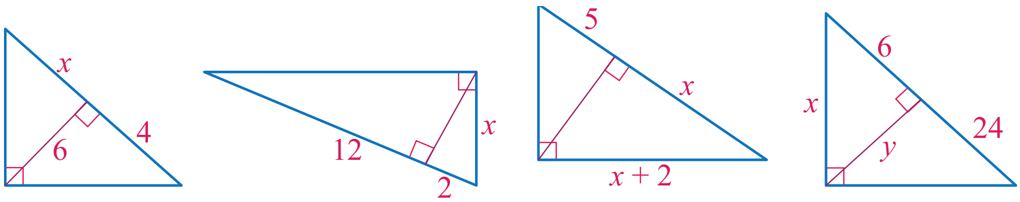
The longest side is 17. So let $c = 17$, $a = 8$ and $b = 15$

$$c^2 = a^2 + b^2 \Rightarrow 17^2 = 8^2 + 15^2 \Rightarrow 298 = 64 + 225 \Rightarrow 289 = 289.$$

By the converse of Pythagoras Theorem the triangle with sides 8, 15 and 17 is a right triangle.

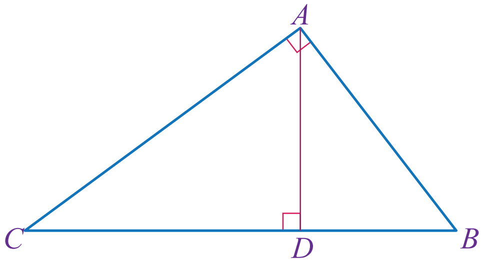
Exercises

1. Find the value of x and y in the following figures



(a) (b) (c) (d)

2. ΔABC is right at A, $\overline{AD}$ is the altitude drawn from A to the hypotenuse $\overline{BC}$. What is the value of $(AC)^2$?



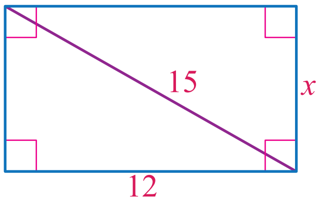
(a) $(AC)^2 = (CD)^2\,(CB)^2$

(b) $(AC)^2 = (AD)^2\,(CD)^2$

(c) $(AC)^2 = (CD)\,(DB)$

(d) $(AC)^2 = (CD)\,(CB)$

3. What the value of x in the figure given below?



4. Which triple numbers (triples) represent a right triangle?

(a) (3, 5, 4) (c) (5, 13, 12)

(b) (10, 8, 6) (d) all

Activity 5

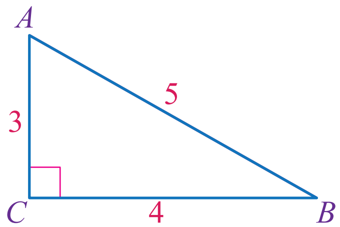
Find any three positive integers (a, b, c, where c is the largest of the three) that satisfy the pythagoras theorem $c^2 = a^2 + b^2$. Can you find any formula of finding such triples?

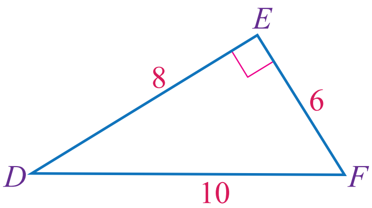
Pythagorean triples are any three positive integers (a, b, c) that completely satisfy the Pythagorean Theorem. The three sides of a right triangle form the Pythagorean triples. 'c' is the 'hypotenuse' or the longest side of the triangle and 'a' and 'b' are the other two legs of the right-angled triangle. The triples like (3, 4, 5) and (6, 8, 10) are Pythagorean triples because they satisfy the Pythagoras theorem and are side lengths of a right triangle

If ΔABC is aright triangle, then $\mathbf{5^2 = 3^2 + 4^2}$

$\mathbf{25 = 25}$

Hence, (3, 4, 5) are Pythagorean triples





If ΔDEF is a right triangle, then

$\mathbf{10^2 = 6^2 + 8^2}$

100 = 100

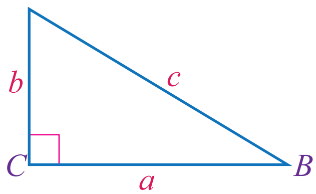
Hence, (6, 8, 10) are Pythagorean triples.

Next, let us learn how to generate a few Pythagorean triples.

Method 1: Scaling up the first known Pythagorean triples (3, 4, and 5)

n	(3*n*, 4*n*, 5*n*)
1	**(3, 4, 5)**
2	**(6, 8, 10)**
3	**(9, 12, 15)**
4	**(12, 16, 20)**
5	**(15, 20, 25)**
6	**(18, 24, 30)**
7	**(21, 28, 37)**
. 10 .	. . **(30, 40, 50)** .

Method 2: Applying the formula $(a, b, c) \Rightarrow \boldsymbol{a} = m^2 - n^2 \quad \boldsymbol{b} = 2\,mn \quad \boldsymbol{c} = m^2 + n^2$



$\boldsymbol{a} = m^2 - n^2$

$\boldsymbol{b} = 2\,mn$

$\boldsymbol{c} = m^2 + n^2.$

where $\boldsymbol{m}$ and $\boldsymbol{n}$ are positive integers and $\boldsymbol{m > n}$

Example 19

Find the Pythagorean triples if

(i) $m = 2$ and $n = 1$

Solution

$m > n$, that is $2 > 1$

$\boldsymbol{a} = m^2 - n^2 = 2^2 - 1^2 = \mathbf{3}$

$\boldsymbol{b} = 2\text{mn} = 2 \times 2 \times 1 = \mathbf{4}$

$\boldsymbol{c} = m^2 + n^2 = 2^2 + 1^2 = \mathbf{5}$. Hence (3, 4, 5) are Pythagorean triples

Check them using Pythagoras theorem $\boldsymbol{c^2 = a^2 + b^2}$

(ii) $m = 3$ and $n = 2$

Solution

$m > n$, that is $3 > 2$

$\boldsymbol{a} = m^2 - n^2 = 3^2 - 2^2 = \mathbf{5}$

$\boldsymbol{b} = 2mn = 2 \times 3 \times 2 = \mathbf{12}$

$\boldsymbol{c} = m^2 + n^2 = 3^2 + 2^2 = \mathbf{13}$. Hence (5, 12, 13) are Pythagorean triples

Check them using Pythagoras theorem $\boldsymbol{c^2 = a^2 + b^2}$

(iii) $m = 10$ and $n = 4$

Solution

$m > n$, that is $10 > 4$

$\boldsymbol{a} = 10^2 - 4^2 = 100 - 16 = \mathbf{84}$

$\boldsymbol{b} = 2mn = 2 \times 10 \times 4 = \mathbf{80}$

$\boldsymbol{c} = m^2 + n^2 = 10^2 + 4^2 = 100 + 16 = \mathbf{116}$

Hence (84, 80, 116) are Pythagorean triples.

Check: By Pythagoras theorem,

$\boldsymbol{c^2 = a^2 + b^2}$

$116^2 = 84^2 + 80^2$

$13456 = 7056 + 6400$

$13456 = 13456$ (True)

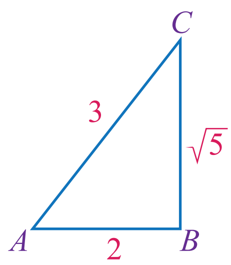
EXERCISES

1. Which of the following is **not** a Pythagorean triple?

   (a) 7, 24, 25 (b) 8, 15, 17 (c) 9, 12, 15 (d) 10, 16, 19

2. If (25, 7, x) are Pythagorean triples, then $x =$ ______

3. Is triangle ABC shown below a right triangle? Why or why not?



We have seen that a triangle is a simple closed curve. A triangle has three sides. All of the three sides are line-segments.

In plane geometry there are several shapes whose sides are *line segments*. Such geometric shapes made up on line segments are called **polygons**.

Definition

A **polygon** is a *simple closed curve* formed by the union of a finite number of line segments, no two of which in succession are collinear.

The line segments are called sides of the polygon, and the end points of the sides are called vertices of the polygon.

Example 20

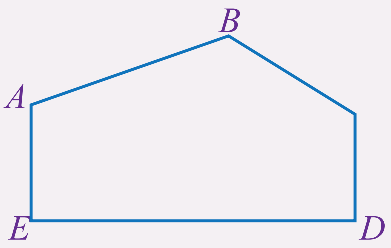


Figure 1.

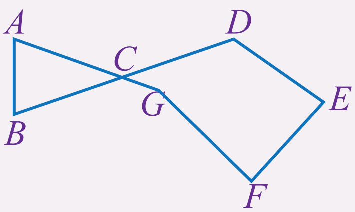


Figure 2.

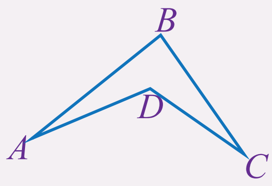


Figure 3.

Figure 1 above shows a polygon with five sides and five vertices.

Figure 2 is <u>not polygon</u> because it is not a *simple closed* curve.

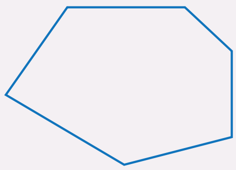
Figure 3 is a polygon with four sides and four vertices.

Definition

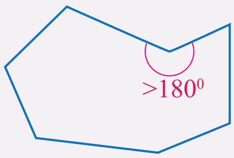
Convex and Concave polygons

A ploygon is a **convex ploygon** if each of its *interior angles* mesures *less than* 180^0. A ploygon is a **concave ploygon** if it has *at least one interior angle with a measure greater than* 180^0

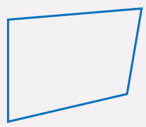
Example 21



8 side *convex* polygon



7 side *convex* polygon



4 side *convex* polygon

Definition

A polygon is called ***an equilateral polygon*** if all of its sides are congruent to each other.

A polygon is called ***equiangular polygon*** if all of its interior angles are congruent to each other.

A polygon whose sides are all congruent and whose vertex angles are all congruent is called ***a regular polygon.***

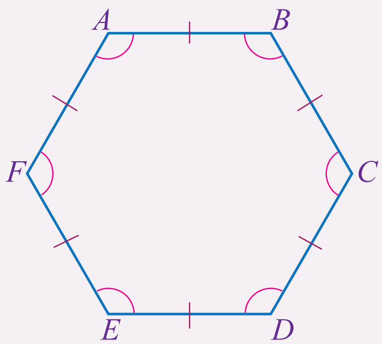
A polygon which is both equilateral and equiangular is ***called a regular polygon.***

Example 22

This 6 sided polygon is a regular polygon because

$\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DE} \cong \overline{EF} \cong \overline{FA}$ and

$\angle A \cong \angle B \cong \angle C \cong \angle D \cong \angle E \cong \angle F.$



Classification of Polygons

We have seen that polygons are simple closed curves with different number of sides. The sides of any polygon are line segments. The number of sides of a given polygon can be three or four or five or six or seven or any positive integer $n \geq 3$.

Therefore, polygons are classified and named by the number of their sides. The following list shows the name given to some polygons having a given number of sides:

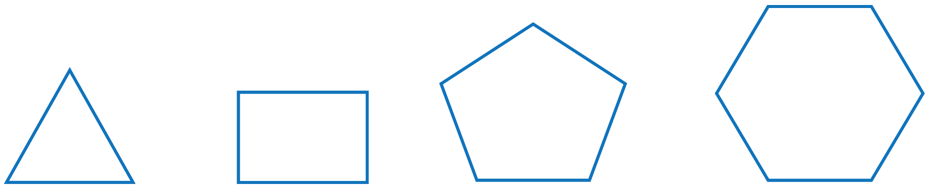
S. No	Name of a polygon	Description
1	Triangle	A three sided polygon
2	Quadrilateral	A four sided polygon
3	Pentagon	A five sided polygon
4	Hexagon	A six sided polygon
5	Heptagon	A seven sided polygon
6	Octagon	An eight sided polygon

S. No	Name of a polygon	Description
7	Nonagon	A nine sided polygon
8	***Decagon***	A ten sided polygon
9	n – gon (polygone)	***n*** sided polygon

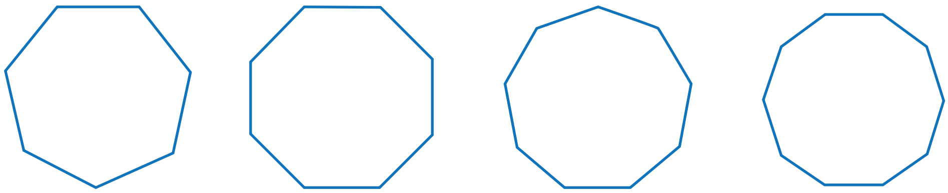
Note that a “triangle” is the simplest polygon with three sides.

Exercises

1. Name the following polygons:



(a) ______ (b) ________ (c) ________ (d) __________



(e) ______ (f) ________ (g) ________ (h) _________

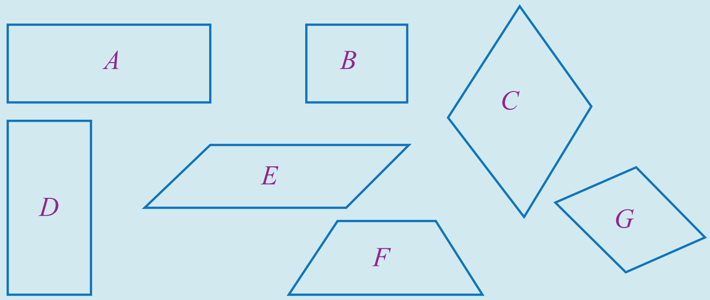
2. Classify the following polygons as convex or concave?

(a) ____ (b) ________ (c) _________ (d) ________ (e) _________

3. Which of the following polygons is not regular?
   - (a) Square
   - (b) equilateral triangle
   - (c) rectangle

Activity 6

1. How many sides has each of the following figures __________
2. How many interior angles has each of the following figures __________
3. What is common to all of them? __________
4. What is a common name to all of them? __________



5. Use your knowledge of quadrilaterals and classify the figures given above as *Quadrilateral, Trapezoid (Trapezium), Parallelogram, Rectangle, Rhombus, Square* and *Kite*. Complete the following table to give your answers

Quadrilateral	Trapezium	Parallelogram	Rectangle	Rhombus	Square	Kite
A						
B						
C						
D						
E						
F						
G						

Any polygon having four sides is called a QUADRILATERAL

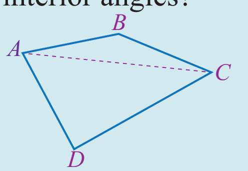
All the above figures have 4 sides. So they are all quadrilaterals

A quadrilateral has also 4 angles

Activity 7

Draw any quadrilateral *ABCD*. How can you find the sum of the four interior angles?

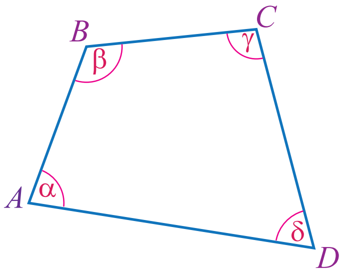
- **Hint**: Draw a diagonal at any one vertex to its opposite angle to divide the quadrilateral in to two triangles. What is the sum of the interior angles of each triangle?



- Is the sum of the six angles of the two triangles equal to the sum of the angles of the quadrilateral?
- What can you conclude about the sum of the four interior angles of a quadrilateral?

Theorem 1

The sum of the measures of the four interior angles of any quadrilateral is 360°



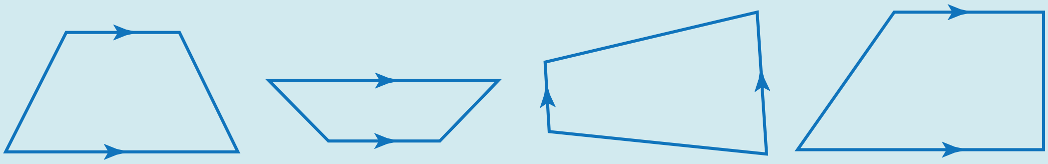
$\alpha + \beta + \gamma + \delta = 360°$

Classification of Quadrilaterals

DEFINITION

Trapezium (UK); Trapezoid (US)

A quadrilateral that has *exactly one pair of parallel opposite sides* is called a **trapezium (trapezoid).**



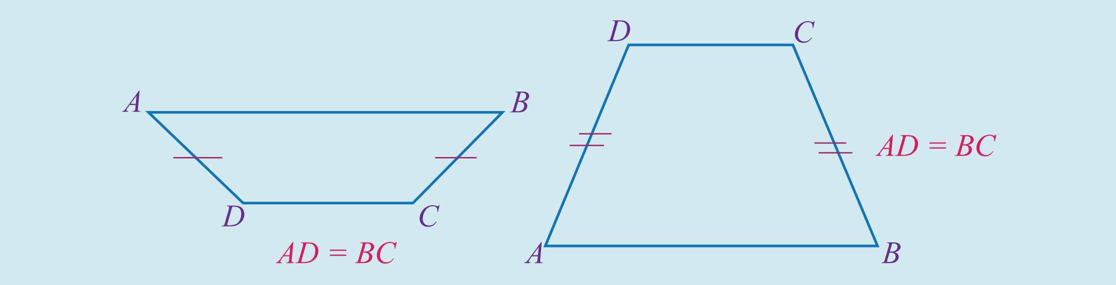
The two *parallel sides* are called **bases**.

The two *non-parallel sides* are called **legs**.

DEFINITION

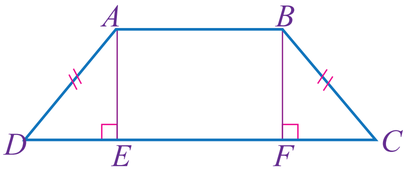
Isosceles trapezium

A trapezium is called an *isosceles trapezium* if its *non- parallel sides (the two legs) are equal*



Theorem 2

The base angles of an isosceles trapezium are congruent.



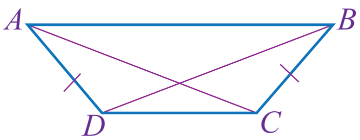
If $ABCD$ is an isosceles trapezium with $\overline{AD} = \overline{BC}$, then

$m\angle D = m\angle C$

$m\angle A = m\angle B$

Theorem 3

The diagonals of an *isosceles trapezium* are congruent.

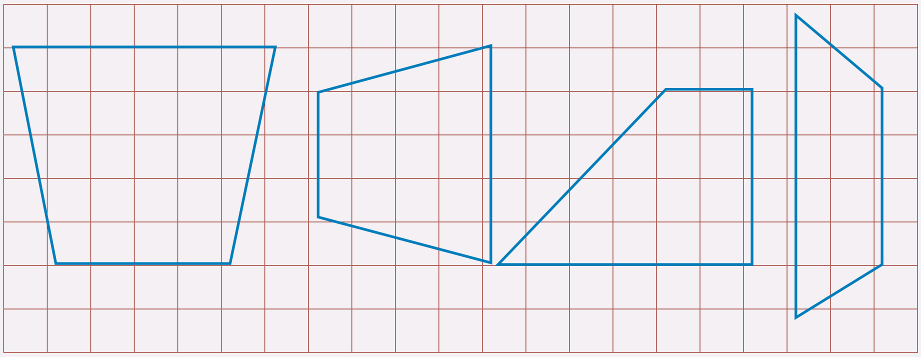


If $ABCD$ is an isosceles trapezium with $\overline{AD} = \overline{BC}$, then $\overline{AC} = \overline{BD}$

Example 23

(a) Why are all the following figures trapeziums?

(b) Which ones do think are Isosceless trapeziums



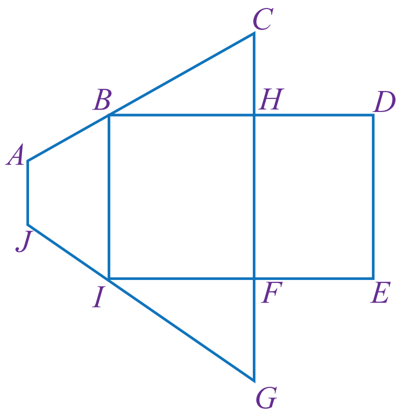
Solution

(a) Because each figure is a quadrilateral and has only one pair of opposite sides parallel. The other pair of opposite sides is not parallel.

(b) All figures except the third (from left to right) are *isosceles trapeziums*.

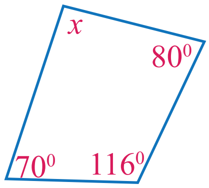
Exercises

1. How many quadrilaterals are there in this figure?

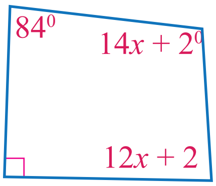


(a) 10
(b) 6
(c) 9
(d) 8

2. What is the value of x in each of the following figures?

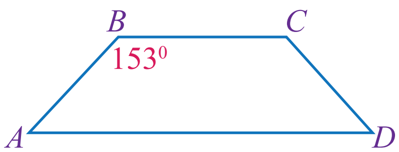


$x =$ ________



$x =$ ________

3. $ABCD$ is an isosceles trapezium and $m\angle B = 153^0$. Find $m\angle A$, $m\angle C$, $m\angle D$. Explain how you know each angle.



$m\angle A =$ ____________

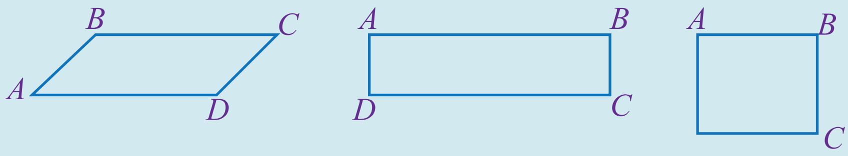
$m\angle C =$ ____________

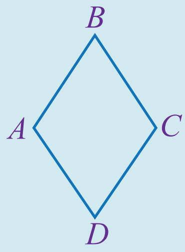
$m\angle D =$ ____________

Definition

Parallelogram

A ***Parallelogram*** is a quadrilateral in which ***both pairs of opposite sides are parallel***





$AB//CD$

$AD//BC$

Remember: parallelograms are not trapeziums (Why?)

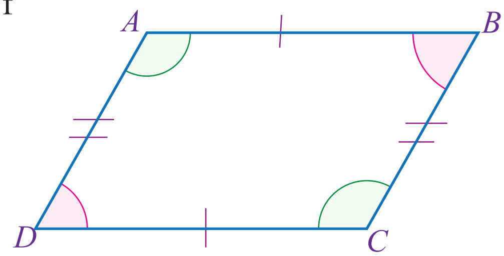
Properties of a parallelogram

Property1

Both pairs of opposite sides of a parallelogram are congruent.

Both pairs of opposite angles of a parallelogram are congruent.

If



$ABCD$ is a parallelogram, then

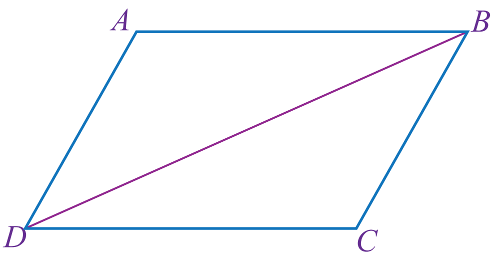
$AB = DC$

$AD = BC$

$m\angle A = m\angle C$

$m\angle D = m\angle B$

Property 2

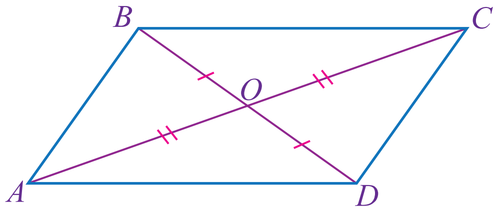


The diagonal of a parallelogram divides the parallelogram in to two congruent triangles.

If $ABCD$ is a parallelogram and BD is its diagonal, then $\Delta ABD \equiv \Delta CDB$

Property 3

The two diagonals of a parallelogram bisect each other



If AC and BD are diagonals of a parallelogram $ABCD$, then $AO = CO$ and $BO = DO$.

Property 4

Any two consecutive (adjacent) angles of a parallelogram are supplementary

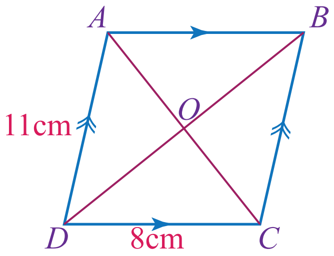
If $ABCD$ is a parallelogram, then

$\alpha + \delta = 180^o$ $\delta + \gamma = 180^o$

$\beta + \gamma = 180^o$ $\alpha + \beta = 180^o$

Exercises

1. Find AB and BC for the parallelogram shown. [Given: AD = 11 cm and DC = 8 cm]



AB = ____________

BC = ____________

2. Consider parallelogram $ABCD$ given above. If $m \angle ADC = 80^o$, and DB = 10 cm, then

(a) $m \angle ABC$ = ________

(b) $m \angle DCB$ = ________

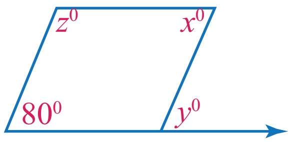
(c) $m \angle BAD$ = ________

(d) $m \angle DCB + m \angle ABC$ = ______

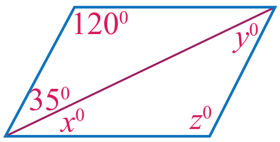
(e) BO = ________ cm

(f) DO = ________ cm

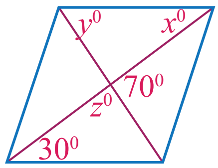
3. For each of the following parallelograms find the measures of x, y and z?



x = ________
y = ________
z = ________



x = ________
y = ________
z = ________

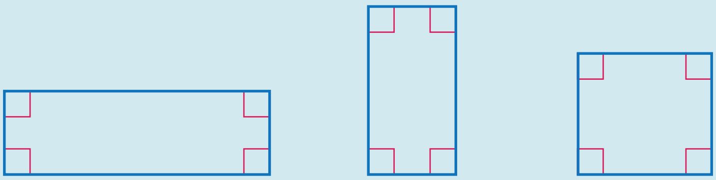


x = ________
y = ________
z = ________

DEFINITION

Rectangle

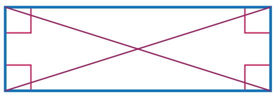
- If each of the four angles of a *parallelogram* is a *right angle* (measures 90^0), then the parallelogram is called a *rectangle.*
- A rectangle is a parallelogram having each of its angles 90^0.



Remember: All rectangles are parallelograms (Why?)

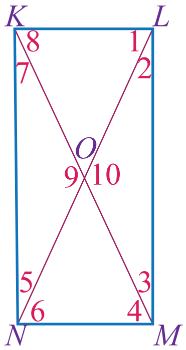
Properties of a Rectangle

1. Has all the properties of a parallelogram, that is
   - Opposite sides are parallel
   - Opposite sides are congruent
   - Opposite angles are congruent
   - Consecutive angles are supplementary
   - Diagonals bisect each other
2. Has four right angles
3. Diagonals are congruent



Exercises

Refer to the rectangle *KLMN* and the information given to solve the following:



If $m\angle 1 = 70^o$ and $m\angle 7 = 20^o$, then

$m\angle 2 =$ ______ $m\angle 6 =$ ______

$m\angle 3 =$ ______ $m\angle 8 =$ ______

$m\angle 4 =$ ______ $m\angle 9 =$ ______

$m\angle 5 =$ ______ $m\angle 10 =$ ______

If $ON = 15$, then

$OM =$ ______

$OL =$ ______

$OK =$ ______

$NL =$ ______

If $KL = 16$, then

$KM =$ ______

$KN =$ ______

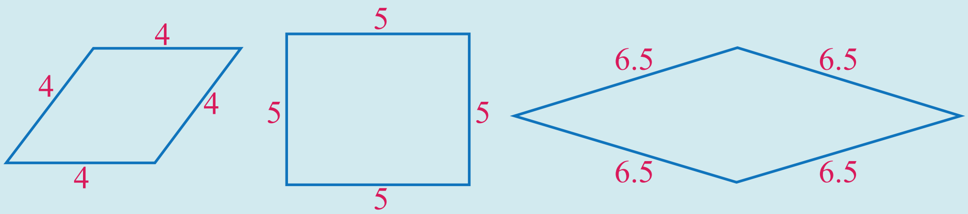
$NM =$ ______

$LM =$ ______

Definition

Rhombus

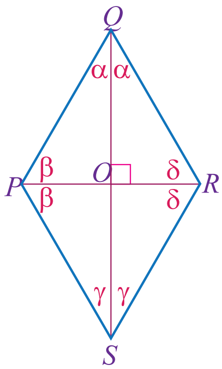
A *rhombus* is a parallelogram *all of whose sides are equal*



Remember: Every rhombus is a parallelogram (Why?)

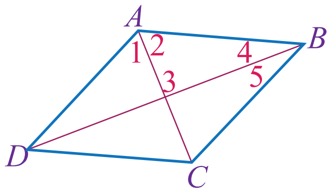
Properties of a rhombus

1. Has all the properties of a parallelogram, that is
2. Has four equal sides
3. Diagonals are perpendicular to each other
4. Diagonals bisect the vertex angle



Exercises

1. Refer to the rhombus *ABCD* and answer the following:



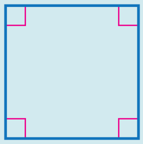
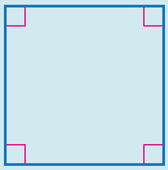
   (a) If $AD = 13$, then $AB =$ ______
   (b) If $m\angle 4 = 25^{\circ}$, then $m\angle 5 =$ ______
   (c) If $m\angle DAB = 130^{\circ}$, then $m\angle ADC =$ ______
   (d) If $m\angle 4 = 3x - 2$ and $m\angle 5 = 2x + 7$, then $x =$ ______
   (e) If $m\angle 1 = 5x + 18$ and $m\angle 5 = 3x - 8$, then $x =$ ______
   (f) If $m\angle 2 = 3y + 9$ and $m\angle 4 = 2y - 4$, then $y =$ ______
   (g) $m\angle 3 =$ ______

Definition

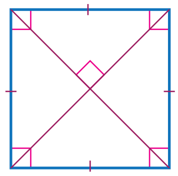
Square

A square is *a <u>rectangle</u> all of whose sides are equal*

Remember: All squares are parallelograms (Why?)
All squares are rectangles (Why?)
All squares are rhombuses (Why?)

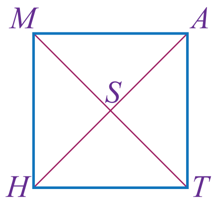
Properties of a square



1. A square has all the properties of a rhombus
2. A square has all the properties of a rectangle
3. A square has all the properties of a parallelogram
4. A square has diagonals that are both congruent and perpendicular

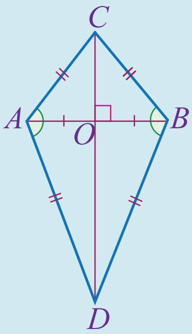
Exercises

Refer to the square *MATH* to answer the following



(a) If $MA = 8$, then $AT =$ ____________
(b) $m\angle HST =$ ____________
(c) $m\angle MAT =$ ____________
(d) If $HS = 2$, then $HA =$ ________ $MT =$ ________
(e) $m\angle HMT =$ _________

DEFINITION



Kite

A kite is a quadrilateral in which two pairs of adjacent sides are equal in length

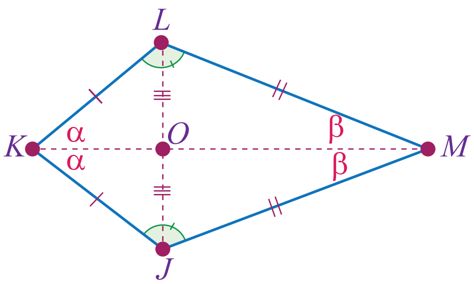
$AC = BC$

$AD = BD$

Remember: A square is a kite (Why?)

A rhombus is a kite (Why?)

Properties of a kite



The two angles where the unequal sides meet are equal. $m\angle L = m\angle J$

It can be viewed as a pair of congruent triangles with a common base.

The diagonals are perpendicular. *JL is perpendicular to KM*

The longer or main diagonal bisects the other diagonal.

The main diagonal bisects a pair of opposite angles (angle *K* and angle *M*).

A kite is symmetrical about its main diagonal.

The shorter diagonal divides the kite into 2 isosceles triangles.

The longer diagonal divides the kite into 2 congruent triangles.

EXERCISES

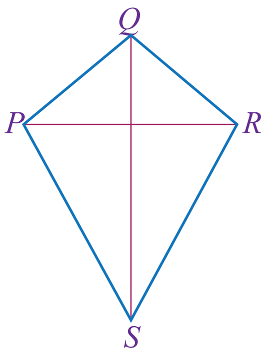
1. Refer to the kite *PQRS* given below; with $PQ = QR = 10$, $PR = 16$, $RS = 12$. Then find

   (a) $TR =$ ________

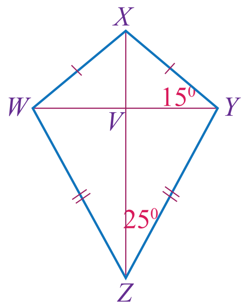
   (b) $QT =$ ________

   (c) If $m\angle QRT = 40^0$, then $m\angle PQR =$ ______

   (d) If $m\angle PSR = 30^0$, then $m\angle TRS =$ ______



2. Refer to the kite $WXYZ$ given below and find the following



(a) $m\angle VYZ =$ ______

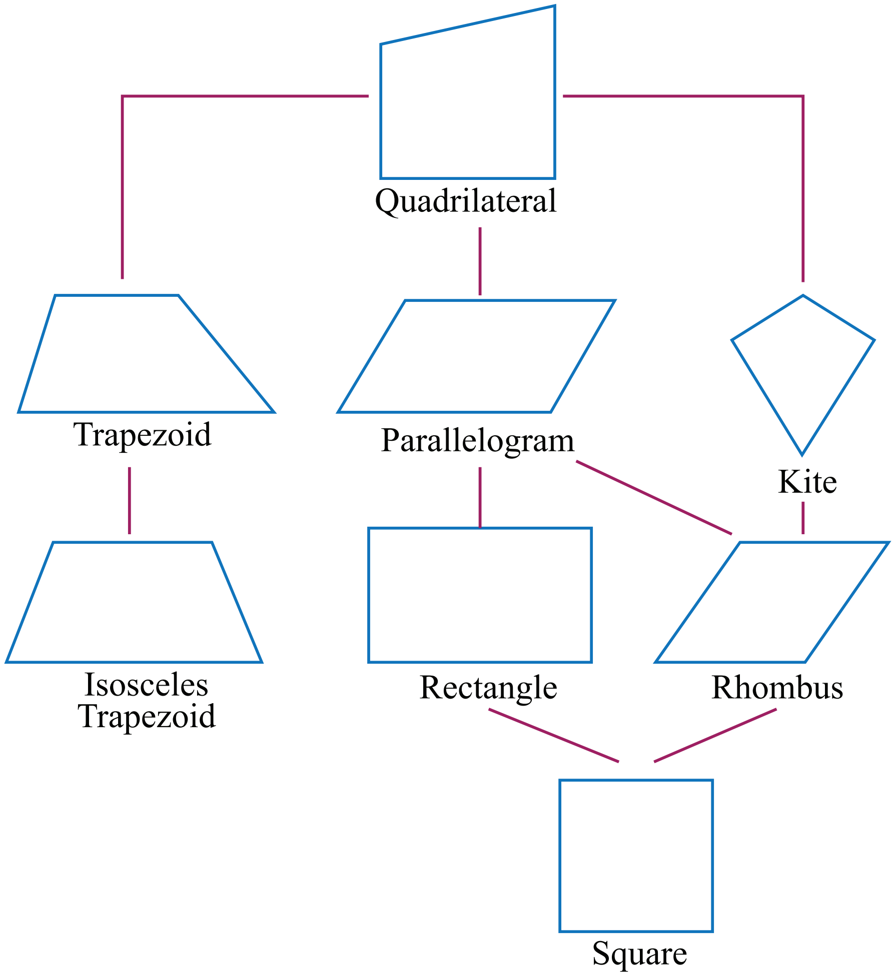
(b) $m\angle XWZ =$ ______

(c) $m\angle VXY =$ ______

(d) $m\angle WXY =$ ______

(e) $m\angle WZY =$ ______

The relationship among quadrilaterals is summarized as follows



Exercises

Answer the following questions. Give a reason to each of your answers.

(a) Is a square a rhombus? ________________

(b) Is a square a rectangle? ________________

(c) Is a square a parallelogram? ________________

(d) Is a square a trapezium? ________________

(e) Is a square a kite? ________________

(f) Is a rhombus a square? ________________

(g) Is a rhombus a rectangle? ________________

(h) Is a rhombus a parallelogram? ________________

(i) Is a rhombus a trapezium? ________________

(j) Is a rhombus a kite? ________________

(k) Is kite a rhombus? ________________

(l) Is kite a parallelogram? ________________

(m) Is a kite a trapezium? ________________

(n) Is a rectangle a parallelogram? ________________

(o) Is a parallelogram a trapezium? ________________

(p) Is a parallelogram a kite? ________________

(q) Is an isosceles trapezium a parallelogram? ________________

(r) Is a rhombus a regular polygon? ________________

(s) Is a square a regular polygon? ________________

Key Terms

- Acute triangle
- Angle
- Alternate interior angles
- Altitude Theorem
- Corresponding angles
- Euclids Theorem
- Equilateral triangle
- Iscosceless triangle
- Kite
- Obtuse triangle
- Parallel lines
- Parallelogram
- Polygon
- Pythagoras Theorem
- Pythagorean Triples
- Quadrilateral

- Ray
- Rectangle
- Right triangle
- Rhombus
- Scalene triangle
- Square
- Trapezium (trapezoid)
- Triangle

Summary

This unit was all about concepts related to plane geometry. Plane Geometry mainly deals about flat shapes like lines, angles, triangles, quadrilaterals and other polygons that can be drawn on a piece of paper.

An angle is defined as a surface created by the union of two rays having a common end point. Depending on their size, angles are classified as acute, right, obtuse, straight, reflex and full rotation angles.

When two parallel lines are cut by a third transversal line, the alternate interior angles formed are equal and the corresponding angles are also equal.

A triangle is the simplest polygon with three sides and three interior angles.

Depending on the lengths of their sides, triangles are classified as equilateral, isosceles and scalene.

Based on the size of their angles triangles are also classified as acute, right and obtuse.

Other groups of polygons discussed in this unit are quadrilaterals.

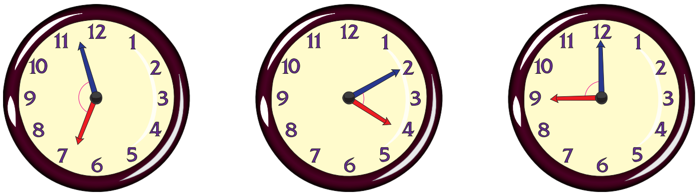
Any four sided geometric figure is called a quadrilateral.

The well-known quadrilaterals are Trapezium, Parallelogram Rectangle, Rhombus, Square and Kite. Each has its own specific meaning and property.

- **Trapezium/Trapezoid**: is a quadrilateral which has **only** one pair of opposite sides parallel
- **Parallelogram**: is a quadrilateral which has two pair of opposite sides parallel
- **Rectangle**: is a parallelogram with four right angles
- **Rhombus**: is a parallelogram with four equal sides
- **Square**: is a parallelogram with four equal sides and four right angles
- **Kite** is a quadrilateral with two pairs of adjacent sides equal

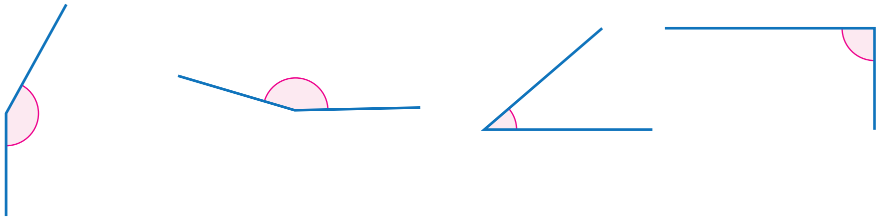
Exercises

1. An angle whose measure is less than that of a right angle is called
   (a) Acute angle
   (b) Right angle
   (c) Obtuse angle
   (d) Scalene angle
2. Which clock shows an obtuse angle by two of its hands?

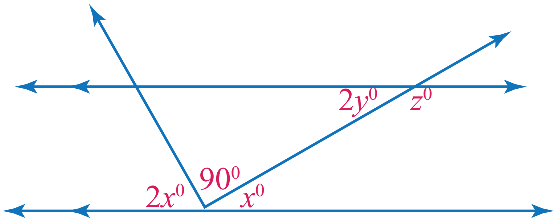


(a) (b) (c)

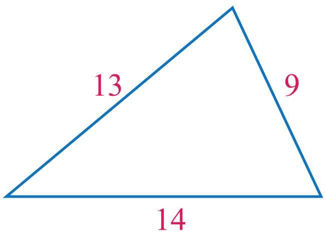
3. Which of the following angles is acute?



4. In the figure given below, the values of x, y and z respectively are

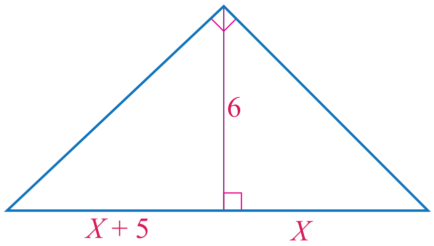


   (a) 30, 15, 150 (b) 15, 150, 30 (c) 150, 15, 30
5. Which one is true about the triangle below?



   (a) It is an equilateral and acute angle triangle
   (b) It is scalene and acute angle triangle
   (c) It is Isosceles and acute angle triangle
   (d) It is scalene and right angle triangle

6. Given is a right triangle with side lengths as indicated in the figure below. What is the value of x?



(a) 9 (c) 4

(b) 6 (d) 5

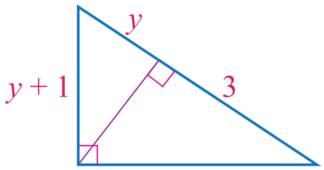
7. Refer to the right triangle given below. What is the value of y?

(a) 1

(b) 2

(c) 3

(d) 4



8. What is the hypotenuse of the right angled triangles whose other sides are 9 and 12

(a) 16 (c) 13

(b) 15 (d) 20

9. Which triples do not represent a right triangle?

(a) (8, 15, 17)

(b) (9, 40, 41)

(c) (11, 60, 61)

(d) (7, 24, 26)

10. In an isosceles trapezium $ABCD$, if the diagonal AC is $2x - 3$ and diagonal BD is $41 - 6x$, which of the following is false?

(a) $x = 5.5$

(b) $AC = BD = 8$

(c) $m\angle A = m\angle D$

(d) $m\angle A = m\angle B$

11.

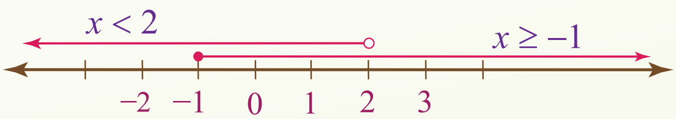
Make a tick(✓) mark if the quadrilateral satisfies the property and mark (×) if it doesn't satisfy						
Property	**Trapezium**	**Parallelogram**	**Rectangle**	**Rhombus**	**Square**	**Kite**
Has four equal sides	×	×	×	✓	✓	×
Has four right angles						
Opposite sides are equal						
Opposite sides are parallel						
Only one pair of opposite sides is parallel						
Two pair of opposite sides are parallel						
two pairs of adjacent sides equal				✓	✓	

M10CH06

CHAPTER 6

LINEAR EQUATIONS AND INEQUALITIES

Chapter Contents



Chapter Outcomes

Learners are able to apply concepts and skills to discuss and solve related problems on linear equations and inequalities

Objectives: Upon completion of this chapter, learners should be able to:

- use equality and equivalence concepts;
- find the solution set of a linear equation;
- formulate word problems as linear equations and solve;
- solve linear inequalities in one variable;
- graph the solutions of linear inequalities;
- formulate word problems as linear inequalities and solve.

Introduction

One of the important tasks in mathematics is solving equations and inequalities. An equation is simply a statement that two algebraic expressions are equal. For instance, $5x + 2$ and $3(x - 1)$ are examples of algebraic expression; and $5x + 2 = 7$ or $5x + 2 = 3(x - 1)$ are examples of equation. The task is to find the value of x that makes the given equation true when x is replaced by it.

Similarly, an algebraic inequality is a mathematical statement that an expression is less than (or greater than) another expression. For instance, $5x + 2 < 7$ or $5x + 2 > 3(x - 1)$ are examples of algebraic inequalities. The related task is to find possible values of x that make the given inequality true when x is replaced by the values. These will be discussed in this unit.

Activity 1

1. A *number on a card* plus 3 is equal to 5. What is the number on the card?
2. Twice of a *number* minus 4 is equal to the number itself. What is the number?
3.
   a. If x represents *'a number on the card'*, write the phrase '*a number on the card* plus 3' as an algebraic expression using x. Then, formulate (write) the sentence above in (a) as an equation.
   b. If x represents the unknown number in (b), write the phrase '*twice of the number*' as an algebraic expression using x. Then, formulate (write) the sentence above in (b) as an equation.

The following activity that you may work on in groups involves opening problems that will help you to better understand the discussions that follow.

In mathematics, we represent an unknown (*a certain number*) by a letter, say, x. So, in the above Group Activity, you may formulate the statement in (a) as $x + 3 = 5$ and the statement in (b) as $2x - 4 = x$. Such a letter x that stands in place of the unknown number is called a **variable**. Moreover, we formally define an equation as follows.

Definition

An **equation** is a mathematical statement that states two expressions are equal. Two expressions that are required to be equal are separated by the equality symbol, '='

Example 1

Each of the following are examples of equation.

(a) $x + 3 = 5$

(b) $2x - 4 = x$

(c) $5(x - 1) = 2x + 7$

(d) $\frac{3x+2}{x-4} = 1$

(e) $x^2 - 2x = 0$

Looking back to the opening questions in Group Activity 6.1, the question in (a) can be restated as 'what is a number x so that $x + 3 = 5$?' and question in (b) is 'What is a number x so that $2x - 4 = x$?'. You can observe that, for (a), $x + 3 = 5$ is true if $x = 2$ since $2 + 3 = 5$. This value of x, (i.e., $x = 2$) is called the solution of the equation.

In general, a value of the variable that makes the equation true is called a **solution** of the equation. A solution is said to satisfy the equation. A set which contains all possible solutions of an equation is called the **solution set** or **truth set** of the equation. For instance, the solution set (or truth set) of $x + 3 = 5$ is $\{2\}$.

To **solve an equation** means to determine the value of the variable that makes the equation true (or satisfies the equation). The concept of equivalence of equations is usually needed to solve an equation. To get sense of equivalent equations, consider the following Activity that you may discuss it in a group.

Activity 2

Ahmed, Diana and John have separately seen a number written on a card and they described it follows.

Ahmed: The number plus 1 is equal to 7.

Diana: Twice of the number is equal to 12.

John: Half of the number is equal to 3.

1. Try to identify the number that each of them described?

   Did they describe different numbers or the same number in different words?

2. Letting x to represent the number on the card, write the statement of each of them as equations.

In the above Activity 2, you may have observed that they described the same number in different words. Letting x to represent the number on the card, Ahmed, Diana and John described $x = 6$ as $x + 1 = 7$, $2x = 12$, and $\frac{x}{2} = 3$, respectively. In other word, the solution of each of the three equations is 6. Such equations that have the same solution are called equivalent equations.

Definition

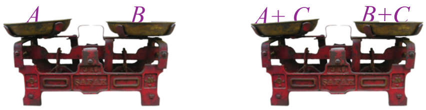
Two equations are said to be **equivalent** if they have the same solution set. That is, the solution of one of the equations is the solution of the other equation as well.

For instance, $x + 1 = 7$, $2x = 12$, and $\frac{x}{2} = 3$ are equivalent equations as discussed above. Similarly, $4x = 20$ and $2x = 10$ are equivalent because $x = 5$ satisfies both equations; that is, the solution set of both equations is $\{5\}$.

To solve an equation, we usually transform (change) the given equation to a simpler equivalent equation step by step until the value of the variable is specified. This is based on the following four **basic rules of equation** (rules for equivalence of equations).

Rule 1

If equal values are added to both sides of equals, the results are equal.



That is, if $A = B$, then $A + C = B + C$

That is, *addition of equal values to both sides of an equation yields its equivalent equation.*

Example 2

Use this rule to solve $x - 11 = 5$.

Solution

Adding 11 to both sides of the equation we get

$$x - 11 + 11 = 5 + 11$$

That is, $x = 16$. (Thus, the solution set is $\{16\}$.)

Recall that adding a negative number is subtraction. That is, $A + (-C) = A - C$. Therefore, this rule can also be stated as follows.

Rule 2

If equal values are subtracted from both sides of equals, the results are equal.

That is, if $A = B$, then $A - C = B - C$.

That is, *subtraction of equal values from sides of an equation yields its equivalent equation.*

Example 3

Use Rule 2 to solve $x + 5 = 3$.

Solution

Subtracting 5 from both sides of the given equation, we get

$$x + 5 - 5 = 3 - 5$$

That is, $x = -2$. Thus, the solution set is $\{-2\}$.

Next, we state the rule for multiplication.

Rule 3

If equals are multiplied by equals, the results are equal.

That is, if $A = B$, then $cA = cB$.

That is, *multiplying both sides of an equation by equal values yields its equivalent equation.*

Example 4

Use Rule 3 to solve $\frac{x}{4} = 6$.

Solution

Multiply both sides of the given equation by 4 to get

$$4 \times \frac{x}{4} = 4 \times 6$$

That is, $x = 24$. Thus, the solution set is $\{24\}$.

Recall that multiplication of a number by $1/n$ is division of the number by n. That is, $\frac{1}{n} \times A = \frac{A}{n}$. Therefore, the third rule can be stated also as follows:

Rule 4

If equals are divided by the same nonzero number, the results are equal.

i.e., if $A=B$ and $c \neq 0$, then $\frac{A}{c} = \frac{B}{c}$

EXAMPLE 5

Use Rule 4 to solve $-2x = 5$.

Solution

Dividing both sides of the given equation by –2, we get

$$\frac{2x}{-2} = \frac{5}{-2}$$

That is, $x = -\frac{5}{2}$. Thus, the solution set is $\left\{-\frac{5}{2}\right\}$.

In general, to solve an equation, you may apply combination of these rules step by step to get simpler equivalent equations until the value of the variable is specified. That is, at each step use appropriate rule to collect or group the variable to the left side and constant numbers to the right side until the variable is separated alone at the left side.

EXAMPLE 6

Solve $5x - 5 = 2x + 10$.

Solution

First, to bring $2x$ to the left side, subtract $2x$ from both sides of the given equation.

$5x - 2x - 5 = 2x - 2x + 10$. ($2x$ is subtracted from both sides of the given equation)

$\Rightarrow 3x - 5 = 10$.

$3x - 5 + 5 = 10 + 5$ (Adding 5 to both sides)

$3x = 15$. (Next, divide both sides by 3 to separate x.)

$\frac{3x}{3} = \frac{15}{3}$ or $x = 5$.

Therefore, the solution set (or truth set) is $\{5\}$.

EXERCISES

Identify whether each of the following pair of equations are equivalent or not.

1. $\frac{3}{2}x - 1 = 0$, $3x = 2$.
2. $3(x + 6) = 9$, $x + 2 = 3$.
3. $2x + 1 = 4$, $x + 1 = 2$.
4. $4(0.5 + 2x) = 1 - 2x$, $10x = -1$.

DEFINITION

An equation in one variable, say x, that can be written in the form

$ab + b = 0,$

Where a and b are specified numbers such that $a \neq 0$, is called **linear equation**.

That is, the power of the variable is exactly 1.

EXAMPLE 7

1. $3x+2 = 0$
2. $-3x + 6 = 0$
3. $\frac{1}{2}x - 3 = 0$

are examples of a linear equation. You can obtain the solution of such linear equation as follows:

$ax + b = 0$ is equivalent to $ax = -b$ (Subtracting b from both sides of the given equation)

$x = \frac{-b}{a}$ (Dividing both sides by a)

Therefore, the solution set of $ax + b = 0$ is $\left\{-\frac{b}{a}\right\}$

EXAMPLE 8

Solve the linear equations given in Example 7 above, (i) - (iii).

Solution

1. $3x + 2 = 0$ implies ($\Rightarrow$) $x = -\frac{2}{3}$. Therefore, S.S. = $\left\{-\frac{2}{3}\right\}$
2. $-3x + 6 = 0 \Rightarrow x = \frac{-6}{-3} = 2$. Therefore, S.S. = $\{2\}$
3. $\frac{1}{2}x - 3 = 0 \Rightarrow \frac{1}{2}x = 3$ (3 added to both sides of the given equation)

   $x = 6$ (Both sides multiplied by 2)

   Therefore, S.S. = $\{6\}$

There are various equations that are not exactly in the form of the definition of linear equation, $ax + b = c$, but may be transformed to this form using the basic rules of equation as illustrated by the following example.

EXAMPLE 9

Solve each of the following equations.

(a) $2x-4=x$

(b) $7x-4=3(5-x)+1$

(c) $\dfrac{3x+2}{x-4}=1$

(d) $\dfrac{4}{3x}+2=1-\dfrac{3}{x}$

Solutions

In each case, apply basic rules of equation to bring the variables to the left side and all constant numbers to the right side of the equation as follows:

$2x-4=x$ (The given equation)

$2x=x+4$ (Add 4 to both sides)

$2x-x=4$ (Subtract x from both sides)

$x=4.$

Therefore, the solution set S.S = $\{4\}$.

(a) $7x-4=3(5-x)+1$ (The given equation)

$7x-4=15-3x+1$ (Removing parenthesis by distribution; i.e., $3(5-x)=15-3x$)

$7x-4=-3x+16$ (Combining like terms; i.e., 15 + 1 = 16)

$7x-4+3x=16$ (Adding $3x$ to both sides to remove $-3x$ from the right side)

$7x+3x=16+4$ (Adding 4 to both sides to remove −4 from the left side)

$10x=20$ (Combining like terms)

$\dfrac{\cancel{10}x}{\cancel{10}}=\dfrac{20}{10}=2.$ That is, $x=2$.

Therefore, the solution set S.S = $\{2\}$.

(b) $\dfrac{3x+2}{x-4}=1$ (The given equation. We should have $x\neq 4$ so that the denominator $x-4\neq 0$)

$3x+2=x-4$ (Multiplying both sides by $x-4$)

$3x-x=-4-2$ (Subtracting x and 2 from both sides)

$2x=-6.$ Thus, $x=-\dfrac{6}{2}=-3.$

Therefore, the solution set S.S = $\{-3\}$.

(c) $\dfrac{4}{3x}+2=1-\dfrac{3}{x}$ (The given equation. We should have $x\neq 0$)

$3x\left(\dfrac{4}{3x}+2\right)=3x\left(1-\dfrac{3}{x}\right)$ (Multiplying both sides by $3x$ to remove denominators)

$4+6x=3x-9$

$6x-3x=-9-4$ (Subtracting $3x$ and 4 from both sides)

$3x = -13$. Thus, $x = -\frac{13}{3}$.

Therefore, the solution set $S.S = \left\{-\frac{13}{3}\right\}$.

Linear equations involving fractions: In some linear equations, the coefficients can be fractions. For example, $\frac{1}{5}x + \frac{3}{5} = \frac{4x-3}{3} - \frac{5}{6}$ is a linear equation involving fractions whose denominators are 5, 3 and 6. Though such equations can be solved directly as discussed in the previous section, it would be better to first simplify the equations as follows:

Compute the **LCM** (Least Common Multiple) of all the **denominators** that appear in the equation; and **multiply both sides** of the equation by the LCM.

This clears the denominators from the equation.

Example 10

Find the solution set of $\frac{1}{5}x + \frac{3}{5} = \frac{4x-3}{3} - \frac{5}{6}$

Solution

The denominators in this equation are 5, 3 and 6; and their LCM is 30. Thus, first multiply both sides of the equation by 30 to clear the denominators:

$$30\left(\frac{1}{5}x + \frac{3}{5}\right) = 30\left(\frac{4x-3}{3} - \frac{5}{6}\right)$$

$$\Rightarrow 6x + 18 = 40x - 30 - 25$$

$$\Rightarrow 6x - 40x = -55 - 18 \quad \text{(Collecting like terms)}$$

$$\Rightarrow -34x = -73 \text{ or } x = \frac{-73}{-34} = \frac{73}{34}.$$ Therefore, $S.S = \left\{\frac{73}{34}\right\}$

Example 11

Find the solution set of $\frac{2}{3}(x-2) - \frac{3}{7}(2-5x) = \frac{5}{2}(2x+1) - 2x$.

Solution

The denominators in this equation are 3, 7 and 2; and their LCM is 42. Thus, first multiply every term of both sides by 42 to clear the denominators:

$$42\left[\frac{2}{3}(x-2)\right]-42\left[\frac{3}{7}(2-5x)\right]=42\left[\frac{5}{2}(2x+1)\right]-42(2x)$$

$\Rightarrow 28(x-2)-18(2-5x)=105(2x+1)-84x$

$\Rightarrow 28x-56-36+90x=210x+105-84x$

$\Rightarrow 118x-92=126x+105$ (Combining like terms in each side)

$\Rightarrow 118x-126x=105+92$ (Collecting like terms- variables to the left and numbers to the right side)

$-8x=197$ or $x=\frac{197}{-8}=-\frac{197}{8}$. Therefore, S.S = $\left\{\frac{-197}{8}\right\}$

Exercises

Find the solution set for each of the following equations.

1. $\frac{3}{2}x-1=0$
2. $3x-2(x-3)+1=x+7$
3. $2x+1=4$
4. $3(2x+1)=2(1-5x)+6x+11$
5. $3(x+6)=9$
6. $4(0.5+2x)=1-2x$
7. $\frac{x+3}{2}-4=3$
8. $\frac{8x+3}{2}-5(x+2)=-3\left(x+\frac{5}{6}\right)$
9. $\frac{x}{3}-3=\frac{2x}{6}+1$
10. $\frac{3}{2}x-2x+\frac{1}{2}=\frac{2}{5}x+\frac{7}{5}$
11. $\frac{24}{x+2}-1=3$
12. $\frac{1}{6}(x+3)-\frac{3}{4}(2-\frac{2}{3}x)=\frac{3}{2}(x+1)$

The solution of a word problem is the solution of the equation that represents the problem. Thus, to solve a word problem you need to represent an unknown whose value is to be determined by a variable, say x, and express the problem as a mathematical equation. The following table presents algebraic expressions for some verbal phrases that frequently appear in word problems.

Verbal Phrases	Algebraic expression of the phrases
1. Five more than certain number Or a number increased by 5 Or a number plus 5	Let x be the number (certain number). Then, we write $x + 5$
2. Three less than certain number Or a number decreased by 3 Or a number minus 3	Let x be the number. Then, we write $x - 3$
3. Twice of a number	$2x$ (Let x represent the unknown)
4. Four more than twice a number	$2x + 4$
5. Five less than twice a number or five below twice a number or twice a number decreased by 5	$2x - 5$
6. Two consecutive natural numbers	x and $x + 1$ (x being the initial natural number)

Example 12

Three less than twice a certain number is 5. Find the number.

Solution

Let x = the number. Then, *twice of the number* $\equiv 2x$ and *three less than it* $\equiv 2x - 3$ Thus, the problem is: $2x - 3 = 5$.

Solving this you can get $x = 4$. Thus, the number is 4.

Example 13

The sum of two consecutive natural numbers is 15. Find the numbers.

Solution

Let x = *the first number* (*the smaller number*).

Then, the next natural number is $x + 1$.

So, the sum of these two numbers is 15 means

$x + (x + 1) = 15$.

Thus, $2x + 1 = 15$.

Solving this, you can get $x = 7$. And, the next natural number is $7+1 = 8$.

Therefore, 7 and 8 are *the two consecutive numbers whose sum is* 15.

Example 14

The age of a woman is 5 years more than twice the age of her daughter. If the woman is 43 years old, what is the age of her daughter?

Solution

Let x = *the age of the daughter.* Then, the age of the woman, which is 5 more than twice of x, is 43 means

$$2x + 5 = 43$$

$$2x = 38$$

$$x = 19.$$

Therefore, the age of the daughter is 19 years.

Example 15

The cost, L$C, of maintaining and operating a machine is given by $C = 50 + 3t$, where t is the number of hours operated. How many hours may the machine be operated when the cost is L$65?

Solution

When the cost $C = 65$, the given formula for the cost becomes

$$50 + 3t = 65$$

$$3t = 15$$

$$t = 5$$

Therefore, the machine is operated for 5 years.

Exercises

1. Five less than twice a certain number is 13. Find the number.
2. If half of a number increased by 1 is $\frac{15}{4}$, what is the number?
3. The sum of three consecutive integers is 102. Find the integers.
4. A shop-keeper buys 20 kg of sugar at L$ p per kg. He sells 16 kg at L$ $\left(p+\frac{3}{4}\right)$ per kg and the rest at L$ $(p + 1)$ per kg. How much is his profit?
5. Five years ago the age of a boy was half as old as he is now. How old is he now?
6. Peter is now five times as old as John. In ten years time, Peter will be three times as old as John. How old are they now?
7. The length of a rectangle is three times its width. If the perimeter of the rectangle is 42 cm, then what is the area of the rectangle?

Learners are expected to recall the interpretation of the inequality symbols $<$, $\leq$, $>$ and $\geq$. If the equality symbol (=) in a linear equation $ax+b=0$ is replaced by any one of the inequality symbols ($<$, $\leq$, $>$, $\geq$), then we get a linear inequality as defined below.

Definition

An inequality in one variable, say say x, that can be written in the form

$\boldsymbol{ax + b < 0}$ (or with any one of $\leq, >$ or $\geq$ at the place of $<$)

where a and b are specified numbers such that $a \neq 0$, is called **linear inequality**.

For instance, $3x+2 \leq 0$, $-3x+6 > 0$, $\frac{-2}{3}x < 0$ are examples of a linear inequality. Note that while an equality such as $x = 5$ represents exactly one value, an inequality, say $x < 5$, represents (satisfied by) several numbers such as 4.99, 4, 3.5, 1, 0, −2, etc. The solution of an inequality is the set of all values of the variable that satisfy the inequality, that is, the set of all values of the variable for which the inequality is true. This set is called the truth set (or solution set) of the inequality. For instance, the truth set of $x < 5$ is $\{x \in \mathbb{R} \mid x < 5\}$. That is, every real number below 5 satisfies the inequality.

Similar to the process of solving a linear equation, a given linear inequality may be solved by transforming it to a simpler equivalent inequality (that has the same solution set) step by step until the values of the variable is specified. The steps use the following basic rules of inequality. To get sense of the next rule, observe the following. Given an inequality, say,

$5 < 8$, the inequality remains true if you add any equal values to both sides, say, $5+4 < 8+4$; or it remains true if you subtract any equal values from both sides, say, $5-6 < 8-6$.

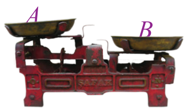
You may practice this with various examples of your own; and do this also for $>, \leq, \geq$.

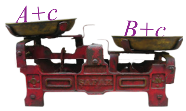
Rule 1

When equal values are added to (or subtracted from) both sides of a given inequality, the result is the same inequality. That is,

If $A < B$, then $A+c < B+c$;

or $A-c < B-c$





The rule is valid if any of the other inequalities, $\leq$, $>$ or $\geq$, is taken in place of $<$.

Example 16

Use the above rule to solve

(a) $x-3<0$ (b) $x+4 \geq 10$.

Solution

(a) Given $x-3<0$, add 3 to both sides to get

$x-3+3<0+3$

$\Rightarrow x<3$

Therefore, the solution set (or truth set) is $\{x \in \mathbb{R} \mid x<3\}$.

(b) Given $x+4 \geq 10$, subtract 4 from both sides to get

$x+4-4 \geq 10-4$.

$\Rightarrow x \geq 6$

Therefore, the solution set (or truth set) is $\{x \in \mathbb{R} \mid x \geq 6\}$.

Next, we state the rule for multiplication and division. Note that given an inequality, say,

8 < 12, it remains true if you multiply both sides by any equal positive values, say, $2 \times 8 < 2 \times 12$; or it remains true if you divide both sides by any equal positive values, say, $\frac{8}{4}<\frac{12}{4}$.

Here, notice that the multiplier should be positive for the inequality to remain true.

Rule 2

When both sides of a given inequality are multiplied (or divided) by the same ***positive*** *number, the result is the same inequality;* i.e., *when $C > 0$.*

If $A < B$, then $cA = cB$,

or $\frac{A}{c}<\frac{B}{c}$

The rule is valid if any of the other inequalities, ≤, > or ≥, is taken in place of <.

Example 17

Use this rule to solve

(a) $\frac{x}{2} \leq 5$ (b) $6x>-3$

Solution

(a) Given $\frac{x}{2} \leq 5$, multiply both sides by 2, to get

$$\not{2}\left(\frac{x}{\not{2}}\right) \leq 2 \times 5$$

That is, $x \leq 10$. So, the truth set T.S. = $\{x \in \mathbb{R} \mid x \leq 10\}$.

(b) Given $6x > -3$, divide both sides by 6, to get

$\frac{6x}{6} > \frac{-3}{6}$ or $x > \frac{-1}{2}$. So, the truth set, T.S = $\{x \in \mathbb{R} \mid x > -\frac{1}{2}\}$

The above rules of inequality are the same as the rules for equation. However, the property of inequality is different when the multiplier is **negative**. For instance, for each of the following inequalities, observe that multiplication of both sides by -1 changes the inequality.

$1 < 5$ but $-1 > -5$ $\qquad$ $10 > 2$ but $-10 < -2$

And so on. In general, multiplying (or dividing) both sides of an inequality by a negative value changes the inequality. Thus, we have the following rule.

Rule 3

Multiplying (or dividing) both sides of a given inequality by the same ***negative*** *number changes (reverses) the inequality.* That is,

if $A < B$ and $c < 0$, then $A > B$ or $\frac{A}{c} > \frac{B}{c}$

The rule is valid if any of the other inequalities ≤, > or ≥, is taken in place of <.

Example 18

Use Rule 3 to solve

(a) $-\frac{x}{4} < \frac{1}{2}$ $\qquad$ (b) $-6x \geq 3$

Solution

(a) Given $-\frac{x}{4} < \frac{1}{2}$, multiply both sides by -4 to get the following.

$$x > -4\left(\frac{1}{2}\right)$$

so that $x > -2$.

Therefore, the truth set T.S. = $\{x \in \mathbb{R} \mid x > -2\}$.

(b) Given $-6x \geq 3$, divide both sides by -6, to get

$$\frac{-6x}{-6} \leq \frac{3}{-6}$$

$$\Rightarrow x \leq -\frac{1}{2}$$

That is, the truth set, T.S = $\left\{x \in \mathbb{R} \mid x \leq -\frac{1}{2}\right\}$

In general, linear inequalities are solved in much the same way as linear equations. However, when both sides of an inequality are multiplied (or divided) by a negative number, the inequality sign is reversed. The following example helps you to make more practice.

Example 19

Find the truth set of each of the following inequalities.

(a) $3x - 5 \geq 10$

(b) $3x - 1 < 7(x + 1)$

(c) $x - 4(x+1) \geq -13 - (x-2)$

(d) $4x + 3 \leq 5x + 4 < 2x + 3$

Solution

(a) $3x - 5 \geq 10$ (The given inequality)

$3x \geq 15$ (Adding 5 to both sides)

$\frac{3x}{3} \geq \frac{15}{3}$ (Divide both sides by 3, which is positive).

$x \geq 5$.

Therefore, T.S = $\{x \in \mathbb{R} \mid x \geq 5\}$.

(b) $3x - 1 < 7(x + 1)$ (The given inequality)

$3x - 1 < 7x + 7$ (You can state the reason)

$3x - 7x < 7 + 1$ (Subtracting $7x$ from both sides, and adding 1 to both sides)

$-4x < 2$

$x > \frac{2}{-4}$ (Dividing both sides by -4 reversed the inequality as $-4 < 0$.)

Therefore, T.S = $\{x \in \mathbb{R} \mid x > -\frac{1}{2}\}$

(c) $x - 4(x+1) \geq -13 - (x-2)$ (The given inequality)

$x - 4x - 4 \geq -13 - x + 2$ (Removing parentheses by distribution)

$-3x - 4 \geq -11 - x$ (Combining like terms; i.e., $x - 4x = -3x$ and $-13+2 = -11$)

$-3x + x \geq -11 + 4$ (Adding x to both sides; and also add 4 to both sides)

$-2x \geq -7$ (Next, division of the inequality by -2 reverses the inequality)

$\frac{\not{-}2x}{\not{-}2} \leq \frac{-7}{-2}$ or $x \leq \frac{7}{2}$;

Therefore, T.S = $\left\{x \in \mathbb{R} \mid x \leq \frac{7}{2}\right\}$.

(d) To solve $4x + 3 \leq 5x + 4 < 2x + 10$, let us recall that $a \leq b < c$ means $a \leq b$ and $b < c$.

Consequently, $4x + 3 \leq 5x + 4 < 2x + 10$ means $4x + 3 \leq 5x + 4$ and $5x + 4 < 2x + 10$. So, we solve these inequalities and get two solution set. Then, the intersection of these solution sets will give the solution set of the given inequality. This is because the solution of the given inequality must satisfy each of the two inequalities.

To solve $4x + 3 \leq 5x + 4$ subtract $5x$ and 3 from both sides to get

$-x \leq 1$ or $x \geq -1$.

Hence, the solution set of the first inequality is $S_1 = \{x \in \mathbb{R} \mid x \geq -1\}$; and to solve $5x + 4 < 2x + 10$, subtract $2x$ and 4 from both sides to get

$3x < 6$ or $x < 2$.

Hence, the solution set of the second inequality is $S_1 = \{x \in \mathbb{R} \mid x < 2\}$.

Therefore, the solution set of $4x + 3 \leq 5x + 4 < 2x + 10$ is

$S_1 \cap S_2 = \{x \in \mathbb{R} \mid -1 \leq x < 2\}$.

Exercises

1. Identify whether each of the following pair of inequalities are equivalent or not.
   (a) $\frac{3}{2}x - 1 \leq 0$, $3x \leq 2$.
   (b) $3(x + 6) > 9$, $x + 2 > 3$.
   (c) $-2x + 1 < -5$, $x < 3$.
   (d) $3(1 - 2x) \geq -9x$, $x \leq -1$.
2. Find the truth ser (solution set) for each of the following inequalities.
   (a) $\frac{3}{2}x - 1 < 0$
   (b) $\frac{x-3}{2} - 4 \leq 3 + x$
   (c) $1 - 2x < 3x + 1$
   (d) $2x \leq 10 - 3x < 3x + 22$
   (e) $3(1 - 6) \geq -5x$
   (f) $6 - 3x \geq x + 6 > 1 - x$

You recall that the real numbers can be represented graphically (geometrically) by points on a horizontal straight line called a number line as follows.

Figure 1. The number line

The solutions of a linear inequality can be presented graphically on the number line. For instance, the following figures, Figure 2(a) and 2(b) show the solutions of $x \geq 0$ and $x > 0$ by a ray along the number line toward the right starting at 0. The solid dot '•' at the beginning of the ray indicates that the end point (here 0) is included; while the hollow dot 'o' indicates that the end point is excluded.

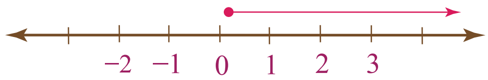
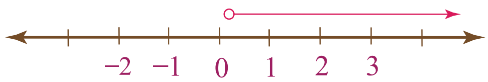
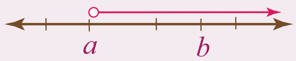
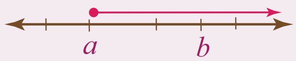
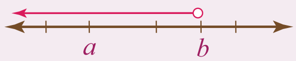
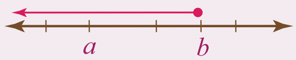
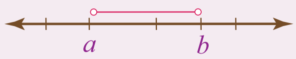
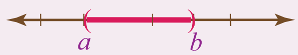
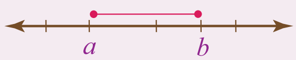
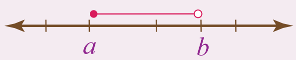
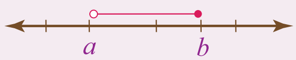


Figure 2. (a) $x \geq 0$



(b) $x > 0$

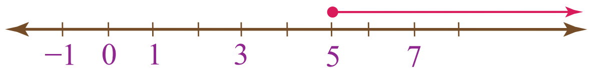
You can similarly show the graph of $x \leq 0$ and $x < 0$ on a number line by a ray toward the left starting at 0. Likewise, the graph of any other inequality such as $x < a$, $x \geq a$, etc., for some real number a, can be presented on a number line as shown on the following table. In this presentation, a and b are any fixed real numbers such that $a < b$.

The given inequality	The inequality as a set	The graph representation of the inequality on the number line
$x > a$	$\{x \in \mathbb{R} \mid x > a\}$	 or 
$x \geq a$	$\{x \in \mathbb{R} \mid x \geq a\}$	 or 
$x < b$	$\{x \in \mathbb{R} \mid x < b\}$	 or 
$x \leq b$	$\{x \in \mathbb{R} \mid x \leq b\}$	 or 
$a < x < b$	$\{x \in \mathbb{R} \mid a < x < b\}$	 or 
$a \leq x \leq b$	$\{x \in \mathbb{R} \mid a \leq x \leq b\}$	 or 
$a \leq x < b$	$\{x \in \mathbb{R} \mid a \leq x < b\}$	 or 
$a < x \leq b$	$\{x \in \mathbb{R} \mid a < x \leq b\}$	 or 

Now you can represent solution of a linear inequality graphically. For instance, let us represent the solutions of the inequalities given in Example 17 graphically.

(i) The solution set of $3x - 5 \geq 10$ is $\{x \in \mathbb{R} \mid x \geq 5\}$.

Graphical representation:

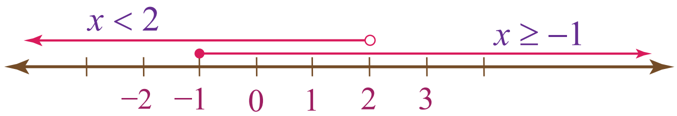


Similarly, you can graphically represent the solution of (ii) and (iii).

(ii) You have seen that the solution set $4x + 3 \leq 5x + 4 < 2x + 10$ is the intersection of the solution sets of two inequalities $4x + 3 \leq 5x + 4$ and $5x + 4 < 2x + 10$. The Graphs of the two solutions on the same number line makes identification of the intersection easier. You have seen that the solution of

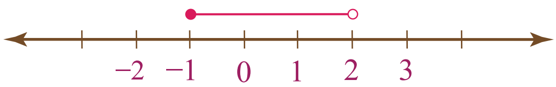
$$4x + 3 \leq 5x + 4 \text{ and } 5x + 4 < 2x + 10$$

are $x \geq -1$ and $x < 2$, respectively. Let us put their graphs (rays) on the same line:



Now, you can see that the common solutions for both inequalities are where the two rays overlap; which is between −1 and 2 (including −1 but excluding 2). Therefore, the solution set of the inequality $4x + 3 \leq 5x + 4 < 2x + 10$ is $\{x \in \mathbb{R} \mid -1 \leq x < 2\}$.

Graphical representation of this solution:



Exercises

Solve each of the following inequalities and represent the solution graphically.

1. $\frac{3}{2}x - 1 < 0$
2. $\frac{x-3}{2} - 4 \leq 3 + x$
3. $1 - 2x < 3x - 9$
4. $3(3 - 2x) < 2(3 + x)$
5. $3(x - 6) \geq -5x$
6. $2x \geq 10 - 3x < 3x + 221 - 7x > x + 3$
7. $6 - 3x \geq x + 6 > 1 - x$

Problems involving linear inequalities are very common in everyday life. You have observed in the previous sections that while a linear equation gives us exactly one value when solved, a linear inequality gives us multiply values. For instance, describe or express the following statements as linear inequalities and think about their possible solutions.

1. It takes at least 30 minutes to get there. (What are the possible amount of times needed to get there?)
2. The trip will cost at most L$ 250. (How much will the possible cost of the trip?)
3. The speed of the car exceeded the 30 km/h limit. (What was the possible speed of the car?)

The table below shows the inequalities representing the above statements and some of their possible solutions.

Statement	Inequality	Some possible solutions
It takes **at least** 30 minutes = Not less than 30 minutes = 30 minutes or more	Let x be the time it takes. $x \geq 30$	30 minutes, 35 minutes, 40 minutes, ...
It costs **at most** L$250. = It costs not more than L$250. = It costs L$250 or below.	Let x be the cost $x \leq 250$	L$250, L$248, L$245, ...
The speed **exceeded** 30 km/h = It is above 30 km/h = It is greater than 30 km/h	Let x be the speed $x > 30$	31 km/h, 35 km/h 40 km/h...

Example 20

A worker receives L$240 for his weekly wage plus L$12 per hour for overtime. How much overtime must he work to be sure of at least L$300 for a week's work?

Solution

Let x be the number of hours he worked overtime.

Then, his weekly pay is $12x$ (for overtime) plus 250 (wage). Thus, he wants to have

$$12x + 240 \geq 300$$

$\Rightarrow 12x \geq 60$

$\Rightarrow x \geq \dfrac{60}{12}$. i.e., $x \geq 5$.

Therefore, he has to work at least 5 hours overtime in order to earn at least L$300 for a week.

Example 21

Chan wants to order DVDs over the internet. Each DVD costs L$21.5 and shipping for the entire order is L$19.3. Chan has no more than L$230 to spend. How many DVDs can Chan order without exceeding his L$230 limit?

Solution

Let x be the number of DVDs he orders.

The cost of these DVDs plus the shipping cost is $21.50x + 19.30$. This should not exceed 230.

That is, $21.5x + 19.30 \leq 230$.

$\Rightarrow 21.5x \leq 230 - 19.3$

$\Rightarrow x \leq \dfrac{210.7}{21.5}$

$\Rightarrow x \leq 9.8$.

Since Chan cannot order 0.8 of a DVD, we round 9.8 down to 9.

Therefore, Chan can order 9 DVDs (or less) without exceeding his limit of L$230.

Example 22

Yellow Cab Taxi charges a L$1.50 flat rate (fixed rate to start) in addition to L$0.25 per km. Lucy can spend at most L$5 on the ride. How many kilo meters can Lucy travel without exceeding her budget?

Solution

The problem is to determine the number of km that Lucy can travel. So, let

x = the number of km that she travels on the ride.

For this, she will pay $0.50x+1.50$ and this must be at most (less than or equal to) 5. That is,

$0.50x + 1.50 \leq 5$

$0.50x \leq 3.50$ (1.50 subtracted from both sides)

$$x \leq \frac{3.50}{0.50}$$

Therefore, $x \leq 7$.

That is, she can travel at most 7 km on the ride without exceeding her budget.

Exercises

1. Five less than the product of −3 and a number is greater than −23. Write and solve an inequality to represent this relationship. Also graph the solution set .
2. The sum of two consecutive integers is at least −17. What are the smallest values of consecutive integers that will make this true?
3. Connor went to the carnival with L$22.50. He bought a sandwich and a drink for L$3.75, and he wanted to spend the rest of his money on ride tickets which cost L$1.25 each. What is the maximum number of ride tickets that he can buy?
4. The area of a rectangular plot of land is less than 2541 ft^2. If one of the dimensions of the plot is 42 ft, what may the other dimension be?
5. The cost of obtaining a machine is L$50 and the cost of operating the machine is L$4 per hour. How many hours may the machine be operated without the cost exceeding L$80?

Key Terms

- basic rules of equation
- basic rules of inequality
- equation
- equivalent equations
- graph of solutions of inequality
- inequality
- linear equations
- linear inequality
- solution of linear equations
- solution of linear inequalities
- solution set
- truth set

Summary

- An **equation** is a mathematical statement that states two expressions are equal.
- **Solution of an equation** is the value of the unknown (variable) that makes the equation true or satisfies the equation. A **solution set** (or **truth set**) of an equation is the set containing all solutions of the equation.

- Two equations are said to be **equivalent** if they have the same solution set.
- **Basic Rule of Equation:**

  If $A = B$, then (1) $A + c = B + c$; $A - c = B - c$ *for any c*

  (2) $cA = cB$; $\frac{A}{c} = \frac{B}{c}$ *for any* $c \neq 0$
- An equation that can be written in the form

  $ax + b = 0$, where $a \neq 0$, is called **linear equation**. The solution of this equation is $x = -\frac{b}{a}$.
- An **inequality** is a mathematical statement that involves $<, \leq, >$ or $\geq$.
- **A solution of an inequality** is the value of the unknown (variable) that makes the inequality true or satisfies the inequality. The **truth set** (**solution set**) of an inequality is a set that contains all solutions of the inequality.
- Two inequalities are said to be **equivalent** if they have the same truth set.
- **Basic Rules of Inequality:**

  If $A \leq B$, then (1) $A + c \leq B + c$, *for any c*; $A - c \leq B - c$, *for any c*

  (2) $cA \leq cB$ *if* $c > 0$; $\frac{A}{c} \leq \frac{B}{c}$ *if* $c > 0$

  (3) $cA \geq cB$ *if* $c < 0$; $\frac{A}{c} \geq \frac{B}{c}$ *if* $c < 0$.
- An inequality that can be written as

  $ax - b < 0$ or $ax - b \leq 0$, where $a \neq 0$, is called **linear inequality**.
- The **truth set** (**solution set**) of the inequality $ax + b < 0$ is $\left\{x \in \mathbb{R} \mid x < -\frac{b}{a}\right\}$ if $a > 0$; and it is $\left\{x \in \mathbb{R} \mid x > -\frac{b}{a}\right\}$ if $a < 0$.

Exercises

1. Determine whether each of the following pair of equations (or inequalities) is equivalent or not.

(a) $\frac{1}{2}x + 1 = 1 - \frac{2}{3}x$; $3x = -4x$.

(b) $1 - x \leq 2$; $x \leq -1$

(c) $\frac{x}{x+2} = \frac{1}{3}$; $x + 2 = 3x$.

(d) $\frac{1}{3} < \frac{1}{x}$; $x < 3$.

(e) $x - 1 = 3(2x + 1)$; $x - 1 = 2x + 1$;

(f) $-x \geq 1$; $x \geq 1$.

2. Find the solution set of each of the following equations.
   (a) $2x - 3 = -1$
   (b) $2(1-5x) = 42 - 3(6+7x)$
   (c) $3x+2 = 5x-6$
   (d) $3.2 + 1.8x = 2.6x$
   (e) $\frac{1}{3}x - \frac{1}{5}(2+x) = x + \frac{7}{3}$
   (f) $\frac{2x+5}{2} + \frac{x-7}{3} = \frac{17}{6}$
   (g) $2 - \frac{1}{3}(2x-1) = \frac{2}{3}x$
   (h) $\frac{1}{2x} - \frac{3-x}{3x} = \frac{1}{12}$
   (i) $\frac{1}{2}x - \frac{x-4}{5} = \frac{23}{10}$
   (j) $\frac{2}{x+2} - 3 = \frac{x}{x+2}$
3. Find the truth set (solution set) of each of the following inequalities and graph it.
   (a) $5x - 6 \leq 3x + 2$
   (b) $4x + 3 \geq 3(2x - 1)$
   (c) $7x + 4 < \frac{1}{2}(4x+3)$
   (d) $\frac{1}{4}x - \frac{1}{3}(x-1) \leq \frac{7}{12}$
   (e) $3(x-3) < x-3$
   (f) $x - 8 < 1-2x \leq x + 4$
   (g) $\frac{5-x}{3} + 2 > \frac{x-2}{2}$
   (h) $3(x+1) \geq \frac{7}{2}x + 2 > \frac{6x+1}{2}$
4. The sum of two-third of a number and four-fifth of the same number is 22. Find the number.

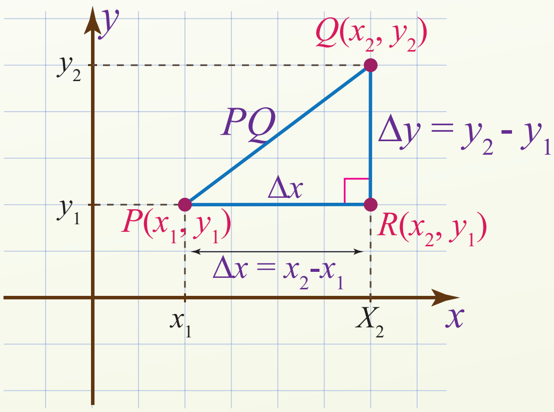
5. When the sum of two consecutive even integers decreased by 5 the result is 29. Find the integers.
6. At present, Maria is three times as old as John. In ten years time, Maria will be twice as old as John. How old are they now?
7. The length of a rectangular field is 1 meter more than twice the width. If the perimeter is 302 meters, determine the length and width of the field.
8. The sum of twice a number and 7 is greater than or equal to the sum of 3 times the same number and 1. Find the range of the values for the number.
9. Khan earned L$7.55 per hour plus an additional L$100 in tips waiting tables on Saturday evening. He earned L$160 in all. To the nearest hour, what is the least number of hours Khan would have to work to earn this much money?
10. Eva has L$500 in her bank account. Every week, she withdraws L$40 for expenses. Without making any deposits, how many weeks can she withdraw this money if she wants to maintain a balance of at least L$200?

M10CH07

CHAPTER 7

RELATIONS AND FUNCTIONS

Chapter Contents



Chapter Outcomes

Learners are able to apply concepts to solve problems on relations, functions, graph relations and functions; and determine the gradient of straight lines and calculate distance between two points.

Upon completion of this chapter, learners should be able to:

- discuss Relations;
- distinguish between various types of relations;
- identify functions;
- change the subject of a relation;
- graph linear functions;
- find the gradient of a straight line;
- calculate the distance between two points;
- graph quadratic functions.

Introduction

This unit has two main topics: *relations* and *functions*. In our daily life, we come across many patterns that characterize certain relations such as mother and son, teacher and student, country and continent, etc. Similarly, in mathematics, we come across different relations such as $5 > 2$, 5 is a factor of 20 and formulas that relate two quantities. In this section, we will focus on mathematical relations and special type of relation which is called function.

A relation describes certain relationship between two things. For example, we say someone is the father of another person, Monrovia is the capital city of Liberia, 10 is a multiple of 2, etc. In mathematics, we usually look for a relation between elements of two sets and use ordered pair to describe related elements. For example, let

A = {Monrovia, Paris, Abuja, Addis Ababa} and

B = {China, Ethiopia, Liberia, Nigeria}.

For $x \in A$ and $y \in B$, if x and y in the ordered pair (x, y) are related by the phrase

"x is the capital city of y",

then the relation can be described by the following set of ordered pairs:

{(Monrovia, Liberia), (Abuja, Nigeria), (Addis Ababa, Ethiopia)}.

Each pair is ordered so as to fit to the phrases (the statement) that describe the relation and should make the statement true. For example, according to the statement of the given relation, the pair (Monrovia, Liberia) means "Monrovia is the capital city of Liberia" which is true.

In general, relation is the set of ordered pairs. Thus, as ordered pairs are elements of the Cartesian product of two sets, it is important to recall the notion of Cartesian product. The following activity is helpful for this purpose.

Activity 1

Recall that the Cartesian product of two sets A and B is given by

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}.$$

1. Suppose $A = \{0, 1\}$, and $B = \{2, 3, 4\}$.
   (a) Find $A \times B$ and $B \times A$
   (b) Plot $A \times B$ and $B \times A$ on the Cartesian coordinate plane.
   (c) Is $A \times B = B \times A$? (Explain).

2. Let $A = \{2, 3, 4, 6, 7\}$ and $B = \{3, 5, 8, 7, 9\ 12\}$. List all ordered pairs (x, y) which satisfy each of the following sentences where $x \in A, y \in B$.
   (a) $x > y$
   (b) y is a multiple of x
   (c) The sum of x and y is odd
   (d) $x + 1 = y$.

DEFINITION

Let A and B be non-empty sets. A relation R from A to B is any subset of $A \times B$.
In other words, R is a relation from A to B if and only if $R \subseteq A \times B$.

EXAMPLE 1

Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 3, 5\}$

(i) $R_1 = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$ is a relation from A to B because $R_1 \subseteq A \times B$. Is R_1 a relation from B to A? Explain your answer.
Notice that we can represent R_1 by the set builder method as
$R_1 = \{(x, y) \mid x \in A, y \in B, x < y\}$.

(ii) $R_2 = \{(2, 3), (4, 5)\}$ is also a relation from A to B because $R_2 \subseteq A \times B$.
In the set builder method, R_2 is represented by
$R_2 = \{(x, y) \mid x \in A, y \in B, y = x + 1\}$.
Note that, even though $2 < 3$ and $4 < 5$, $R_2 \neq \{(x, y) \mid x \in A, y \in B, x < y\}$ (Explain the reason)

Alternatively, we may use Venn diagram with arrows to express a relation. In this case, an arrow from a to b represents the ordered pair (a, b). For instance, the above two relations R_1 and R_2 in Example 1(i) and (ii), can be represented by then Venn diagrams as follows.

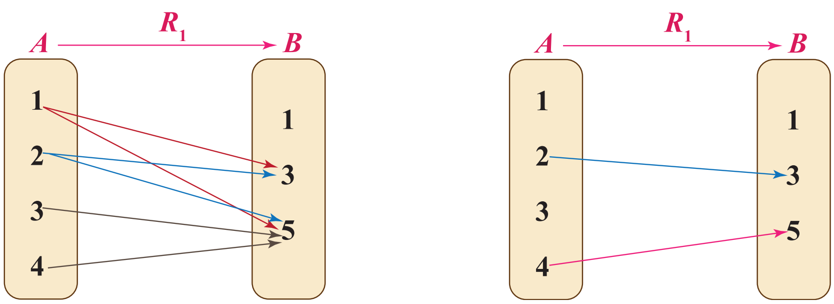


Figure 1.

Remark

The above discussion illustrates that a relation R may be represented either by

- the roster (listing) method; i.e., listing elements of R;
- the set builder method; i.e., stating a rule or formula that describes elements of R; or
- by a Venn diagram in which an arrow from a to b shows $(a, b)\in R$.

Next, we will discuss some basic terminologies and notions that frequently appear in the topics of relations.

1. Image and pre-image of elements of a relation

If $(x, y)\in R$ for a relation R, we say x is related (paired) to y. If so, y is called the *image* of x and x is called the *pre-image* of y. In a relation, x may have more than one image.

2. The domain and range of a relation

DEFINITION

Let R be a relation from a A to B. Then

(i) Domain of R, denoted by dom(R), is $\{x \mid (x, y) \in R \text{ for some } y\}$.

i.e., dom(R) is the set of all first coordinates (first components) of its elements.

(ii) Range of R, denoted by rang(R) is $\{y \mid (x, y) \in R \text{ for some } x\}$

i.e., rang(R) is the set of all second coordinates (second components) of its elements.

Therefore, dom(R) $\subseteq A$; and rang(R) $\subseteq B$.

EXAMPLE 2

If $R = \{(1, 3), (2, 3), (2, 5), (4, 3), (7, 1)\}$, then dom($R$) = $\{1, 2, 4, 7\}$; and rang(R) = $\{3, 5, 1\}$.

EXAMPLE 3

Let $A = \{1, 2, 4, 6, 7\}$, $B = \{3, 5, 7, 8, 9\}$ and $R = \{(x, y) \mid x\in A, y\in B, x > y\}$.

Find the domain and range of R.

Solution

Note that $R = \{(4, 3), (6, 3), (6, 5), (7, 3), (7, 5)\}$.

Therefore, dom(R) = $\{4, 6, 7\}$ and rang(R) = $\{3, 5\}$.

3. Relation on a given set.

A relation R can be from a set A to A itself. In this case, we say R is a **relation on** A. That is, R is a relation on A if $R \subseteq A \times A$.

Example 4

Let $A = \{0, 1, 2, 3, 4, 5, 6\}$. Suppose R is a relation on A defined by

$R = \{(a, b) \mid a, b \in A,\ \ b = 2a\}$. Find the domain and range of R.

Solution

Observe that $R = \{(0, 0), (1, 2), (2, 4), (3, 6)\}$.

Therefore, $\text{dom}(R) = \{0, 1, 2, 3\}$ and $\text{rang}(R) = \{0, 2, 4, 6\}$.

4. Inverse of a relation

Consider the following two relations represented by the respective Venn diagrams.

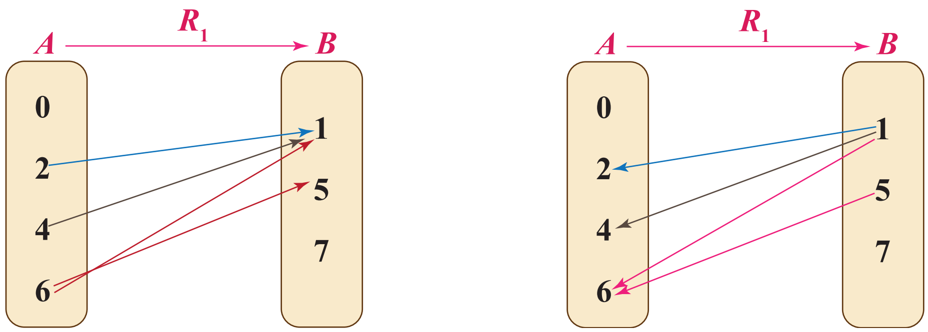


Figure 2.

From the diagram, $R_1 = \{(2, 1), (4, 1), (6, 1), (6, 5)\}$ and

$R_2 = \{(1, 2), (1, 4), (1, 6), (5, 6)\}$.

There is a close relationship between the two relations: if $(a, b) \in R_1$, then $(b, a) \in R_2$ and vice versa. Such relations are called inverse of one another.

Definition

Let R be a relation from A to B.

The **inverse** of R, which is denoted by R^{-1}, is a relation from B to A defined by

$R^{-1} = \{(y, x) \mid (x, y) \in R\}$.

Hence, the inverse of a relation is the set of ordered pairs obtained by interchanging the coordinates of elements of R. From the definition, observe that

$\text{dom}(R) = \text{rang}(R^{-1})$. and $\text{rang}(R) = \text{dom}(R^{-1})$.

Example 5

If $R = \{ (2, 1), (6, 3), (10, 5), (14, 7) \}$, then

$R^{-1} = \{ (1, 2), (3, 6), (5, 10), (7, 14) \}$.

Moreover, $\text{dom}(R).= \{2, 6, 10, 14\} = \text{ran}(R^{-1})$ and $\text{rang}(R) = \{1, 3, 5, 7\}= \text{dom}(R^{-1})$.

Exercises

1. Let $A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$
   (a) Is $R = \{(2, 1), (2, 3), (2, 5), (1, 2), (3, 4), (5, 6)\}$ a relation from A to B? Give the reason for your answer
   (b) Let $R = \{(2, 2), (4, 4), (6, 6)\}$ is a relation on A. Express R using the set builder method.
   (c) If R is a relation from A to B given by $R = \{(x, y): y = x - 1\}$, then
      (i) list the elements of R
      (ii) list the elements of R^{-1}.
      (iii) Express R^{-1}. using the set builder method.
2. If $R = \{(x, y): y = 2x +1\}$ is a relation on A, where $A = \{1, 2, 3, 4, 5, 6\}$, then find the domain and range of R.
3. Let $A = \{1, 2, 3, 4\}$ and $R = \{(x, y): y \geq x +1; x, y \in A\}$.
   (a) List all elements of R; and determine the domain and the range of R.
   (b) List all elements of R^{-1} and express R^{-1} using the set builder method.

In this section, we will describe some types of relations whose elements have some special properties.

Recall that a relation from A to B is a subset of $A \times B$. A subset can be empty or can be the entire set. Likewise, it is possible for a relation to be $\varnothing$ (empty set) or the entire $A \times B$.

Definition

Let R be a relation R from A to B.

(i) R is said to be an **empty relation** or **void** if $R = \varnothing$.

(ii) R is said to be the **universal relation** if $R = A \times B$.

Example 6

Given $A = \{1, 2\}$, $B = \{4, 5, 6\}$,

let R_1, R_2 and R_3 be relations from A to B defined as follows:

(i) $R_1 = \{(x, y) \mid x \in A, y \in \text{B}, x > y\}$

(ii) $R_2 = \{(x, y) \mid x \in A, y \in \text{B}, x < y\}$

(iii) $R_3 = \{(x, y) \mid x \in A, y \in \text{B}, x \text{ is a prime factor of } y\}$.

Identify whether each of these relations is *void*, *universal* or neither.

Solution

(i) There is no element of A which is greater than any of the elements of B.
Hence, $R_1 = \varnothing$; i.e., R_1 is void.

(ii) Every element of A is less than every element of B; i.e., $(x, y) \in R_2$ for every $x \in A$, $y \in B$.
Hence, $R_2 = A \times B$; i.e., R_2 is the universal relation from A to B.

(iii) 2 is the only prime number in A and it is a factor of 4 and 6. Hence, $R_3 = \{(2, 4), (2, 6)\}$.
So, $R_3 \neq \varnothing$ and $R_3 \neq A \times B$. Therefore, R_3 is neither void nor universal relation.

Definition

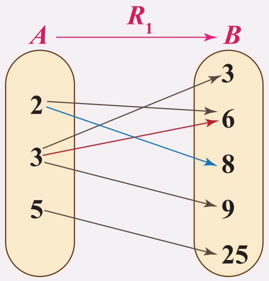
Let R be a relation from A to B. R is said to be:

(i) **One-to-many** if there is an element of A which is paired with more than one elements of B; i.e., there are two or more ordered pairs that have the same first component.

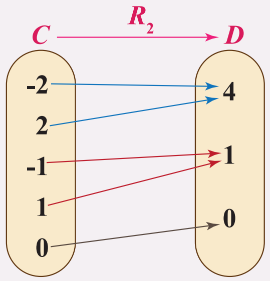
(ii) **Many-to-one** if there are two or more elements of A which are paired with one element of B. i.e., two or more ordered pairs have the same second component.

Example 7

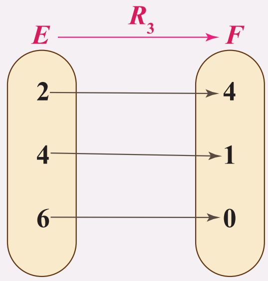
Let R_1, R_2 and R_3 be relations defined by the following Venn diagram. Identify whether each of these relations is one-to-many, many-to-one or neither.



R_1={(x, y) | x∈A, y∈B, x is a prime factor of y}



R_2={(x, y) | $y = x^2$, x∈C, y∈D}



R_3 ={(x, y) | y =2x, x∈E, y∈F}

Solution

R_1 is one-to-many because, for instance, (2, 6) and (2, 8) are in R_1.

- R_2 is many-to-one because, for instance, (−2, 4) and (2, 4) are in R_2.
- R_3 is neither one-to-many nor many-to one. Such relation is called one-to-one.

Definition

Let R be a relation on a set A. R is said to be:

(i) **Reflexive** if $(x, x) \in R$ for every $x \in A$. i.e. every element of A is related with itself.

(ii) **Symmetric** if $(x, y) \in R$ implies $(y, x) \in R$

(iii) **Transitive** if $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$

Example 8

Let R be a relation on $\mathbb{N}$ (the set of natural numbers) given by

$R = \{(x, y) \mid x \leq y,\ x, y \in \mathbb{N}\}$. Then,

(i) R is reflexive. i.e., $(x, x) \in R$ for all $x \in \mathbb{N}$ because $x \leq x$ for all $x \in \mathbb{N}$.

(ii) R is not symmetric because $x \leq y$ does not imply $y \leq x$.

For instance, $(2,3) \in R$ but $(3,2) \notin R$.

(iii) R is transitive because if $x \leq y$ and $y \leq z$, then $x \leq z$.

i.e., if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

Example 9

Let R be a relation on $\mathbb{Z}$ (the set of integers) given by

$$R = \{(x, y) \mid y - x \text{ is an integral multiple of } 3\}.$$

Show that R is reflexive, symmetric and transitive.

Solution

Note that the integral multiples of 3 are 0, ±3, ±6, ±9, ±12, etc, i.e., $3k$ for $k \in \mathbb{Z}$.

That is, $(x, y) \in R$ if $y - x = 3k$ for some $k \in \mathbb{Z}$.

1. To show R is reflexive: For any $x \in \mathbb{Z}$, $x - x = 0$ which is a multiple of 3.

   Hence, $(x, x) \in R$ for all $x \in \mathbb{Z}$. Therefore, R is reflexive.

2. To show R is symmetric: Let $(x, y) \in R$

   $\Rightarrow y - x = 3k$, for some $k \in \mathbb{Z}$.

   $\Rightarrow x - y = -3k$, which is a multiple of 3.

   $\Rightarrow (y, x) \in R$.

   Thus, if $(x, y) \in R$ then $(y, x) \in R$. Therefore, R is symmetric.

3. To show R is transitive: Let $(x, y) \in R$ and $(y, z) \in R$.

   $\Rightarrow y - x = 3k$ and $z - y = 3m$ for some $k, m \in \mathbb{Z}$.

   $\Rightarrow (y - x) + (z - y) = 3k + 3m$

   $\Rightarrow z - x = 3(k + m)$, which is a multiple of 3.

   $\Rightarrow (x, z) \in R$.

Hence, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$. Therefore, R is transitive.

Exercises

1. Determine whether each of the following relations defined on $A = \{0, 1, 2, 3, 4\}$ is void, universal or neither.

   (a) $R_1 = \{(x, y) \mid x \in A, y \in A, xy \geq 0\}$

   (b) $R_2 = \{(x, y) \mid x \in A, y \in A, x^2 = y\}$

   (c) $R_3 = \{(x, y) \mid x \in A, y \in A, x^2 = y + 17\}$

2. Let $A = \{x \in \mathbb{Z} \mid -9 \leq x \leq 9\}$. Determine whether the following relation is one-to many, many-to-one or neither of these.

   (a) $R_1 = \{(x, y) \mid x, y \in A, y = x^2\}$

   (b) $R_2 = \{(x, y) \mid x, y \in A, y^2 = x\}$

   (c) $R_3 = \{(x, y) \mid x, y \in A, y = x^3\}$

3. Determine whether each of the following relation defined on $A = \{1, 2, 3, 4\}$ is reflexive, symmetric, transitive or neither of these.

   (a) $R_1 = \{(1, 1), (2, 2), (3,3), (4, 4), (2, 4), (4, 2)\}$

(b) $R_2 = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 4), (4, 3)\}$

(c) $R_3 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$

4. Determine whether each of the following relation is reflexive, symmetric, transitive or neither of these.

(a) $R_1 = \{(x, y) \mid x, y \in \mathbb{R}, y > x\}$

(b) $R_2 = \{(x, y) \mid x, y \in \mathbb{R}, y \leq x^2\}$

(c) $R_3 = \{(x, y) \mid x, y \in \mathbb{Z}, y - x$ is an integral multiple of 4.$\}$

(d) $R_4 = \{(x, y) \mid x, y \in \mathbb{Z}, y = x + 5m$, for some $m \in \mathbb{Z}\}$

You have seen different types of relations in the previous section. A relation which is **not** one-to-many is a function. That is, a function is a special type of relation where **no** element of its domain is paired with more than one element of its range**.** Thus, function is defined as follows.

Definition

A function f from A to B is a relation from A to B in which each element of A is related (paired) to only one of the elements of B.

That is, for every $x \in A$ there is only one $y \in$ B such that $(x, y) \in f$.

So, a relation fails to bet a function if it has two ordered pairs (x, y) and (x, z) such that $y \neq z$.

Example 10

Consider the relation $R = \{(1, 2), (7, 8), (4, 3), (7, 6)\}$

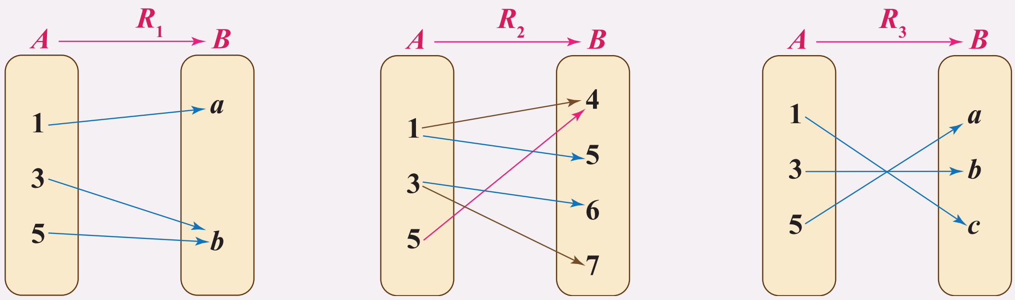
R is not a function because $(7, 8)$ and $(7, 6)$ are in R but $8 \neq 6$,

Example 11

Let $R = \{(1, 2), (7, 8), (4, 3)\}$. This relation is a function because no *first*-coordinate is paired (related) with more than one element of the *second*-coordinate.

Example 12

Consider the following arrow diagrams.



Which of these relations are functions?

Solution

R_1 is a function. (Why?)

R_2 is not a function because 1 and 3 are both mapped to (paired with) two numbers.

R_3 is a function. (Why?)

Example 13

The relation $R = \{(x, y): y \text{ is the father of } x\}$ is a function because no child has more than one father.

Example 14

Consider the relation $R = \{(x, y): y \text{ is a grandmother of } x\}$.

This relation is not a function since everybody (x) has two grandmothers.

Notations and terminologies :

1. A function from A to B is usually denoted by f and written as $f: A \to B$. (So, the notation $f: A \to B$ is read as 'f is a function from A to B')
2. Given $f: A \to B$, if $(x, y) \in f$, we write $f(x) = y$. (Read $f(x)$ as 'f of x') That is, $f = \{(x, y) \mid y = f(x)\}$.
3. Various terminologies are used to express $f(x) = y$, such as,
   - the value of f at x is y.
   - f maps x to y,
   - f assigns y to x, and
   - y is the image of x under f

Example 15

Let $A = \{1, 2, 3, 4\}$, $B = \{3, 5, 7, 9, 11\}$ and suppose

$f: A \to B$ such that $f = \{(1, 3), (2, 5), (3, 7), (4, 9)\}$.

Then, this can be written as $f(1) = 3$, $f(2) = 5$, $f(3) = 7$, and $f(4) = 9$.

A function from A to B is usually defined by a rule (or formula) that assigns to each $x \in A$ certain $y \in B$ uniquely. For instance, the function f in Example 15 above can be defined by the rule: $f(x) = 2x + 1$, for each $x \in A$. Thus, a function can also be defined as follows:

A function f from A to B is a rule (or a formula) that assigns to each $x \in A$ exactly one element, denoted by $f(x)$ in B.

Note: The following directly follows from the definition of a function $f: A \to B$:

$$\text{dom}(f) = A \text{ ; and } \text{rang}(f) = \{ y \in B \mid y \text{ is the image of some } x \in A \text{ under } f \}$$
$$= \{ y \in B \mid y = f(x), \text{ or each } x \in A \}$$

Example 16

Let $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$.

1. Describe R as a set of ordered pairs using the set builder method.
2. Determine domain and range of f.

Solution

1. $f = \{ (x, y) \mid y = f(x), \text{ for any } x \in \mathbb{R} \}$.

$$= \{(x, y) \mid y = x^2, \text{ for any } x \in \mathbb{R}\}.$$
$$= \{(x, x^2) \mid x \in \mathbb{R}\}.$$

2. $\text{dom}(f) = \mathbb{R}$ (the set of all real numbers);

$\text{rang}(f) = \{y \mid y = x^2, \text{ for } x \in \mathbb{R}\}$

$= \{y \mid y \geq 0\}$ since $x^2 \geq 0$ for every $x \in \mathbb{R}$.

That is, rang (f) is the set of all non-negative real numbers.

In general, when a function f is specified by a rule, its domain is the set of all possible numbers on which the rule is defined (or valid); and the range of the function is $\text{rang}(f) = \{ y \mid y = f(x), x \in \text{dom}(f) \}$

Example 17

Consider $f(x) = -2x$. Find the domain and range of f.

Solution

Since $f(x) = -2x$ is defined for all $x \in \mathbb{R}$, dom(f) is the set of all real numbers.

The range is also $\mathbb{R}$ since for every $y \in \mathbb{R}$, there is a real number x $\left(\text{say, } x = -\frac{y}{2}\right)$ such that $y = f(x) = -2x$.

Example 18

Let $f(x) = \sqrt{x-3}$. Find the domain and range of f.

Solution

Since the expression in the radical must be non-negative, $x - 3 \geq 0$.

This implies $x \geq 3$. So dom(f) = $\{x \mid x \geq 3\}$.

Since the value of $\sqrt{x-3}$ is always non-negative, rang(f) = $\{y \mid y \geq 0\}$.

Example 19

Let $f(x) = x^2+5$. Find the domain and range of f.

Solution

Since x^2+5 is defined (valid) for every real number, dom (f) = $\mathbb{R}$.

On the other hand, rang(f) = $\{ y \mid y = x^2+5, x \in \text{dom}(f) \}$. To find this, note that $x^2 \geq 0$ implies $x^2+5 \geq 5$, for every $x \in \mathbb{R}$. Thus, $y = x^2+5 \geq 5$ for every $x \in \mathbb{R}$.

Therefore, rang(f) = $\{y \mid y \geq 5\}$.

Example 20

Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 7, 9\}$

If $f: A \rightarrow B$ is given by $f(x) = 2x + 1$, then find the domain and the range of f.

Solution

Since $f(1) = 3 \in B, f(2) = 5 \in B, f(3) = 7 \in B$ and $f(4) = 9 \in B$,

dom(f) = $\{1, 2, 3, 4\}$ and rang(f) = $\{3, 5, 7, 9\}$.

Remark: If $f: A \rightarrow B$ is a function, then, for any $x \in A$ the image of x under f, $f(x)$, is also called the functional value of f at x. For example, if $f(x) = x - 3$, then the functional value of f at $x = 5$ is $f(5) = 5 - 3 = 2$.

Finding the functional value of f at x is also called evaluating f at x.

EXAMPLE 21

Take $f(x)=\sqrt{x-3}$ and evaluate:

(a) $f(3)$ (b) $f(12)$

Solution

(a) $f(3)=\sqrt{3-3}=\sqrt{0}=0$.

(b) $f(x)=\sqrt{12-3}=\sqrt{9}=3$.

EXAMPLE 22

For the function $f(x) = 1-x^2$

(a) Find the domain and the range of f.

(b) Evaluate $f(2)$ and $f(-1)$.

Solution

(a) dom $(f) = \{x| x\in\mathbb{R}\}$, the set of all real numbers, since $1-x^2$ is defined for all $x\in\mathbb{R}$.

On the other hand, rang$(f) = \{ y \mid y = 1-x^2, x\in \mathbb{R} \}$. To find this, note that $-x^2\leq 0$ implies $1-x^2\leq 1$, for every $x\in\mathbb{R}$. Thus, $y = 1-x^2 \leq 1$ for every $x\in\mathbb{R}$.

Therefore, rang$(f) = \{y\in\mathbb{R} \mid y\leq 1 \}$.

(b) $f(2) = 1-(2)^2 = 1-4 = -3$ and $f(-1) = 1-(-1)^2 = 1-1 = 0$.

EXERCISES

1. Determine whether each of the following relations is a function or not, and give reasons for those that are not functions.
   (a) $R = \{(-1, 2), (1, 3), (-1, 3)\}$
   (b) $R = \{(x, y): y$ is a multiple of $x\}$
   (c) $R = \{(1, 1), (1, 3), (-1, 3), (2, 1)\}$
   (d) $R = \{(x, y): y = x^2 + 3\}$
   (e) $R = \{(x, y): y$ is the area of triangle $x\}$
   (f) $R = \{(x, y): y < x\}$
   (g) $R = \{(x, y): x$ is the area of triangle $y\}$
   (h) $R = \{(x, y): x$ is the son of $y\}$

2. Find the domain and the range of each of the following functions:

   (a) $f(x) = 3$

   (b) $f(x) = 1 - 3x$

   (c) $f(x) = x^2 - 3$

   (d) $f(x) = \sqrt{x+4}$

   (e) $f(x) = \dfrac{1}{x-1}$

   (f) $f(x) = \sqrt{3-2x}$

3. If $f(x) = 2x + \sqrt{x+4}$, evaluate each of the following:

   (a) $f(-4)$

   (b) $f(5)$

4. Match each of the functions in column A with its corresponding domain in column *B*:

A	*B*
(i) $f(x) = \sqrt{2-x}$	(a) $\{x: x \geq 3\}$
(ii) $f(x) = 2x - 1$	(b) $\{x: x \leq 2\}$
(iii) $f(x) = \sqrt{x-3}$	(c) $\{x: x \in \mathbb{R}\}$

5. Match each of the functions in column A with its corresponding range in column *B*.

A	*B*
(i) $f(x) = \sqrt{2-x}$	(a) $\{y: y \geq 0\}$
(ii) $f(x) = 2x - 1$	(b) $\{y: y \in \mathbb{R}\}$
(iii) $f(x) = \sqrt{x-3}$	(c) $\{y: y \geq 10\}$

A functions whose range is a subset of the set of real numbers is called a *real valued* function. Two real valued functions f and g can be combined to form new functions $f+g, f-g, fg$, and $\dfrac{f}{g}$ in a manner similar to the way we add, subtract, multiply and divide real numbers. These are formally defined next.

Definition

Suppose f and g are two real valued functions. Let $\mathbf{S} = \text{dom}(f) \cap \text{dom}(g)$.

(i) **Sum of functions**: The sum of f and g, $f+g$, is a function given by $(f+g)(x) = f(x) + g(x)$, for every $x \in \mathbf{S}$.

(ii) **Difference of functions**: The difference of f and g, $f-g$, is a function given by $(f-g)(x) = f(x) - g(x)$, for each $x \in \mathbf{S}$.

(iii) **Product of functions**: The product of f and g, fg, is a function given by $(fg)(x) = f(x)g(x)$, for each $x \in \mathbf{S}$. (Here, $f(x)g(x) = f(x) \times g(x)$.)

(iv) **Quotient of functions**: The quotient of f and g, $\frac{f}{g}$, is a function given by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, for each $x \in \mathbf{S} \setminus \{x \mid g(x) = 0\}$.

Remark

The domain of $f+g$, $f-g$ and fg is dom $(f) \cap$ dom (g); and the domain of $\frac{f}{g}$ is dom $(f) \cap$ dom (g) excluding those x where $g(x) = 0$.

Example 23

Let $f(x) = 3x$ and $g(x) = x-1$. Note that $\text{dom}(f) \cap \text{dom}(g) = \mathbb{R}$.

(i) The sum of f and g is $(f+g)(x) = 3x + (x-1) = 4x - 1$, $\text{dom}(f+g) = \mathbb{R}$.

(ii) The difference of f and g is $(f-g)(x) = 3x - (x-1) = 2x + 1$, $\text{dom}(f-g) = \mathbb{R}$.

(iii) The product of f and g is $(fg)(x) = 3x(x-1) = 3x^2 - 3x$, $\text{dom}(fg) = \mathbb{R}$.

(iv) The quotient of f and g is $\left(\frac{f}{g}\right)(x) = \frac{3x}{x-1}$, $\text{dom}\left(\frac{f}{g}\right) = \mathbb{R} \setminus \{x \mid x-1=0\} = \mathbb{R} \setminus \{1\}$.

Example 24

Let $f(x) = 2x$ and $g(x) = \sqrt{x-5}$.

Find

(a) $f+g$ (b) $f-g$ (c) fg (d) $\frac{f}{g}$

and determine the domain of each function.

Solution

$\text{mod}(f) = \mathbb{R}$ and $\text{dom}(g) = \{x \geq 5\}$ since $\sqrt{x-5}$ is defined only when $x-5 \geq 0$.

So, dom $(f) \cap$ dom $(g) = \mathbb{R} \cap \{x \mid x \geq 5\} = \{x \mid x \geq 5\}$ which is domain of $f+g$, $f-g$, and fg.

Moreover,

(a) $(f+g)(x) = f(x) + g(x) = 2x + \sqrt{x-5}$, $\text{dom}(f+g) = \{x \mid x \geq 5\}$.

(b) $(f-g)(x) = f(x) - g(x) = 2x - \sqrt{x-5}$, dom$(f-g) = \{x \mid x \geq 5\}$.

(c) $(fg)(x) = f(x)g(x) = 2x\sqrt{x-5}$, dom$(fg) = \{x \mid x \geq 5\}$.

(d) $\left(\frac{f}{g}\right)(x) = \frac{2x}{\sqrt{x-5}}$, $\text{dom}\left(\frac{f}{g}\right) = (\text{dom}(f) \cap \text{dom}(g)) \setminus \{x \mid g(x) = 0\}$

$$= \{x \mid x \geq 5\} \setminus \{x \mid \sqrt{x-5} = 0\}$$
$$= \{x \mid x \geq 5\} \setminus \{x \mid x = 5\}$$
$$= \{x \mid x > 5\}.$$

Therefore, $\text{dom}\left(\frac{f}{g}\right) = \{x \mid x > 5\}$.

Example 25

Let $f(x) = 2 - 3x$ and $g(x) = x - 3$. Evaluate $\left(\frac{f}{g}\right)(4)$ and $(f+g)(4)$

Solution

$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{2-3x}{x-3}$. So, $\left(\frac{f}{g}\right)(4) = \frac{2-3(4)}{4-3} = -10$

$(f+g)(x) = f(x) + g(x) = -2x - 1$. So, $(f+g)(4) = -2(4) - 1 = -9$.

Example 26

Let $f(x) = x - 1$ and $g(x) = 3x$. Determine:

(a) $(2f+3g)(1)$

(b) $\frac{f}{2g}(3)$

Solution

(a) $(2f+3g)(1) = 2(1-1) + 3(3(1)) = 9$.

(b) $\frac{f}{2g}(3) = \frac{3-1}{2(3)(3)} = \frac{2}{18} = \frac{1}{9}$.

Exercises

1. If $f = \{(1, 2), (-3, 2), (2, 5)\}$ and $g = \{(2, 4), (1, 5), (3, 2)\}$. Find:
   (a) $f+g$ and $f-g$
   (b) the domain of $(f+g)$
2. Let $f = \{(2, 3), (4, 9), (3, -8)\}$ and $g = \{(1, 2), (2, 5), (3, 10), (4, 0)\}$. Determine:
   (a) $-2f$
   (b) $f+g$
   (c) $fg(2)$
   (d) g^2
   (e) $\frac{f}{g}$

3. Let $f(x) = \dfrac{2}{x-1}$ and $g(x) = \dfrac{2x-2}{3x+3}$. Find:
   (a) $f+g$ (b) fg
   (c) domain of $(f+g)$ and fg
4. Let $f(x) = 3x - 3$ and $g(x) = \dfrac{2}{x-1}$. Evaluate:
   (a) $2fg(2)$ (b) $\left(\dfrac{f}{g} - 2f\right)(3)$ (c) $(f-g)(4)$

In an equation, if a variable (symbol) is expressed in terms of others, the variable is called the subject of the equation. For instance, in the equation $y = 4x - 8$, y is the subject of the equation. Similarly, in the formula (equation) $I = \dfrac{V}{R}$, I is the subject of the formula.

Note that $I = \dfrac{V}{R}$ can be rearranged to express V in terms of I and R. That is, $V = IR$ for which V is the subject. Similarly, $y = 2x - 6$ can be rearranged to an equivalent equation $x = \dfrac{1}{2}y + 3$ for which x is the subject.

Given an equation, when a variable other than the current subject is required to be a subject, then we have to rearrange the equation to get the new subject. The task of making a new subject by rearranging a given equation is called change of subject.

In other words, making a given variable the subject of an equation means expressing that variable in terms of the others.

Example 27

Make x the subject of $y = \dfrac{1}{2}x - 1$.

Solution

We need to express x in terms y by rearranging the equation. To do this, first adding 1 to both sides of the given equation, we get

$$y + 1 = \frac{1}{2}x \quad \text{(Multiply this by 2)}$$

$$\Rightarrow 2(y+1) = x \quad \text{(Now, rearrange to get } x \text{ on left hand side (LHS))}$$

That is, $x = 2(y+1)$ (This is an equivalent equation in which x is the subject).

In general, a change of subject is simply an aspect that isolates a desired variable on one side of the equation. There are no new rules to do this. The same rules as were used for simple equations in Unit 6 are used; i.e., the balance of an equation must be maintained at each step**:** *whatever is done to one side of an equation must be done to the other side.*

We illustrate this using various examples below.

Example 28

Given that $3x + 5y = 30$,

(a) make y the subject and find y when $x = 5$.

(b) make x the subject and find x when $y = -3$.

Solution

(a) To make y the subject, we should express y in terms of x. That is, isolate y to the left side by rearranging the equation step by step using suitable rules of equation. Thus,

given $3x + 5y = 30$, (subtract $3x$ from both sides)

$5y = -3x + 30$, (divide both sides by 5)

$$\Rightarrow y = -\frac{3}{5}x + 6$$

So, y is the subject of the last equation.

Now, when $x = 5$, $y = -\frac{3}{5}(5) + 6 = -3 + 6 = 3$.

(b) To make x the subject, we should now express x in terms of y. That is, isolate x to the left side of the equation by taking every other terms to the right. Thus,

given $3x + 5y = 30$, (Subtracting $5y$ from both sides to get);

$3x = -5y + 30$, (dividing both sides by 3 to get);

$$x = -\frac{5}{3}y + 10$$

Thus, x is the subject of the last equation.

Now, when $y = -3$, $x = -\frac{5}{3}(-3) + 10 = 5 + 10 = 15$.

Example 29

Make x the subject of the following relations(equations).

1. $b = 5y + x$
2. $c = \frac{x}{5} - a$
3. $y + ax = bx + c$

Solution

In each, we rearrange the given equation to isolate x to the left hand side (LHS)

1. Given $b = 5y + x$ (Subtract $5y$ from both sides)

   $\Rightarrow b - 5y = x$ (rearrange to get x on LHS)

   Therefore, $x = b - 5y$.

2. Given $c = \frac{x}{5} - a$ (add a to both sides)

   $\Rightarrow c + a = \frac{x}{5}$ (multiply both sides by 5)

   $\Rightarrow 5(c + a) = x$ (rearrange to get x on LHS)

   Therefore, $x = 5(c + a)$

3. Given $y + ax = bx + c$ (take bx to the left and y to the right - you can specify the reason)

   $\Rightarrow ax - bx = c - y$

   $\Rightarrow (a - b)x = c - y$ (divide both sides by $a - b$)

   $\Rightarrow x = \frac{c - y}{a - b}$

Example 30

Make t the subject of the relation $x = \frac{at + b}{c}$.

Solution

Multiplying both sides of the given equation by c, we get

$cx - b = at$

$\Rightarrow cx - b = at$

$\Rightarrow \frac{cx - b}{a} = t$

Therefore, $t = \frac{cx - b}{a}$

Example 31

If $p = \frac{q - m}{1 + mq}$, express q in terms of p.

Solution

Multiplying both sides of the given equation by $1 + mq$, we get

$$p(1 + mq) = q - m$$

$$\Rightarrow p + mpq = q - m$$

$$\Rightarrow p + m = q - mpq$$

$$\Rightarrow p + m = (1 - mp)q$$

Therefore, $q = \dfrac{p+m}{1-mp}$

Example 32

If $a = \dfrac{2}{b} - \dfrac{1}{c}$, find an expression for c in terms of a and b.

Solution

Multiplying both sides of the given equation by bc, we get

$abc = 2c - b$ (because, at the right side, $\dfrac{2}{b}(bc) - \dfrac{1}{c}(bc) = 2c - b$)

$\Rightarrow b = 2c - abc$ (Give the reason)

$\Rightarrow b = c(2 - ab)$

Therefore, $c = \dfrac{b}{2-ab}$

Example 33

Make z the subject of $r = m\sqrt{\dfrac{z}{n}}$.

Solution

Dividing both sides of the given equation by m, we get

$\dfrac{r}{m} = \sqrt{\dfrac{z}{n}}$ (square both sides)

$$\Rightarrow \frac{r^2}{m^2} = \frac{z}{n}$$

Therefore, $z = \dfrac{nr^2}{m^2}$

Exercises

1. Make z the subject of the following relations (equations).

   (a) $y=\frac{a}{b}z+1$ (b) $y=\frac{3z+1}{5+z}$ (c) $y=\frac{az+b}{c+4z}$

2. Make x the subject of the following relations

   (a) $m(3+x)=n(x+1)$ (c) $\frac{1}{x}+\frac{1}{y}=\frac{1}{a}$

   (b) $\frac{x-1}{a}+\frac{y}{b}=1$

3. Make x the subject of the following relations

   (a) $m(3+x)=n(x+1)$ (c) $\frac{1}{x}+\frac{1}{y}=\frac{1}{a}$

   (b) $\frac{x-1}{a}+\frac{y}{b}=1$

4. If $t=\sqrt{\frac{10-x}{10+x}}$, then

   (a) find x in terms of t.

   (b) find the value of x when $t=3$.

5. For a lens, the focal length f, the object distance u and the image distance v are related by the formula: $\frac{1}{f}=\frac{1}{u}+\frac{1}{v}$.

   (a) Make f the subject of the equation

   (b) Find f when $u=30$ cm and $v=20$ cm.

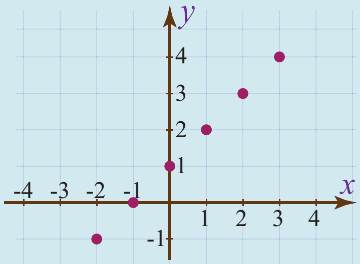
A graph of function is a visual representation of elements of the function on the coordinate system. In this section, we will consider plotting elements (ordered pairs) of a special relation, called linear function.

Activity 2

1. Suppose a relation R is the set of ordered pairs of the points shown by the dots on the adjacent xy-coordinate system.

   (a) List the elements of R by the roster (complete listing) method.

   (b) Describe the elements of R by the set builder method.



2. Plot the following relations (set of ordered pairs) in the *xy*-coordinate plane.
   (i.e., show the points corresponding to the ordered pairs by dots (•) or crosses(×).)
   (a) $R_1 = \{(-1, -2), (0, 0), (1, 2), (2,4), (3, 6)\}$
   (b) $R_2 = \{(x, y) \mid y = 2x + 1, x \in A\}$ where $A = \{-2, -1, ..., 2\}$
3. Let $R = A \times B$, where $A = \{x \mid x \in \mathbb{R}, 0 \le x \le 2\}$ and $B = \{2\}$. i.e., $R = \{(x, 2) \mid x \in A\}$.
   (a) List as many elements of R as you can (at least five) and plot them on the xy-coordinate system.
   (b) Observe that R has infinitely many elements. What do you get if you try to plot all elements of R?
4. Let R be a relation on $A = \{x \mid x \in \mathbb{R}, 0 \le x \le 2\}$ given by $R = \{(x, y) \mid y = x\}$.
   (a) List as many elements of R as you can (at least five) and plot them on the Cartesian coordinate plane.
   (b) What do you get if you try to plot all elements of R?

As you have seen in the above activity, the graph of a function (relation) on the *xy*- coordinate plane is the set of points on the plane whose coordinate pairs are the ordered pairs of the functions. In this section, we will focus on the graph a linear function defined below.

Definition

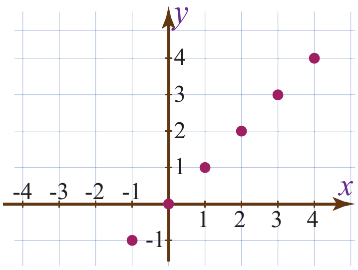
If a and b are fixed real numbers, $a \neq 0$, then $f(x) = ax + b$ for $x \in \mathbb{R}$ is called a linear function.

Sometimes linear functions are written as $y = ax + b$ because, as a set of ordered pairs, the linear function can be written as

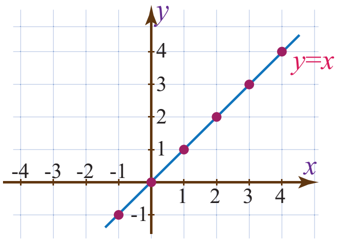
- $f = \{(x, y) \mid y = ax + b, x \in \mathbb{R}\}$.
- If $a = 0$, $f(x) = b$ is called a constant function.

Graphing $f(x) = ax + b$ means plotting, on the *xy*-coordinate plane, all coordinate pairs who are the ordered pairs of $f = \{(x, y) \mid y = ax + b, x \in \mathbb{R}\}$.

For instance, suppose $f(x) = x$. In order to graph f, we need to find some ordered pairs in $f = \{(x, y) \mid y = x, x \in \mathbb{R}\} = \{(x, x) \mid x \in \mathbb{R}\}$.



Some of these are $(-1, -1)$, $(-2, -2)$, $(0, 0)$, $(1, 1)$, $(2, 2)$, and $(3, 3)$. We plot these points (ordered pairs) as shown next.



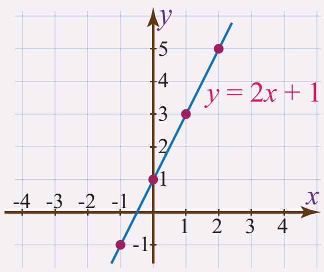
But this is not the graph of f as, for example, $\left(\frac{1}{4},\frac{1}{4}\right)$, $\left(\frac{1}{2},\frac{1}{2}\right)$, $\left(\frac{3}{2},\frac{3}{2}\right)$, etc., are in f but missed on the above graph. In fact, as the elements of f are infinitely many, it is impossible to list and plot all elements of f. However, the points that are missed on the graph can be covered by drawing a line segment through the plotted points. Then, use your imagination to extend the line segment straight in both directions. This is shown on the next graph.

In general, to graph a function f, we plot some sample points of $\{(x, y)| \ y = f(x)\}$ and then join the plotted points by a smooth curve which suggests the shape of the graph. This is a straight line when f is a linear function.

Example 34

Draw the graph of $f(x) = 2x + 1$

Solution



To get sample points, we first form table of ordered pairs (x, y), where $y = f(x) = 2x + 1$ for some x, say, when $x = -2, -1, 0, 1, 2$.

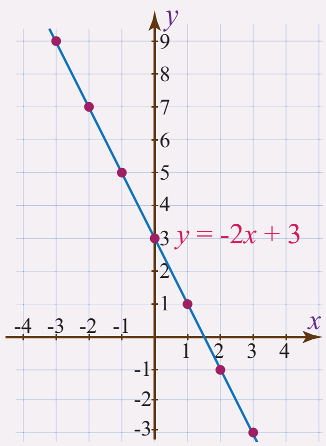
x	-2	-1	0	1	2
$y = 2x + 1$	-3	-1	1	3	5

Thus, $(-2,-3)$, $(-1, -1)$, $(0, 1)$, $(1, 3)$, and $(2, 5)$ are samples points of the graph. That is, the straight line through these points is the graph f as shown in the figure below.

Example 35

Sketch the graph of $f(x) = -2x - 3$

Solution



First construct a table of values from the domain to get sample points:

x	-3	-2	-1	0	1	2	3
$f(x)$	9	7	5	3	1	-1	-3

This table is pairing the values of x and $f(x)$. i.e., it gives us the sample points:

$(-3, 9)$, $(-2, 7)$, $(-1, 5)$, $(0, 3)$, $(1, 1)$, $(2, -1)$, $(3, -3)\}$

Now you can plot these points in a coordinate system and the straight line through these points as shown in the figure below.

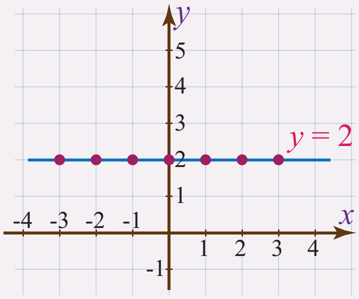
Example 36

Draw the graph of the constant function $f(x) = 2$.

Solution

Construct a table of values of the function, plot the ordered pair and draw a line through the points to get the required graph as shown below.

x	–3	–2	–1	0	1	2	3
$f(x)$	2	2	2	2	2	2	2



Indeed,, the graph of $f(x) =2$ is the set of all points of $(x, 2)$, for every $x \in \mathbb{R}$. This is the horizontal line through (0, 2) as shown on the adjacent graph.

Note: The graph of a constant function $f(x) = b$ is a horizontal line (parallel to x-axis) that intersects the y-axis at $(0, b)$.

Exercises

Draw the graph of the following functions

(a) $f(x) = 3$

(b) $f(x) = 3x - 2$

(c) $f(x) = -2x + 3$

(d) $f(x) = -2$

(e) $f(x) = -x + 2$

(f) $f(x) = 4x - 3$

(g) $f(x) = \frac{1}{2}x$

(h) $f(x) = \frac{1}{2}x + 3$

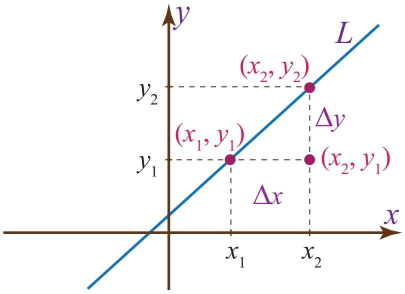
(i) $f(x) = -\frac{1}{2}x + 4$

In the previous section, we have seen that the graph of a linear function $f(x) = ax+b$ is a straight line which is inclined with certain sort of steepness. A measure of this steepness is the property called gradient or slope. In this section, we will discuss a way of finding the gradient of a line.

Note: In the sequel, we may use the term 'slope' for gradient, a' line' for straight line, and the symbol Δ *(read 'delta')* means "*the change in*". For instance, 'Δx' means *'the change in x'*.

From geometry, you know that two distinct points in a plane determine a unique line. So, any two distinct points (x_1, y_1) and (x_2, y_2) that lies on a line can be used to describe the line and to define the slope of the line.

For any two distinct points (x_1, y_1) and (x_2, y_2) on a line, we define



$\Delta x = x_2 - x_1$ (change in the values (coordinates) of x)

$\Delta y = y_2 - y_1$ (change in the values (coordinates) of y)

The slope, denoted by m, of a line L that passes through point (x_1, y_1) and (x_2, y_2) is defined by

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 37

Find the slope of the line that passes through each of the following pair of points, and draw the line.

(a) (1, 3) and (2, 5)

(b) (0, 3) and (2, 1)

(c) (−1, 3) and (2, 3)

(d) (2, 1) and (2,4)

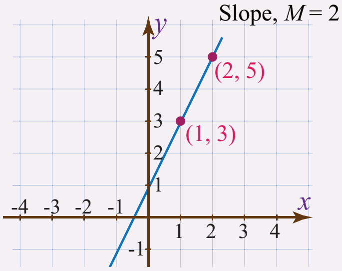
Solution

The slope of the line through a given pair of points (x_1, y_1) and (x_2, y_2). is $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$. You can draw the line that passes through (x_1, y_1) and (x_2, y_2) using a ruler or a straight edge. So,

(a) The slope of the line through (1, 3) and (2, 5) is

$$m = \frac{\Delta y}{\Delta x} = \frac{5-3}{2-1} = \frac{2}{1} = 2$$

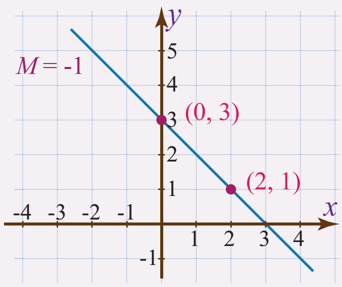
The adjacent graph shows the line .



(b) The slope of the line through (0, 3) and (2, 1) is

$$m = \frac{\Delta y}{\Delta x} = \frac{1-3}{2-0} = \frac{-2}{2} = -1$$

The adjacent graph shows the line .

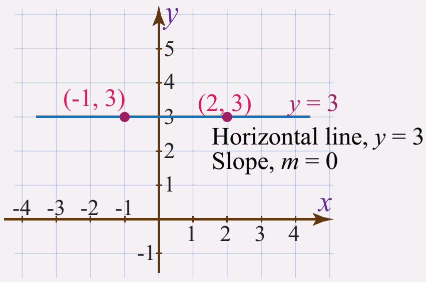


(c) The slope of the line through (−1, 3) and (2, 3) is

$$m=\frac{\Delta y}{\Delta x}=\frac{3-3}{2-(-1)}=\frac{0}{3}=0$$

The line is horizontal (parallel to the x-axis)

In general, the **slope of a horizontal line** is 0 because $\Delta y = 0$.

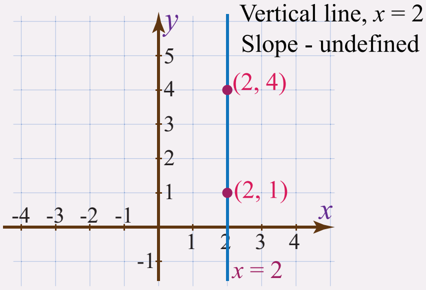


(d) The slope of the line through (2,1) and (2,4) is undefined because

$$m=\frac{\Delta y}{\Delta x}=\frac{4-1}{2-2}=\frac{3}{0}\ \text{(undefined)}$$

The line is vertical (parallel to the y-axis)

In general, the **slope of a vertical line** is undefined, because $\Delta x = 0$.



Note

1. The slope of any horizontal line is **0**; because its Dy is 0.
2. The slope of any vertical line is undefined; because its Dx is 0.
3. The slope of an inclined line is either a *positive number* or a *negative number*
   - If its slope $m > 0$, the line is rising up (from 3rd quadrant to1st quadrant),
   - If its slope $m < 0$, the line is falling (from 2nd quadrant toward 4th quadrant).

Equation of straight lines

In the previous section, you have seen that the graph of a linear function $f(x) = ax + b$ is a straight line L. The ordered pairs satisfying the equation of this linear function (i.e., ordered pairs of points on L) are (x, y) such that $y = ax + b$ for every $x \in \mathbb{R}$.

Thus, $y = ax + b$ is called equation of the line L.

What is the slope of this line? To answer this, we choose any two points (x_1, y_1) and (x_2, y_2) on L and compute its slope. For instance,

When $x = 0$, $y = f(0) = b \Rightarrow (0, b)$ is on L. So, let $(x_1, y_1) = (0, b)$

When $x = 1$, $y = f(1) = a + b \subseteq (0, a + b)$ is on L. So, let $(x_2, y_2) = (1, a + b)$

Therefore, the slope of the line is $m = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{(a+b)-b}{1-0} = a$.

Consequently, the equation of the line is usually written as

$y = mx + b$, where m is the slope of the line.

The line intersects with the y-axis at $(0, b)$ because $(0, b)$ is on the y-axis and also on the line. We say that $(0, b)$ is the ***y*-intercept** of the line.

Note also that, when $y = 0$, $mx + b = 0$ implies $x = -\dfrac{b}{m}$, for $m \neq 0$. So, $\left(-\dfrac{b}{m}, 0\right)$ is on the line and it is also on the x-axis. We say that $\left(-\dfrac{b}{m}, 0\right)$ is the ***x*-intercept** of the line.

Summary

$y - mx + b$ is an equation of the line with

slope m, y-intercept $(0, b)$ and x-intercept $\left(-\dfrac{b}{m}, 0\right)$.

- To draw this line, you can use the following two points of the line:
  1. $(0, b)$ and $(1, m+b)$. (i.e., y-intercept and point obtained using the slope m) or
  2. $(0, b)$ and $\left(-\dfrac{b}{m}, 0\right)$, if $b \neq 0$. (i.e., y- and x-intercepts if they are not $(0, 0)$)

Example 38

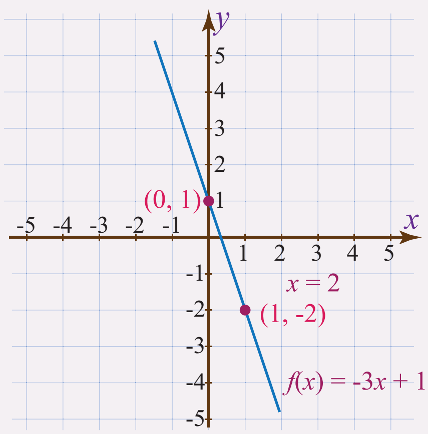
Draw the line given by $y = -3x + 1$ (i.e., graph of $f(x) = -3x + 1$)

Solution

- The y-intercept of the line, when $x = 0$, is $(0, 1)$.
- And using the slope of the line, $m = -3$, the second point on the line is $(1, -3+1) = (1, -2)$.

So, the line passes through $(0, 1)$ and $(1, -2)$.

The line is shown on the adjacent figure.



Example 39

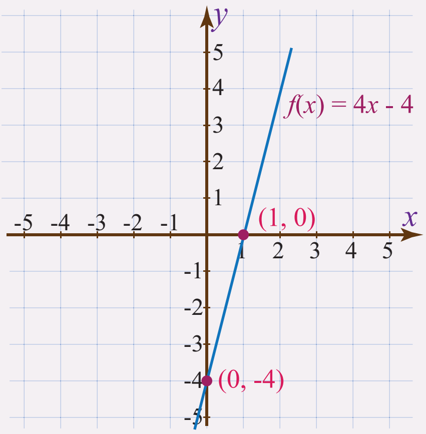
Draw the line given by $y = 4x - 4$.
(i.e., graph of $f(x) = 4x - 4$)

Solution

Here let us use the two intercepts.

- The y-intercept, when $x = 0$, is $(0, -4)$.
- The x-intercept, when $y = 0$, is $(1, 0)$.

So, the line passes through $(0, -4)$ and $(1, 0)$.
The line is shown on the adjacent figure.



The general form of equation of a line

We have seen that an equation in two variables x and y given by $y = mx + c$ is an equation of a line. The equation $y = mx + c$ is called *slope-intercept from* of equation of the line. This equation may appear in other equivalent forms, such as, $-mx + y = c$ or $mx - y + c = 0$. In general, equation of a line can be written as

$$ax + by = c,$$

where a and b are not both 0. This is called the *general form* of equation of a line. Equation of any type of a line can be obtained from this general form. In particular, note the followings.

(i) If $a = 0$, then $by = c \Rightarrow y = \frac{c}{b}$. This is equation of a horizontal line with y-intercept $\left(0, \frac{c}{b}\right)$.

(ii) If $b = 0$, then $ax = c \Rightarrow x = \frac{c}{a}$. This is equation of a vertical line with x-intercept $\left(\frac{c}{a}, 0\right)$.

(iii) Suppose $a \neq 0$ and $b \neq 0$. Then, we can change $ax + by = c$ to the slope-intercept form by making y the subject of the equation. That is, $ax + by = c$ is equivalent to $y = -\frac{a}{b}x + \frac{c}{b}$ (which is equation of a line with slope $m = -\frac{a}{b}$ and y-intercept $\left(0, \frac{c}{b}\right)$).

Example 40

Suppose the lines L_1 and L_2 are given by the equations $x + y = 5$ and $-2x + y = -1$ respectively.

(a) Find the slope and y-intercept of each line.

(b) Draw both lines on the same coordinate plane.

(c) Find the intersection point of the lines.

Solution

(a) Changing the equations into the slope-intercept form, we get that

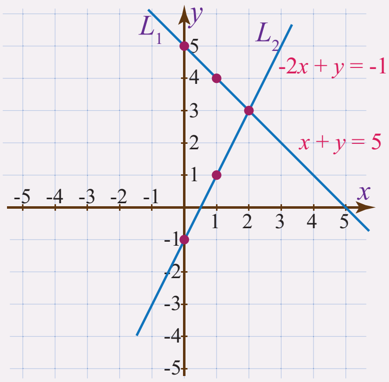
L_1: $y = -x + 5$, slope $m_1 = -1$, y-intercept (0, 5)

L_2: $y = 2x - 1$,. slope $m_2 = 2$, y-intercept (0, –1)

(b) So, L_1 passes through (0, 5) and (1, 4), and

L_2 passes through (0, –1) and (1, 1).

The lines are shown on the adjacent figure.



(c) Observe that, (2, 3) is the intersection of L_1 and L_2.

Example 41

Suppose the lines L_1 and L_2 are given by the equations $-x + y = 1$ and $x - y = 1$ respectively.

(a) Find the slope and y-intercept of each line.

(b) Draw both lines on the same coordinate plane.

(c) Find the intersection point of the lines, if any.

Solution

(a) Changing the equations into the slope-intercept form, we get that

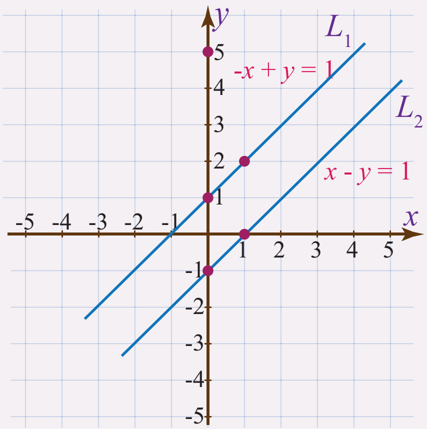
L_1: $y = x + 1$, slope $m = 1$, y-intercept (0, 1)

L_2: $y = x - 1$, slope $m = 1$, y-intercept (0, –1)

(b) So, L_1 passes through (0, 1) and (1, 2), and

L_2 passes through (0, –1) and (1, 0).

The lines are shown on the adjacent figure.



(c) L_1 and L_2 have the same steepness (have equal slope). Hence, L_1 and L_2 are parallel.

Therefore, the lines have no intersection point.

Remark :

1. Two lines with equal slopes are parallel. Distinct parallel lines never intersect. Therefore, two distinct lines with equal slopes have **no** intersection point.
2. Two lines with different slopes intersect at one and only one point.

Exercises

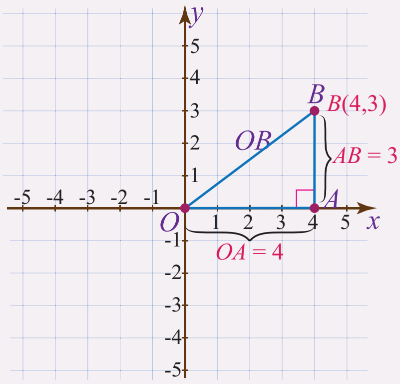
1. Find the gradient (slope) of the line that passes through each of the following pair of points and draw the line.
   (a) (0, 2) and (3, 2)
   (b) The origin and (1, 2)
   (c) (–1, 3) and (1, 6)
   (d) (2, 0) and (2, 3)
   (e) The origin and (1, –3)
   (f) (–3, –2) and (2, –2)
2. Determine the slope, y-intercept and x-intercept of the line determined by each of the following equations. Also draw the lines.
   (a) $x + y - 1 = 0$
   (b) $y = 3x - 4$
   (c) $3x - 2 = 2y$
   (d) $y - 3 = x$
   (e) $y - 5 = 3x$
   (f) $y - 2x + 5 = 1$
3. Determine whether the lines given by each of the following pair of equations are parallel or not.
   (a) $3y - 3x - 5 = 4;\ -x + y = 4$
   (b) $y = 4x + 2;\ 8x - 2y + 5 = 0$
   (c) $y - 3x - 5 = 0;\ 3x + y = 5$
   (d) $3y = -2x + 3;\ 4x + 6y + 3 = 0$
   (e) $x + 5y + 3 = 0\ ;\ 5x + 3y + 2 = 0$
   (f) $\frac{x}{3} + \frac{y}{2} = 1\ ;\ 2x + 3y - 4 = 0$

The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is the length of the line segment PQ. In this section, we will develop a formula for the distance between any two points in the coordinate system. Before considering this for arbitrary points, we will do it for particular points in the following example.

Example 42

Find the distance between $O(0, 0)$ and $B(3, 4)$.

Solution



Let us plot $O(0, 0)$, $B(4, 3)$ and form the right angled triangle OAB where A is at $(4, 0)$, as shown in the adjacent figure.

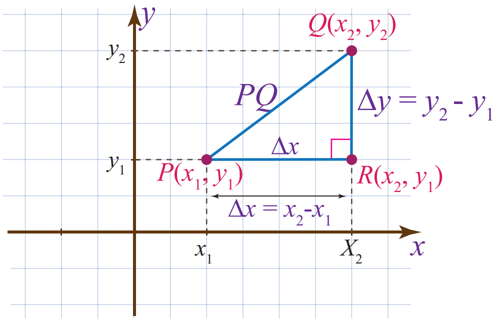
Consequently, OB is the hypotenuse of ΔOAB while OA and AB are the lengths of the horizontal and vertical sides of the triangle such that $OA = 4$ and $AB = 3$.

Hence, by Pythagoras' theorem, $OB = \sqrt{4^2 + 3^2} = 5$.

That is, the distance between (0, 0) and (4, 3) is 5.

We will follow the same approach as discussed in the above example and use the Pythagoras' theorem to develop a formula for the distance between any two points. This is described below

Suppose two points P and Q have coordinates (x_1, y_1) and (x_2, y_2), respectively.



To find the distance PQ between the two points, we plot the two points and form the right angled triangle PRQ, where the coordinates of R is (x_2, y_1), as shown in the adjacent figure. Then we can see that the lengths of the sides of the triangle are

$PR = \Delta x = x_2 - x_1$, and

$RQ = \Delta y = y_2 - y_1$.

Then, using Pythagoras' theorem,

$$\Rightarrow \text{PQ} = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Therefore, the distance d between points (x_1, y_1) and (x_2, y_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This is known as the **distance formula**. Although we have derived the formula for points in the quadrant I, the formula is true for any location of the points.

Example 43

Find the distance between (–5, –4) and (3, 2).

Solution.

$\Delta x = x_2 - x_1 = 3 - (-5) = 8$; and $\Delta y = y_2 - y_1 = 2 - (-4) = 6$.

Hence, the distance between the two points is $d = \sqrt{8^2 + 6^2} = \sqrt{100} = 10$

Example 44

Suppose the vertices of ΔABC are at A (–2, –1), B(1, 4) and C(4, –1). Show that the triangle is isosceles.

Solution

To show ΔABC is isosceles, we need to show that two of its sides are equal.

So, to find side AB, its

$\Delta x = 1 - (-2) = 3$, and

$\Delta y = 4 - (-1) = 5$.

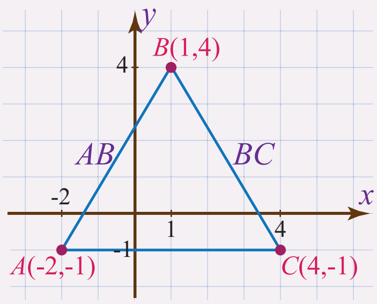
Thus, $AB = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{3^2 + 5^2} = \sqrt{34}$

Next, to find side BC, its

$\Delta x = 4 - 1 = 3$, and

$\Delta y = -1 - 4 = -5$.

Thus, $BC = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{3^2 + (-5)^2} = \sqrt{34}$



Therefore, $AB = \sqrt{34} = BC$ implies that ΔABC is isosceles.

Exercises

1. Given A(1, 0), B (7, 8), C (–2, –3), D(3, 5), E(3, –4) and F(–3, –4), find the distance between
   - (a) A and B
   - (b) B and C
   - (c) C and D
   - (d) D and E
   - (e) E and F
   - (f) F and A
   - (g) B and F
   - (h) C and F
2. Show that a triangle with vertices at (–2, 0), (–1, 4) and (2, 1) is isosceles.
3. Suppose the vertices of ΔABC are at A(–1, 1), B(1, 5) and C(5, 3). Show that
   - (a) ΔABC is isosceles.
   - (b) ΔABC is right angled triangle. (Hint: Show that its sides satisfy Pythagoras' theorem)

In section 7.6, we have discussed linear functions and their graphs. In this section, we will discuss about quadratic functions and their graphs.

Definition

A function defined by $f(x) = ax^2 + bx + c$, where $a, b, c \in R$ and $a \neq 0$, is called a quadratic function.

- a is called the leading coefficient, and c is called the constant term.
- The domain of a quadratic function is the set of all real numbers.

Example 45

$f(x) = 2x^2 - 3x + 2$ is a quadratic function with $a = 2$, $b = -3$, and $c = 2$.

Note: Any function that can be reduced to the form $f(x) = ax^2 + bx + c$ is a quadratic function.

Example 46

(a) $f(x) = (x - 3)^2 + 5$ can be expressed as $f(x) = x^2 - 6x + 14$

So, $f(x) = (x - 3)^2 + 5$ is a quadratic function with $a = 1$, $b = -6$, and $c = 14$.

(b) $f(x) = (x - 2)(x + 2)$ can be expressed as $f(x) = x^2 - 4$.

So, $f(x) = (x - 2)(x + 2)$ is a quadratic function with $a = 1$, $b = 0$, and $c = -4$.

Let us now draw graphs of a simple quadratic function by constructing tables of values.

Activity 3

1. Construct a table of values for each of the following quadratic functions, for $x = -2, -1, 0, 1, 2$.

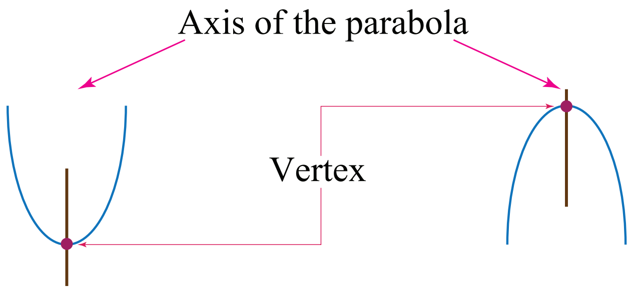
   (a) $f(x) = x^2$ (b) $f(x) = x^2 + 2$ (c) $f(x) = -x^2 + 5$
2. Using the tables in Question 1a plot the points (x, x^2) on xy-coordinate systems. Connect those points by smooth curves (not straight line segments).
3. Discuss the type of graphs you obtained.

The graph of a quadratic function is a curve known as **parabola.**

Before a further discussion on quadratic functions, let us describe some properties of a parabola which are helpful for graphing quadratic functions.

Some properties of a parabola

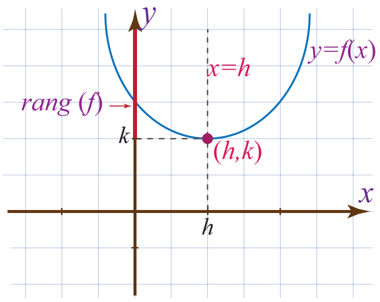
As a graph of a function, a parabola is a smooth curve which opens upward or opens downward as shown in the figure below. The turning point on the parabola is called the **vertex** of the parabola. The vertical line through the vertex is called **axis** of the parabola. Axis of a parabola is its line of symmetry. That is, the portions of the curve that lies ‘to the left side’ and ‘to the right side’ of the axis are mirror images of one another through the axis. The parabola intersects its axis at its vertex.



a. Parabola, opens upward *b. Parabola, opens Downward*

The following features of parabolas are very helpful in graphing a quadratic functions and to determine their ranges.

1. Suppose the graph of $f(x)$ is the parabola shown on the adjacent figure whose vertex is (h, k) and opens upward. Then, from the graph you can see the followings.



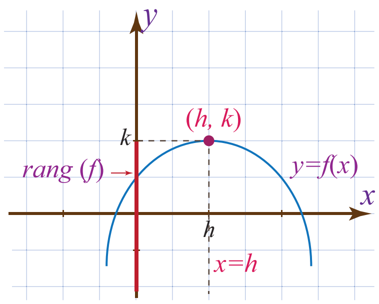
   (a) $y = f(x) \geq k$ for all x . This implies that

   $\text{rang}(f) = \{ y \in \mathbb{R} \mid y \geq k \}$

   That is, k is the minimum value of f which occurs when $x = h$.

   (b) The vertical line $x = h$ is the axis of the parabola.

2. Suppose the graph of $f(x)$ is the parabola shown on the adjacent graph whose **vertex** is (h, k) and **opens downward**. Then, from the graph you can see the followings.



   (a) $y = f(x) \leq k$ for all x. This implies that

   $\text{rang}(f) = \{ y \in \mathbb{R} \mid y \leq k \}$

That is, k is the maximum value of f which occurs when $x = h$.

(b) The vertical line $x = h$ is the axis of the parabola.

Note also that if a vertical line $x = h$ is the line of symmetry for the graph (parabola) of f then the values of f at r' units to the left from h' and 'r units to the right from h' are equal. i.e.,

A vertical line $x = h$ is the line of symmetry for the parabola (graph) of f if and only if $f(h - r) = f(h + r)$ for every $r \in \mathbb{R}$.

Now we return to the discussion of the graph of quadratic functions

Graph of quadratic functions

As discussed earlier, the graph of a quadratic function is a parabola. In order to sketch the graph of a parabola, it is important to know its *line of symmetry*, *vertex* and *the direction toward which it opens*. We will determine these and sketch the desired graph starting with a simple quadratic function $f(x) = ax^2$. You will see that the graph of any other quadratic function can be obtained from the graph of this simple quadratic function.

I. Consider $f(x) = ax^2$

Since $f(-r) = f(r)$, for every $r \in \mathbb{R}$, its *line of symmetry* is the vertical line $x = 0$ (i.e., y-axis) . Moreover,

$f(-1) = a$, $f(0) = 0$ and $f(1) = a$ imply that the parabola (graph) of f passes through

$(-1, a)$, $(0, 0)$ and $(1, a)$.

So, the graph of f is constructed by drawing a smooth curve (parabola) that passes through these three points. Note that (0,0) is on both the graph and on the line of symmetry. Therefore, the *vertex* is (0,0).

Now it remains to determine whether the parabola opens upward or down ward. This depends on whether a is positive or negative. In particular,

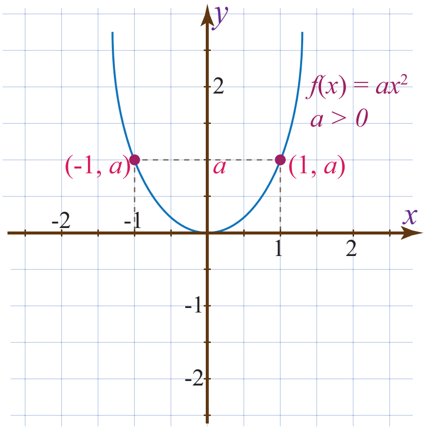
If $a > 0$, the parabola *opens upward* since $(-1, a)$ and $(1, a)$ are above the x-axis.

Indeed, $x^2 \geq 0$ and $a > 0$ implies $ax^2 \geq 0$ for all x. So, rang(f) = $\{y \in \mathbb{R} \mid y \geq 0\}$.

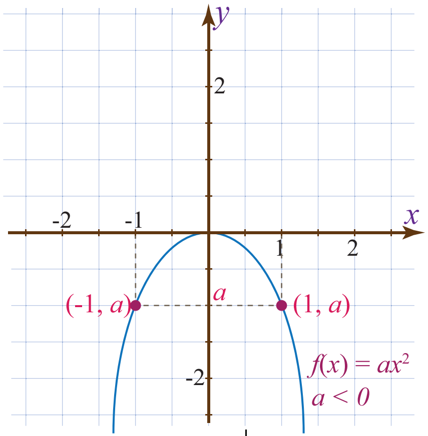
If $a < 0$, the parabola *opens downward* since $(-1, a)$ and $(1, a)$ are below the x-axis.

Indeed, $x^2 \geq 0$ and $a < 0$ implies $ax^2 \leq 0$ for all x. So, rang(f) = $\{y \in \mathbb{R} \mid y \leq 0\}$.

Hence, the graph of $f(x) = ax^2$ is a parabola with line of symmetry $x = 0$, vertex (0, 0) which opens upward when $a > 0$ and opens downward when $a < 0$, as shown in the figure below.



$rang\ (f) = \{y \in \mathbb{R} \mid y \geq 0\}$



$rang\ (f) = \{y \in \mathbb{R} \mid y \leq 0\}$

Three points are enough to approximate the shape of a parabola when one of the points is its vertex. However, it is helpful to have some more points that lie on the parabola in order to get better approximation of the shape of the parabola. The points can be obtained using table of values of the function.

Example 47

Draw the graphs of

(a) $f(x) = x^2$.

(b) $f(x) = 2x^2$.

Solution

(a) $f(x) = x^2$ with $a = 1$. The parabola passes through (–1, 1), (0, 0), and (1, 1). Its *line of symmetry* is $x = 0$, *vertex* (0, 0) and *opens upward*. The three points determine the parabola. However, to get better approximation of the graph, we include more points on the graph using the following table of values:

x	–2	–1	0	1	2
$f(x) = x^2$	4	1	0	1	4

From the table, (–2, 4) and (2, 4) are also on the graph. The graph is in the figure (a) below.

(b) (b) $f(x) = 2x^2$ with $a = 2$. The parabola passes through (–1, 2), (0, 0), and (1, 2). Its *line of symmetry* is $x = 0$, *vertex* (0, 0) and *opens upward*. For a better approximation of the graph, you can include two more points obtained by evaluating the function, say, at $x = -2$, and $x = 2$. The evaluation yields that the graph passes also through (–2, 8) and (2, 8). The graph is shown in the figure (b) below

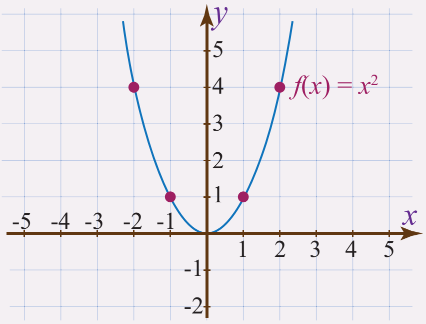


Figure (*a*)

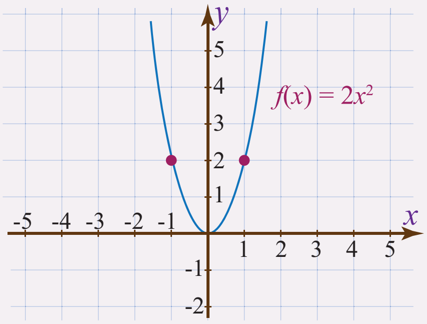


Figure (*b*)

Example 48

Draw the graphs of

(a) $f(x) = -x^2$. (b) $f(x) = -2x^2$.

Solution

(a) $f(x) = -x^2$ with $a = -1$. The parabola passes through (–1, –1), (0, 0), and (1, –1). Its *line of symmetry* is $x = 0$, *vertex* (0, 0) and *opens downward*. The graph is in figure (a) below.

(As $f(-2) = -4$ and $f(2) = 4$; the graph passes also through (–2, –4) and (2, –4).)

(b) $f(x) = -2x^2$ with $a = -2$. The parabola passes through (–1, –2), (0, 0), and (1, –2). Its *line of symmetry* is $x = 0$, *vertex* (0, 0) and *opens downward*. The graph is in figure (b) below.

(As $f(-2) = -8$ and $f(2) = -8$; the graph passes also through (–2, –8) and (2, –8).)

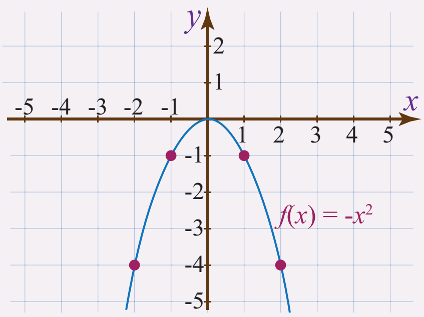


Figure (*a*)

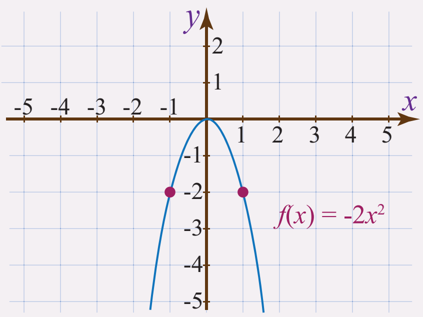


Figure (*b*)

II. Next, we consider the graph of $f(x) = ax^2 + c$

Since $f(-r) = ar^2 + c = f(r)$, for every $r \in \mathbb{R}$, its ***line of symmetry*** is the vertical line $x=0$ (i.e., y-axis). Moreover,

$f(-1) = a + c$, $f(0) = c$ and $f(1) = a+c$ implies that the graph of f passes through $(-1, a + c)$, $(0, c)$ and $(1, a + c)$ with the ***vertex*** at $(0, c)$.

Note that if $c > 0$, the graph of $f(x) = ax^2 + c$ is just the graph $f(x) = ax^2$ shifted c units vertically upward, because each y-coordinate is increased by the same number c. Likewise, if $c < 0$, the graph of $f(x) = ax^2 + c$ is just the graph $f(x) = ax^2$ shifted $|c|$ units vertically downward because each y-coordinate is decreased by the same number $|c|$.

Summary

(**Vertical shift** of the graph of $y = ax^2$ to obtain the graph of $f(x) = ax^2 + c$). The graph of $f(x) = ax^2 + c$ can be obtained by *vertically shifting* the graph of $y = ax^2$ by c units upward if $c > 0$ or $|c|$ units downward if $c < 0$.

In particular, the parabola (graph) of f

- passes through $(-1, a + c)$, $(0, c)$ and $(1, a + c)$.
- Its *line of symmetry* is $x = 0$ (the y-axis), its *vertex* is $(0, c)$; and
- *opens upward* if $a > 0$ (So, rang$(f) = \{y \in \mathbb{R} \mid y \geq c\}$, minimum value is c at $x = 0$)
- *opens downward* if $a < 0$ (So, rang$(f) = \{y \in \mathbb{R} \mid y \leq c\}$, maximum value is c at $x = 0$)

Example 49

Sketch the graphs of

(a) $f(x) = 2x^2 + 3$

(b) $f(x) = 2x^2 - 3$

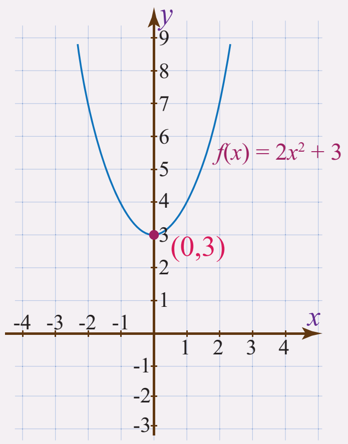
Solution

(a) The graph of $f(x) = 2x^2 + 3$ is obtained by vertically shifting the graph of $y = 2x^2$ by 3 units upward so that its *vertex* is $(0, 3)$ and its *line of symmetry* is $x = 0$, and *opens upward* (so, rang$(f) = \{y \in \mathbb{R} \mid y \geq 3\}$).

Moreover, the graph passes through

$(-1, 5)$, $(0, 3)$ and $(1,5)$.

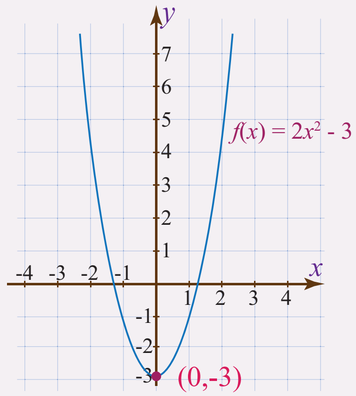
The graph is shown on the adjacent figure.



(b) The graph of $f(x) = 2x^2 - 3$ is obtained by vertically shifting the graph of $y = 2x^2$ by 3 units downward so that its *vertex* is $(0, -3)$, *line of symmetry* $x = 0$, and opens upward (so, $\text{rang}(f) = \{y \in \mathbb{R} \mid y \geq -3\}$) Moreover, the graph passes through

$(-1, -1)$, $(0, -3)$ and $(1, 1)$

The graph is shown on the adjacent figure.



III. Next, we consider the graphs of $f(x) = a(x - h)^2$ when $h \in \mathbb{R}$ (can be positive or negative)

You can show that, $f(h - r) = f(h + r)$, for all $r \in \mathbb{R}$. That is the line of symmetry is $x = h$ (Note that the line of symmetry is shifted from the y-axis horizontally by the distance of h units to the right if $h > 0$ and $|h|$ units to the left if $h < 0$.).

Moreover,

- If $a > 0$, then $a(x - h)^2 \geq 0$ for all x and $a(x - h)^2 = 0$ when $x = h$. That is, $\text{rang}(f) = \{y \in \mathbb{R} \mid y \geq 0\}$ so that the minimum value of f is 0 which occurs when $x = h$. Consequently, the parabola *opens upward* with *vertex* at $(h, 0)$.
- If $a < 0$, then $a(x - h)^2 \leq 0$ for all x and $a(x - h)^2 = 0$ when $x = k$. That is, $\text{rang}(f) = \{y \in \mathbb{R} \mid y \leq 0\}$ so that the maximum value of f is 0, which occurs when $x = h$. Consequently, the parabola *opens downward* with *vertex* at $(h, 0)$.

Summary

(**Horizontal shift** of the graph of $y = ax^2$ to obtain the graph of $f(x) = a(x - h)^2$)
Consider $f(x) = a(x - h)^2$ for $h \in \mathbb{R}$. The graph of $f(x) = a(x - h)^2$ can be obtained by *horizontally shifting* the graph of $y = ax^2$ by h units to the right if $h > 0$ or $|h|$ units to the left if $h < 0$ so that its *vertex* is at $(h, 0)$, *line of symmetry* is $x = h$; and

- *opens upward* if $a > 0$ (So, $\text{rang}(f) = \{y \in \mathbb{R} \mid y \geq 0\}$, minimum value is 0 at $x = h$); or
- *opens downward* if $a < 0$ (So, $\text{rang}(f) = \{y \in \mathbb{R} \mid y \leq 0\}$, maximum value is 0 at $x = h$)

Example 50

Sketch the graph of

(a) $f(x) = (x - 3)^2$

(b) $f(x) = (x + 3)^2$ and contrast them with the graph of $f(x) = x^2$.

Solution

(a) $f(x) = (x - 3)^2$ with $h = 3$. To obtain the graph of f, we shift the graph of $f(x) = x^2$ by 3 units to the right so that its vertex is (3, 0). This is shown in figure (a) below.

(b) $f(x) = (x + 3)^2 = (x - (-3))^2$ with $h = -3$. To obtain the graph of f, we shift the graph of $f(x) = x^2$ by 3 units to the left so that its vertex is (–3, 0). This is shown in figure (b) below.

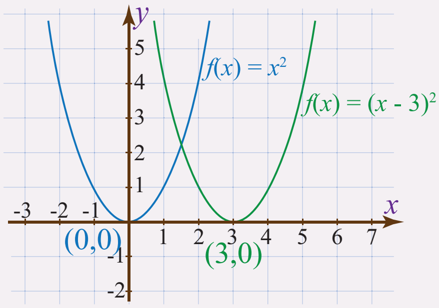


Figure (a)

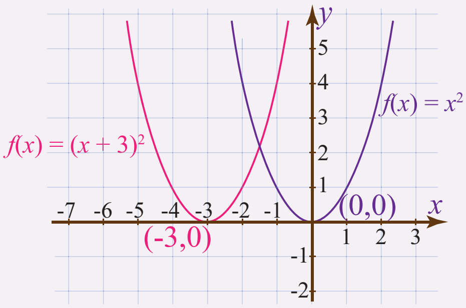


Figure (b)

The general form of a quadratic function is $f(x) = ax^2 + bx + c$. Using the technique of completing the square, this can be always changed to the form $f(x) = a(x - h)^2 + k$ for some $h, k \in \mathbb{R}$. In fact it can be shown that

$$f(x) = ax^2 + bx + c = a(x - h)^2 + k, \text{ where } h = -\frac{b}{2a} \text{ and } k = -\frac{4ac - b^2}{4a}.$$

Now, we examine the graph of the general form of a quadratic function.

IV. Consider $f(x) = a(x - h)^2 + k$, where $h, k \in \mathbb{R}$

The above discussions suggest that the graph of this function can be obtained from the graph of $y = ax^2$ by applying combination of horizontal and vertical shifts as follows.

Step 1: (Horizontal shift): The *horizontal shift* of the graph of $y = ax^2$, h units to the right if $h > 0$ or $|h|$ units to the left if $h < 0$, yields the graph of $y = a(x - h)^2$ whose vertex is $(h, 0)$ with line of symmetry $x = h$.

Step 2 (Vertical shift): Then, the *vertical shift* of the graph of $y = a(x - h)^2$, k units upward if $k > 0$ or $|k|$ units downward if $k < 0$, yields the graph of $f(x) = a(x - h)^2 + k$ whose *vertex* is (h, k) and *line of symmetry* $x = h$.

Moreover, the parabola (graph) of $f(x) = a(x - h)^2 + k$

- *Opens upward* if $a > 0$ (So, rang(f) = $\{y \in \mathbb{R} \mid y \geq k\}$, minimum value is k when $x = h$); or
- *Opens downward* if $a < 0$ (So, rang(f) = $\{y \in \mathbb{R} \mid y \leq k\}$, maximum value is k when $x = h$)

Example 51

Determine the line of symmetry and turning point (vertex) of the graph of each of the following quadratic functions and sketch their graphs.

(a) $f(x) = (x + 3)^2 + 2$ (b) $f(x) = (x - 3)^2 - 2$ (c) $f(x) = x^2+4x+2$

Solution

(a) Note that $f(x) = (x + 3)^2 + 2 = (x - (-3))^2 + 2$ (so, $h = -3$ and $k = 2$)

Step 1: (*Horizontal shift*) First sketch the graph of $f(x) = x^2$ and horizontally shift it 3 units to the left to obtain the graph of $f(x) = (x + 3)^2$ for which the vertex is $(-3,0)$ and line of symmetry is $x = -3$.

Step 2: (*Vertical shift*) Then, obtain the graph of $f(x) = (x + 3)^2 + 2$ by vertically shifting the graph of $f(x) = (x + 3)^2$ by 2 units upward so that its *vertex* is $(-3, 2)$.

Moreover, the parabola *opens upward*. So, rang(f) = $\{y \in \mathbb{R} \mid y \geq 2\}$, minimum value is 2 at $x = -3$.

The graph is shown in the figure (c) below.

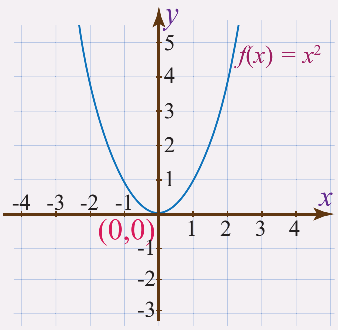


Figure (a)

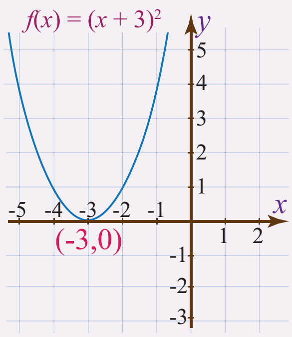


Figure (b)

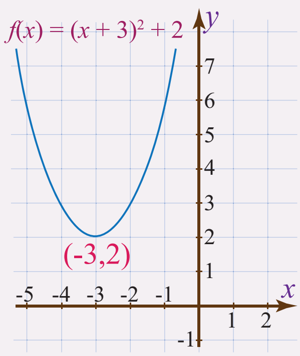


Figure (c)

(b) To sketch the graph of $f(x) = (x - 3)^2 - 2$ (with $h = 3$, $k = -2$)

Step 1 (*Horizontal shift*): First sketch the graph of $f(x) = x^2$ and horizontally shift it 3 units to the right to obtain the graph of $f(x) = (x - 3)^2$ whose vertex is at $(3, 0)$ and line of symmetry $x = 3$.

Step 2 (*Vertical shift*): Next, obtain the graph of $f(x) = (x - 3)^2 - 2$, by vertically shifting the graph of $f(x) = (x - 3)^2$ by 2 units downward so that its vertex is at $(3, -2)$.

Moreover, the parabola *opens upward*. So, rang(f) = $\{y\in\mathbb{R} \mid y \geq -2\}$, minimum value is -2, which occurs when $x = 3$. The graph is shown in the figure (c) below.

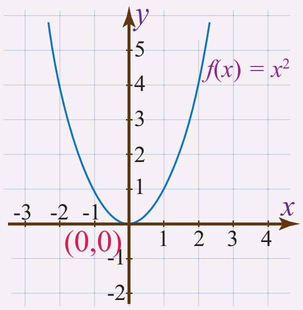


Figure (*a*)

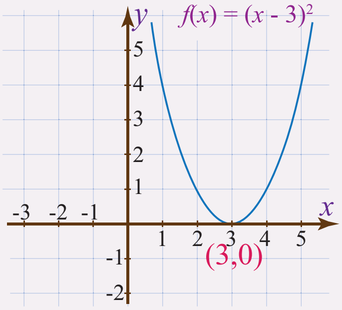


Figure (*b*)

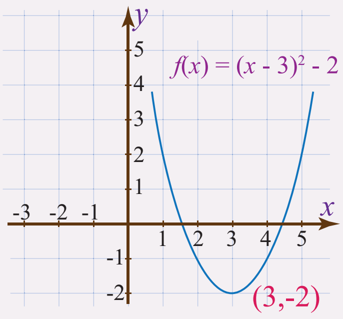


Figure (*c*)

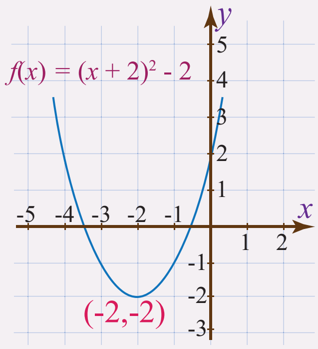
(c) In order to sketch the graph of $f(x) = x^2 + 4x + 2$, first we need to transform this function into the form of $f(x) = (x - h)^2 + k$ by completing the square. Consequently,

$f(x) = x^2 + 4x + 2$ can be expressed as

$f(x) = (x + 2)^2 - 2$ (with $h = -2$, $k = -2$)

Now you can sketch the graph of $f(x) = (x + 2)^2 - 2$ by first shifting the graph of $f(x) = x^2$ by 2 units to the left and then by 2 units downward. The graph is shown in the adjacent figure.

Note also that rang(f) = $\{y\in\mathbb{R} \mid y \geq -2\}$, minimum value is -2 when $x = -2$.



Summary

$f(x) = ax^2 + bx + c$ can be written as

$$f(x) = a(x - h)^2 + k, \text{ where } h = -\frac{b}{2a} \text{ and } k = -\frac{4ac - b^2}{4a}$$

- The graph of this quadratic function is a parabola with vertex (h, k) and axis (line) of symmetry $x = h$.
- If $a > 0$, the parabola opens upward. In this case, rang(f) = $\{y\in\mathbb{R} \mid y \geq k\}$; i.e., the minimum value of $f(x)$ is k which occurs at $x=h$.
- If $a < 0$, the parabola opens downward. In this case, rang(f) = $\{y\in\mathbb{R} \mid y \leq k\}$; i.e., the maximum value of $f(x)$ is k which occurs at $x=h$.

Exercises

1. For each of the following quadratic functions, determine a, b and c such that a is the leading coefficient and c is the constant term.
   (a) $f(x) = 2 + 3x - 2x^2$
   (b) $f(x) = 3x^2 - 4x + 1$
   (c) $f(x) = (x - 3)(2 - x)$
2. For each of the following quadratic functions prepare a table of values in the interval $-3 \le x \le 3$.
   (a) $f(x) = -4x^2$
   (b) $f(x) = 3x^2 + 2$
   (c) $f(x) = 2x^2 - 3x + 2$
3. Sketch the graph for each of the following quadratic functions:
   (a) $f(x) = -3x^2$
   (b) $f(x) = 7x^2 - 3$
   (c) $f(x) = 2x^2 + 6x + 1$
4. Find the domain and range of each of the following functions:
   (a) $f(x) = 3 + 4x - x^2$
   (b) $f(x) = x^2 + 2x + 1$
   (c) $f(x) = (x - 3)(x - 2)$
   (d) $f(x) = -3x^2 - 2$
   (e) $f(x) = 3x^2 + 2$
5. Sketch the graph of each of the following quadratic functions by using the shifting rules:
   (a) $f(x) = 9x^2 + 1$
   (b) $f(x) = x^2 - 3$
   (c) $f(x) = (x - 5)^2$
   (d) $f(x) = (x - 2)^2 + 13$
   (e) $f(x) = (x + 1)^2 - 7$
   (f) $f(x) = 4x^2 + 7x + 3$
6. Find the vertex and the axis of symmetry of the following functions:
   (a) $f(x) = x^2 - 5x + 8$
   (b) $f(x) = (x - 4)^2 - 3$
   (c) $f(x) = x^2 - 8x + 3$
7. Determine the minimum or the maximum value of each of the following functions and draw the graphs:
   (a) $f(x) = x^2 + 7x - 10$
   (b) $f(x) = x^2 + 4x + 1$
   (c) $f(x) = 2x^2 - 4x + 3$
   (d) $f(x) = 4x^2 + 2x + 4$
   (e) $f(x) = -x^2 - 4x$
   (f) $f(x) = -6 - x^2 - 4x$

Key Terms

- Axis of symmetry
- Combination of functions
- Constant function
- Coordinate system
- Distance between points
- Domain
- Function
- Leading coefficient
- Linear functions
- Quadratic function
- Range
- Relation
- Slope
- Subject of a relation
- Turning point
- Vertex
- x-intercept
- Y-intercept

Summary

- In a relation, two things are related to each other by a relating phrase.
- Mathematically, a relation is a set of ordered pairs. If A and B are two non-empty sets, then the relation from A to B is a subset of $A \times B$ that satisfies the relating phrase.
- If A and B are any sets and $R \subseteq A \times B$, we call R a binary relation from A to B or a binary relation between A and B.A relation $R \subseteq A \times A$ is called a relation in A or a reaction on A.
- The set $\{x: (x, y) \in R \text{ for some } y\}$ is called the domain of the relation R.

  The set $\{y: (x, y) \in R \text{ for some } x\}$ is called the range of the relation R.
- A function is a special type of a relation in which each x-coordinate is paired with exactly one unique y-coordinate.
- A function from A to B can sometimes be denoted as f: $A \rightarrow B$, where the domain of f is A and the range of f is a subset of B, in which case B contains the images of the elements of A by the function f.
- Let f and g be functions. We define the sum $f + g$, the difference $f - g$, the product fg, and the quotient $\frac{f}{g}$ as:

  $(f + g)(x) = f(x) + g(x)$ $\quad$ $(fg)(x) = f(x)g(x)$

  $(f - g)(x) = f(x) - g(x)$ $\quad$ $\frac{f}{g}(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$

- If a and b are fixed real numbers, $a \neq 0$, then $f(x) = ax + b$ for $x \in \mathbb{R}$ is called a linear function. If $a = 0$ then $f(x) = b$ is called a constant function. Sometimes linear functions are written as $y = ax + b$.
- In $f(x) = ax + b$ for $a \neq 0$, $x \in \mathbb{R}$, a represents the slope, $(0, b)$ represents the y-intercept and $\left(-\frac{b}{a}, 0\right)$ represents the x-intercept.
- Two lines with equal slopes are parallel. Distinct parallel lines never intersect.
- Two lines with different slopes intersects at one and only one point.
- The distance d between points (x_1, y_1) and (x_2, y_2) is given by
  $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- A function defined by $f(x) = ax^2 + bx + c$, $(a, b, c \in \mathbb{R}$ and $a \neq 0)$ is called quadratic function. a is called the leading coefficient.
- We can sketch the graph of a linear function by using either a table of values, or the x- and y-intercepts.
- We can sketch the graph of a quadratic function by using either a table of values or the shifting rules.
- A quadratic function $f(x) = ax^2 + bx + c$ can be written as
  $f(x) = a(x - h)^2 + k$, where $h = \frac{b}{2}$ and $k = \frac{4ac - b^2}{4a}$. Moreover, the graph of this quadratic function is a parabola with vertex (h,k), line (axis) of symmetry $x=h$.
  - If $a > 0$, the parabola opens upward. In this case rang$(f) = \{y \in \mathbb{R} \mid y \geq k\}$; i.e., the minimum value of $f(x)$ is k which occurs at $x=h$.
  - If $a < 0$, the parabola opens downward In this case rang$(f) = \{y \in \mathbb{R} \mid y \leq k\}$; i.e., the maximum value of $f(x)$ is k which occurs at $x=h$.

Exercises

1. For the relation {(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)} find the domain and the range.
2. If the domain of the relation $R = \{(x, y): y = x+3\}$ is $A = \{1, 2, 3, 4\}$ then list all the ordered pairs that are members of the relation and find the range.
3. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{a, b, c\}$
   (a) Find $A \times B$

(b) Determine relations as subsets of $A \times B$ such that:

(i) $R_1 = \{(x, y): x \text{ is odd}\}$

(ii) $R_2 = \{(x, y): 1 \le x \le 3\}$

4. Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 5\}$

(a) If R is a relation from A to B then, is it true that R is also a relation from B to A? Explain your answer.

(b) If $R \subseteq (A \times B)$ such that $R = \{(2, 4), (2, 2), (4, 4), (4, 2)\}$, then is R also a relation from B to A?

(c) What can you conclude from (b)?

5. Let $R = \{(x, y): x \text{ is taller than } y\}$.

(a) Is R reflexive? Explain.

(b) Is R symmetric? Explain

(c) Is R transitive?

6. Let $R = \{(x, y): y = x\}$. Show that R is reflexive, symmetric and transitive.

7. Find the domain and the range of each of the following relations:

(a) $R = \{(x, y): y = 2x\}$

(b) $R = \{(x, y): y = |x|\}$

(c) $R = \{(x, y): x, y \in \{1, 2, 3, 4, 5\} \text{ and } y = 2x - 1\}$

(d) $R = \{(x, y): y = \sqrt{x^2 - 4}\}$

8. Determine whether each of the following relations is a function. If it is not, give a reason.

(a) $R = \{(a, 1), (b, 2), (c, 3)\}$

(b) $R = \{(1,3), (2, 3), (3, 3), (4, 3), (5, 3)\}$

(c) $R = \{(1, 4), (1, 5), (1, 6), (5, 4), (5, 5)\}$

9. If $A = \{2, 5, 7\}$ and $B = \{2, 3, 4, 6\}$, then is $A \times B$ a function? Explain your answer.

10. Let $f = \{(1, 2), (2, 3), (5, 6), (7, 8)\}$

(a) Find the domain and range of f

(b) Evaluate $f(2)$ and $f(5)$

11. Let $f(x) = 2x + 1$ and $g(x) = -3x - 4$

(i) Determine:

(a) $f + g$ (b) $f - g$ (c) fg (d) $\frac{f}{g}$

(ii) Evaluate:

(a) $(2f + 3g)(1)$ (b) $(3fg)(3)$ (c) $\frac{3f}{2g}(4)$

(iii) Find the domain of $\frac{f}{g}$.

12. Let $f(x) = \frac{x+4}{2x}$ and $g(x) = \frac{2x+4}{x+1}$

(i) Determine:

(a) fg (b) $\frac{g}{f}$ (c) $2f - \frac{f}{g}$

(ii) Find the domains of:

(a) fg (b) $\frac{g}{f}$ (c) $2f - \frac{f}{g}$

(iii) Evaluate:

(a) $(f - g)(1)$ (b) $\frac{g}{f}(2)$ (c) $\left(2f - \frac{f}{g}\right)(3)$

13. Construct tables of values and sketch the graph of each of the following:

(a) $f(x) = 3x + 2$ (d) $f(x) = -3x^2 - 1$

(b) $x - 2y = 1$ (e) $f(x) = 3 - 2x + x^2$

(c) $f(x) = 2 - 7x$

14. Sketch the graph of each of the following by using x- and y-intercepts:

(a) $f(x) = 7 + 2x$ (c) $3x - y = 4$

(b) $f(x) = 3x - 5$

15. By using shifting rule, sketch the graph of each of the following:

(a) $f(x) = 4x^2 - 2x$ (c) $f(x) = 4x + 6 - 3x^2$

(b) $f(x) = x^2 - 8x + 7$

16. For the function $f(x) = 3x^2 - 5x + 7$, determine:

(a) Whether it turns upward or downward

(b) The vertex

(c) The axis of symmetry

17. Determine the range of each of the following functions:

(a) $f(x) = (x + 5)^2 + 3$ (c) $f(x) = -8 - x^2 - 6x$

(b) $f(x) = x^2 - 9x + 10$ (d) $f(x) = -x^2 + 2x + 4$

18. Determine the range and the minimum (or the maximum) value of the following functions:

(a) $f(x) = (x - 4)^2 - 5$

(b) $f(x) = (x + 5)^2 + 3$

(c) $f(x) = 3x^2 - 5x + 8$

(d) $f(x) = -x^2 + 6x - 5$

(e) $f(x) = -2 + 4x - 2x^2$

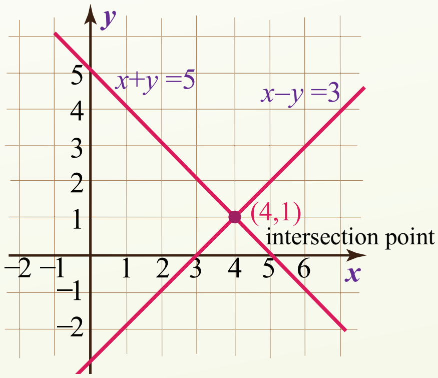
(f) $f(x) = 2x^2 - 6x + 7$

M10CH08

CHAPTER 8

SIMULTANEOUS LINEAR EQUATIONS

Chapter Contents



Chapter Outcomes

Learners are able to solve simultaneous linear equations with two variables using the methods of graphs, substitution and eliminations; and they are able to solve word problems under this topic.

Objectives: Upon completion of this chapter, learners should be able to:

- solve simultaneous linear equations in two variables;
- define and discuss truth sets for simultaneous linear equations;
- solve simultaneous equations using three different methods, namely;
- graph, substitution and elimination;
- solve word problems in simultaneous linear equations.

Introduction

We have discussed in unit 6 about linear equations with one variable. There are also many practical problems that involve more than one variables (unknowns) and more than one equations. In particular, two equations in two variables that should be solved together at the same time are called **simultaneous equations**. In this unit, we shall discuss and solve simultaneous linear equations in two variables.

The following activity is a useful preparation for the discussions that follow.

Activity 1

Consider two numbers x and y described by the following two statements:

(i) The **sum** of the two number is 5.

(ii) The **difference** of these numbers is 3 (that is, the first number *minus* the second number is 3).

1.
   - Observe that several pair of numbers such as 0 & 5, 0.5 & 4.5, 3 & 2, 4 & 1 can make statement (i) true. Does each of these pairs satisfy also the statement in (ii)?
   - Observe also that several other points such as 7 & 4, 3.5 & 0.5, 4 & 1, 6 & 3 can make statement (ii) true. Does each of these pairs satisfy also the statement in (i)?
   - Find pair of numbers that makes both of the statements true at the same time.
2. You may observed that the statements in (i) and (ii) are given by two equations $x+y=5$ and $x-y=3$, respectively.
   - Draw the straight lines given by $x+y=5$ and $x-y=3$ on the same coordinate plane.
   - Identify the intersection point of the two lines?
   - How do you relate this intersection point with a solution of the two equations?

A linear equation with two variables, say x and y, is of the form $ax + by = c$, where a, b and c are constants. Here, a and b are called *coefficients*. The coefficients should be nonzero for the equation to have two variables. Our aim in this unit is to find solutions that satisfy not just one equation but two linear equations at the same time.

Two linear equations in two variables taken together are called **simultaneous linear equations**.

DEFINITION

A **simultaneous linear equations** are system of two linear equations in two variables, say x and y, given as $a_1x+b_1y=c_1$ and $a_2x+b_2y=c_2$, where a_i, b_i and c_i are constants. This is usually written as $\begin{cases} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{cases}$.

For instance, $\begin{cases} x+y=5 \\ x-y=3 \end{cases}$ and $\begin{cases} -2x+y=7 \\ 3x-2y=0 \end{cases}$ are examples of simultaneous linear equations.

Solving a simultaneous equation means finding pair of values for the unknown variables x and y which satisfy both equations at the same time. We write such a pair of values (solutions) as an ordered pair (x, y). This is stated in the following definition.

DEFINITION

A solution to a simultaneous system of two equations in two variables, say x and y, is an ordered pair (x, y) that satisfies both equations. The set of all such solutions is called **truth set** or **solution set**.

For instance, as you may have observed in Activity 1, the solutions of $\begin{cases} x+y=5 \\ x-y=3 \end{cases}$ are $x=4$ and $y=1$. That is, $(x, y)=(4,1)$ is the common solution for both equations. Hence, its truth set (or solution set) is $\{(4, 1)\}$. Methods of solving such system of two equations are discussed in the next section.

There are two common algebraic methods for solving system of linear equations: *substitution* and *elimination*. Graphical method is also considered for the case of system of two linear equations as it is helpful for better understanding of the concept of solution of a system.

Graphical method

Here, given system of two linear equations, we use the graphs of both equations to identify the solution of the system. As you have seen in Unit 7, the graph of $ax + by = c$ is a straight line. In particular, rearrange this equation, you can see that

$$ax + by = c \quad \text{is equivalent to} \quad y = \frac{a}{b}x + \frac{c}{a},$$

Which is an equation of the line with slope $m = -\frac{a}{b}$ and y-intercept $\left(0, \frac{c}{q}\right)$. Recall also that every ordered pair (x, y) of points that lie on this line satisfy the equation of the line.

So, a system of two linear equations $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ has two straight lines given by $y = -\frac{a_1}{b_1}x + \frac{c_1}{a_1}$ and $y = -\frac{a_2}{b_2}x + \frac{c_2}{a_2}$, respectively; and an ordered pair (x, y) corresponding to the **point of intersection** of these lines satisfy both equations. Therefore, this ordered pair is a solution of the system.

So, you can draw the straight line for the two equations and identify their intersection point (if any). Then, the pair of values that correspond to the point of intersection is the solution of the given system of simultaneous equations.

Example 1

Solve the simultaneous linear equations $\begin{cases} x + y = 5 \\ x - y = 3 \end{cases}$.

Solution

First draw the graphs of each equations. Then, identify their point of intersection. Here, the intersection point is (4,1) as shown in the adjacent graph.

Therefore, the solution of the system is

$(x, y) = (4, 1)$.

i.e., its truth set is $\{(4, 1)\}$.

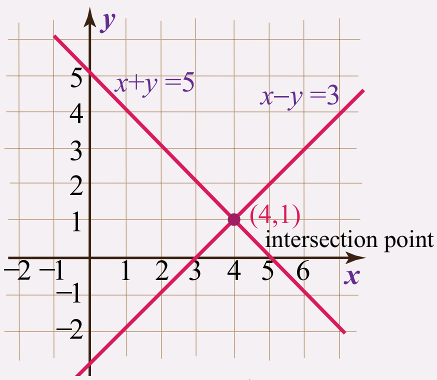


Figure 1.

Recall that if two lines have different slopes, then they have one and only one intersection point. Therefore, a system of two linear equations has exactly one pair of solution if the slopes of their lines are different.

On the other hand, if the two lines are distinct and parallel then the lines have no intersection point. In this case, there is no ordered pair that satisfy both equations. This means, the given system of simultaneous equations has no solution.

Example 2

Solve the simultaneous linear equations $\begin{cases} x - y = -1 \\ 2x - 2y = 2 \end{cases}$

Solution

When we draw the line of each of the component equations, we see that the lines are parallel (slope of both is 1). This means the lines do not intersect. Therefore, there is no solution satisfying both equations at the same time. Hence, the system has no solution.

i.e., its truth set is $\varnothing$ (empty set).

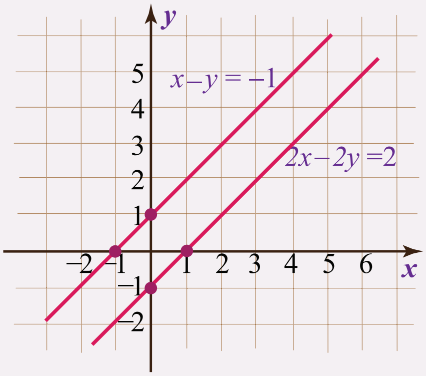


Figure 2.

Notice also that if two lines have equal slopes and have the same y-intercept, then the two lines are fully overlap (coincide). In this case, both equations are actually equivalent equations of one line. If so, all ordered pairs of points that lie on this line satisfy both equations; i.e., the system has infinitely many solutions.

Example 3

Solve the simultaneous linear equations $\begin{cases} 2x - y = 2 \\ -4x + 2y = -4 \end{cases}$

Solution

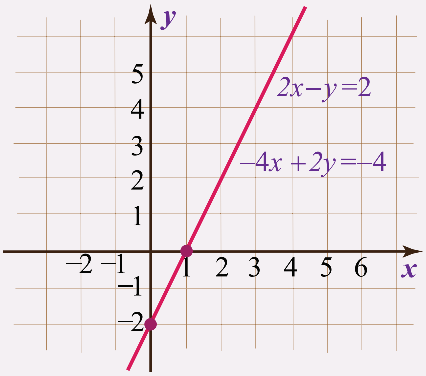


Figure 3.

The lines of these two equations are fully overlapping (the slope of both is 2 and the y-intercept for both is (0, –2)). In fact, note that both equations are equations of the line given by $y = 2x - 2$.

Thus, every point (ordered pair) on this line, such as (–2, 0), (1,0), (2, 2), etc., are solutions of the system. Thus, the system has infinitely many solutions.

From the above discussions, we conclude the following.

Consider a system of two linear equations given by $\begin{cases} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_1 \end{cases}$

Solving both equations for y in terms of x, they can be written as: $\begin{cases} y=-\dfrac{a_1}{b_1}x+\dfrac{c_1}{b_1} \\ y=-\dfrac{a_2}{b_2}x+\dfrac{c_2}{b_2} \end{cases}$

1. If $\dfrac{a_1}{b_1} \neq \dfrac{a_2}{b_2}$ (i.e., their lines have different slopes), then the system has one unique solution. In this case, the system is said to be **consistent**.
2. If $\dfrac{a_1}{b_1} \neq \dfrac{a_2}{b_2}$ but $\dfrac{c_1}{b_1} \neq \dfrac{c_2}{b_2}$ (i.e., their lines have equal slopes but different y-intercepts), then the system has **no** solution. In this case, the system is said to be **inconsistent**.
3. If $\dfrac{a_1}{b_1} \neq \dfrac{a_2}{b_2}$ and $\dfrac{c_1}{b_1} \neq \dfrac{c_2}{b_2}$ (i.e., their lines have equal slopes and the same y-intercept), then the system has infinitely many solutions. If so, the system is called **dependent**.

EXERCISES

Using graphical method determine whether the given system is consistent (has unique solution), inconsistent (has no solution) or dependent (has infinitely many solutions). If it is consistent, determine the solution of the system.

1. $\begin{cases} 2x-2y=4 \\ 3x+4y=6 \end{cases}$
2. $\begin{cases} x+2y=4 \\ 3x+6y=6 \end{cases}$
3. $\begin{cases} 3x+y=5 \\ 6x-2y=10 \end{cases}$

Method of substitution

The method of substitution involves substituting one equation into another in order to eliminate one of the variables. This changes the equation into a linear equation of one variable that can be solved easily. The steps to do so are the following.

To solve a system of two linear equations by the substitution method, you follow the following steps.

1. Take one of the linear equations from the system and write one of the variables in terms of the other; i.e., make one of the variable the subject of the equation.

2. Substitute your result in place of the subject variable in the other equation. This yields an equation with only the second variable. Solve this equation for the second variable.
3. Substitute the second variable by its value in one of the equations and solve for the first variable.

Example 4

Solve the simultaneous linear equations $\begin{cases} x+y=5 \\ x-y=3 \end{cases}$.

Solution

It is helpful to check first whether the system has solution or not. The lines of the two equations have different slopes, 1 and –1. Hence, the system is consistent (has a unique solution). Next to find the solution, let us label the given equations for reference:

$x+y=5$ . . . (1)

$x-y=3$ . . . (2)

Step 1: Choosing equation (2) , we solve for x (i.e., make x the subject of Eq. (2)) so that

$x=y+3$ . . . (3) (Note: choosing equation 2 and solving for x is just a matter of choice. You can choose Eq.(1) or solve for y. The end result will be the same.)

Step 2: Now substitute (replace) x with $y+3$ in equation (1) to get

$(y+3)+y=5$

$\Rightarrow 2y+3=5$

$\Rightarrow 2y=2$

$\Rightarrow y=1$

Step 3: Next, replace y by its value (i.e., by 1) in equation (3) to get

$x=1+3=4$

Thus, we have obtained, $x=4$ and $y=1$. That is, the solution is $(x, y) = (4, 1)$.

Therefore, the truth set of the system is {(4, 1)}.

Example 5

Solve the system of simultaneous equations $\begin{cases} 6x+3y=3 \\ 2x-\frac{3}{2}y=1 \end{cases}$.

Solution

The system is consistent, i.e., has a unique solution, because the slopes of the corresponding two lines are different (Check that!). To find the solution, let us label the given equations for reference:

$6x + 3y = 3. \ . \ . \ . \ . \ . \ . \quad (1)$

$2x - \frac{3}{2}y = 1 \ . \ . \ . \ . \ . \ . \ . \quad (2)$

Step 1: Let us choose equation (1) and solve for y (make y the subject of (1)), so that

$3y = 3 - 6x$

$\Rightarrow y = 1 - 2x \ . \ . \ . \ . \ . \ . \ . \quad (3)$

Step 2: Substituting (replacing) y by $1 - 2x$ in equation (2) to get

$$2x - \frac{3}{2}(1 - 2x) = 1$$

$$\Rightarrow 2x - \frac{3}{2} + 3x = 1$$

$$\Rightarrow 2x + 3x = 1 + \frac{3}{2}$$

$$\Rightarrow 5x = \frac{5}{2}$$

$$\Rightarrow x = \frac{1}{2}$$

Step 3: Now, replacing x by $\frac{1}{2}$ in equation (3) we get

$$y = 1 - 2\left(\frac{1}{2}\right) = 1 - 1 = 0$$

Therefore, we have got $x - \frac{1}{2}$ and $y = 0$.

That is, the truth set (solution set) of the system is $\left\{\left(\frac{1}{2}, 0\right)\right\}$.

Example 6

Solve the system of simultaneous equations $\begin{cases} -2x - y = 7 \\ 4x + 2y = 5 \end{cases}$.

Solution

First check whether the system has a unique solution (consistent) or not. The slopes of their lines are equal since $\frac{-2}{1} = \frac{4}{-2} = -2$; but their y-intercepts are different as. Hence, the system is inconsistent. That is, the system has no solution.

Note: If you apply the substitution method directly to solve this system (Example 6), you will end up with an invalid equation like $-14 = 5$ which is always false. This indicates that the system is inconsistent (has no solution).

Example 7

Solve the system of simultaneous equations $\begin{cases} 2x - y = -7 \\ -4x + 2y = 14 \end{cases}$.

Solution

First check whether the system has a unique solution (consistent) or not. The slopes of the lines of the two equations are equal as $\frac{2}{-1} = \frac{-4}{2} = -2$; and also their y-intercepts are the same since $\frac{-7}{-1} = \frac{14}{2} = 7$. Hence, the system is dependent. That is, the system has infinitely many solutions. All ordered pairs that satisfy $2x - y = -7$ are solutions of the system.

Note: If you directly apply the substitution method to solve this system, you will eventually obtain an equation of type 14 = 14 or 0 = 0 which is always true. This indicates that the system has infinitely many solutions.

Exercises

For each of the following system of equations, determine the system is consistent (has unique solution), inconsistent (has no solution) or dependent (has infinitely many solutions).

If it is consistent, find the solution of the system using the substitution method.

1. $\begin{cases} 2x - y = 1 \\ 3x - 2y = -4 \end{cases}$
2. $\begin{cases} 2x - 4y = 5 \\ -6x - 12y = -15 \end{cases}$
3. $\begin{cases} 2x - 3y = 5 \\ 5x - 3y = 9 \end{cases}$
4. $\begin{cases} 4x - 3y = 8 \\ -2x - \frac{3}{2}y = -6 \end{cases}$

Method of elimination

Alike the substitution method, the elimination method also starts with reducing two equations into an equation of only one variable by eliminating one of the variables. Here, however, the technique to eliminate a variable is not by substitution but by adding the given equations together. For instance, consider the following system of two equations: $\begin{cases} 3x-2y=4 \\ x+2y=4 \end{cases}$.

Add the two equations: $+\begin{array}{l} 3x-2y=4 \\ x+2y=4 \end{array}$

$4x \quad = 8$ ... (y is eliminated (concealed out) as $-2y + 2y = 0$.)

$\Rightarrow \quad x = 2.$

Now, substituting the value of x in any of the equations, say in the 2[nd] equation, we get

$$2 + 2y = 4 \Rightarrow 2y = 2 \Rightarrow y = 1.$$

Therefore, the solution of the system is (2, 1).

In general, the elimination method involves the following steps.

To solve a system of two linear equations by the elimination method, you follow the following steps.

1. Select one of the variables to eliminate and make the coefficients of the selected variable equal but opposite in sign in the two equations. This is done by multiplying one of the equations (or both) by a suitable number (s).
2. Add the resulting two equations to eliminate the selected variable and solve for the resulting variable.
3. Substitute this result again into one of the equations and solve for the remaining variable.

Example 8

Solve the system of linear equations given by $\begin{cases} 2x-y=5 \\ 2x+3y=9 \end{cases}$

Solution

Step 1: Select one of the variables eliminate, say y, and make the coefficients of y equal but opposite in sign by multiplying the first equation by 3. That is,

$$\begin{cases} 2x-y=3 \quad (\times 3) \\ \quad 2x+3y=9 \end{cases} \Rightarrow \begin{cases} 6x-3y=15 \\ 2x+3y=9 \end{cases}$$

$$8x \quad = 24$$

$$\Rightarrow x \frac{24}{8} = 3$$

Step 2: Add these two equations:

Step 3: Substitute $x = 3$ into one of the original equations and solve for y.

Choosing $2x - y = 5$ and replacing $x = 3$, we get

$$2(3) - y = 5$$

$$\Rightarrow -y = 5 - 6$$

$$\Rightarrow -y = -1 \quad \text{or} \quad y = 1.$$

Therefore, the solution is (3, 1).

Example 9

Solve each of the following systems of linear equations.

(a) $\begin{cases} 7x + 5y = 11 \\ -3x + 3y = -3 \end{cases}$ (b) $\begin{cases} 2x - 4y = 8 \\ x - 2y = 4 \end{cases}$ (c) $\begin{cases} 2x - 7y = 9 \\ -6x - 21y = 6 \end{cases}$

Solution

(a) $\begin{cases} 7x - 5y = 11 \\ -3x - 3y = -3 \end{cases}$

Step 1: Let us choose x to eliminate. So, make the coefficients of x equal but opposite in the two equations by multiplying the first equation by 3 and the second equation by 7. That is,

$$\begin{cases} 7x - 5y = 1 & (\times 3) \\ -3x - 3y = -3 & (\times 7) \end{cases} \Rightarrow \begin{cases} 21x + 15y = 33 \\ -21x + 21y = -21 \end{cases}$$

$$36y = 12$$

$$\Rightarrow y = \frac{12}{36} = \frac{1}{3}$$

Step 2: Adding the two equations:

Step 3: Now substituting $y = \frac{1}{3}$ in one of the equations, say in $7x + 5y = 11$, we get

$$7x + 5\left(\frac{1}{3}\right) = 1$$

$$7x = 1 - \frac{5}{3}$$

$$7x = \frac{28}{3}$$

$$x = \frac{28}{21} = \frac{4}{3}.$$

Therefore the solution is $\left(\frac{4}{3}, \frac{1}{3}\right)$.

(b) $\begin{cases} 2x - 4y = 8 \\ x - 2y = 4 \end{cases}$

Here, multiplying the second equation by –2, we get,

$\begin{cases} 2x - 4y = 8 \\ -2x + 4y = -8 \end{cases}$

Adding the two equations: $0 + 0 = 0$

i.e., we get $0 = 0$ which is always true.

Therefore, the system has infinite solutions.

(You can confirm this by checking the slope and *y*-intercept of their lines.)

(c) $\begin{cases} 2x - 7y = 9 \\ -6x + 21y = 6 \end{cases}$

Multiply the first equation by 3 to make the coefficients of the variables opposite.

We get $\begin{cases} 6x - 21y = 27 \\ -6x + 21y = 6 \end{cases}$

Adding the two equations $0 + 0 = 33$.

i.e., we got $0 = 33$ which is always false.

Therefore, the system has no solution.

(You can confirm also this by checking the slope and *y*-intercept of their lines)

Exercises

1.-4. Solve the simultaneous linear equations giving in previous exercise (1 to 4) using the method of elimination.

5. Solve each of the following systems of equations by the elimination method.

(a) $\begin{cases} -3x + y = 5 \\ 3x + y = 5 \end{cases}$ (b) $\begin{cases} 4x - 3y = 6 \\ 2x + 3y = 12 \end{cases}$ (c) $\begin{cases} \frac{2}{3}x - \frac{1}{3}y = 2 \\ -x + \frac{1}{3}y = -3 \end{cases}$

(d) $\begin{cases} \frac{1}{2}x - 2y = 5 \\ 7x + 4y = 6 \end{cases}$ (e) $\begin{cases} x + 3y = 1 \\ 2x + 5y = 2 \end{cases}$

There are practical or real life problems that need to be formulated as simultaneous linear equations that can be solved by the methods discussed above. Here are some examples.

Example 10

A farmer collected a total of L$11,000 by selling 3 cows and 5 sheep. Another farmer collected L$7,000 by selling one cow and 10 sheep. What is the price for a cow and a sheep? (Assume all cows have the same price and also the price of every sheep is the same).

Solution

Let x be the price of a cow and y be the price of a sheep.

Farmer I sold 3 cows for $3x$ and 5 sheep for $5y$ collecting a total of 11,000.

Which means, $3x + 5y = 11{,}000$

Farmer II sold 1 cow for x and 10 sheep for $10y$ collecting a total of 7,000.

Which means, $x + 10y = 7{,}000$

When we consider these equations simultaneously, we get the following system of equations.

$$\begin{cases} 3x + 5y = 11{,}000 \\ x + 10y = 7{,}000 \end{cases}$$

Multiplying the first equation by –2 to make the coefficients of y opposite, we get

$$\begin{cases} -6x - 10y = -22{,}000 \\ x + 10y = 7{,}000 \end{cases}$$

Adding the equations we get: $-5x = -15{,}000$

$$\Rightarrow x = 3{,}000$$

Substituting $x = 3{,}000$ in one of the equations, say the second equation, we get,

$$3{,}000 + 10y = 7{,}000$$

$$10y = 4{,}000$$

$$y = 400.$$

Therefore, the solution is (3000, 400) showing that the price for a cow is L$3,000 and the price for a sheep is L$400 in the given unit of currency.

Example 11

Simon has twin younger brothers. The sum of the ages of the three brothers is 48 and the difference between his age and the age of one of his younger brothers is 3. How old is Simon?

Solution

Let x be the age of Simon and y be the age of each of his younger brothers.

The sum of the ages of the three brothers is 48.

So, $x + y + y = 48$

i.e., $x + 2y = 48$.

The difference between his age and the age of one of his younger brothers is 3 implying

$x - y = 3$.

Thus, to find Simon's age, we need to solve the system $\begin{cases} x + 2y = 48 \\ x - y = 3 \end{cases}$

Multiplying the second equation by 2 to make the coefficients of y opposite, we get

$$\begin{cases} x + 2y = 48 \\ 2x - 2y = 6 \end{cases}$$

Adding the equations, we get: $3x = 54$

$$\Rightarrow x = \frac{54}{3} = 18 .$$

Therefore, Simon is 18 years old.

Example 12

Two math books and one biology book cost L$35. Moreover, 3 math books and 4 biology books cost L$65. Find the cost of each math book and biology book.

Solution

Let x be the cost of math book and y be the cost of biology book. Then, the first and second statements are respectively formulated as:

$$2x + y = 35 \quad \text{and}$$

$$3x + 4y = 65$$

Then, solving this simultaneous linear equations using either the method of elimination or substitution you can get $x = 15$ and $y = 5$.

Therefore, the cost of one math book is L$15 and the cost of one biology book is L$5.

Example 13

The product of two positive number is 40. If the sum of the larger number and twice the smaller number is 18, find the two numbers.

Solution

Let the numbers be x and y with $y > x$.

Then, $xy = 40$ and $y + 2x = 18$

From the equation $y + 2x = 18$, we get $y = 18 - 2x$. Substituting y by $18 - 2x$ in the equation $xy = 40$, gives

$$x(18 - 2x) = 40$$

$$18x - 2x^2 = 40$$

$$2x^2 - 18x + 40 = 0$$

Dividing both sides by 2, gives

$$x^2 - 9x + 20 = 0$$

Using factorization of quadratic expression, we have

$$x^2 - 4x - 5x + 20 = 0$$

$$\Rightarrow x(x - 4) - 5(x - 4) = 0$$

$$\Rightarrow (x - 4)(x - 5) = 0$$

$$\Rightarrow x - 4 = 0 \text{ or } x - 5 = 0$$

$$\Rightarrow x = 4 \text{ or } x = 5.$$

Substituting x by 4 in the equation $xy = 40$ or $y + 2x = 18$, gives $y = 10$.

Also, substituting x by 5 gives $y = 8$.

Therefore, the two number are 4 and 10 or 5 and 8.

Example 14

The sum of a positive number and 4 times another positive is 20. If the larger number is one less than the square of the smaller number, find the two numbers.

Solution

Let the numbers be x and y with $y > x$.

Hence, $y + 4x = 20$ and $y = x^2 - 1$

Using the method of substitution, we have

$$x^2 - 1 + 4x = 20$$

$$x^2 + 4x - 21 = 0.$$

Rewriting the equation, using quadratic factorization, shows,

$$x^2 + 7x - 3x - 21 = 0$$

$$x(x + 7) - 3(x + 7) = 0$$

$$(x + 7)(x - 3) = 0$$

$$x + 7 = 0 \text{ or } x - 3 = 0$$

$$x = -7 \text{ or } x = 3.$$

Therefore, $x = 3$ since x is a positive number.

Substituting x by 3 in one of the equation $y + 4x = 20$ or $y = x^2 - 1$ gives $y = 8$.

Therefore, the numbers are 3 and 8.

Exercises

1. The sum of two number is 14 and their difference is 2. Find the numbers.
2. If 2 is added to the numerator and denominator it becomes $\frac{9}{10}$ and if 3 is subtracted from the numerator and denominator it become $\frac{4}{5}$. Find the fractions.
3. If twice the age of son is added to age of father, the sum is 56. But if twice the age of the father is added to the age of son, the sum is 82. Find the ages of father and son.
4. The sum of two digits number and the number obtained by reversing the digits is 110. The difference between the digits is 4. Find the number.
5. The area of a rectangle gets reduced by 10 square units if its length is reduced by 4 units and breadth is increase by 2 units. If we increased the length by 3 units and breadth by 4 units, the area is increased by 96 square units. Find the length and breadth of the rectangle.

Key Terms

- Consistent system
- Dependent system
- Elimination method
- Graphical method
- Inconsistent system
- Simultaneous linear equations
- Solution of simultaneous equations
- Solution set of system of equations
- Substitution method
- Truth set of system of equations

Summary

- A system of two linear equations (simultaneous linear equations) in two variables is equations that can be represented as

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}.$$

- A solution of system of two equations in two variables, say x and y, is an ordered pair (x, y) that satisfy both equations. The set of all possible ordered pairs that satisfy both equations is called the truth set (or solution set) of the system.
- A system of simultaneous linear equations can have either: **one unique solution**, **infinitely many solutions** or **no solutions**. In particular, considering the two lines of the two equations,
  - If the two lines intersect at one point, the system has **one unique solution**. In this case the ordered pair corresponding to the intersection point is the solution of the system.
  - If the two lines coincide (overlap), then the system has **infinitely many solutions**. I.e., the ordered pairs of every point that lie on the line are solutions of the system.
  - If the two lines are distinct parallel lines, then the system has **no solution**.

- Rewriting $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$ as $\begin{cases} y = \dfrac{a_1}{b_1}x + \dfrac{c_1}{b_1} \\ y = -\dfrac{a_1}{b_2}x + \dfrac{c_2}{b_2} \end{cases}$, we can conclude also the following:
  - If $\dfrac{a_1}{b_1} \neq \dfrac{a_2}{b_2}$, the system has exactly **one solution**. (The lines have different slopes)
  - If $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2}$ and $\dfrac{c_1}{b_1} = \dfrac{c_2}{b_2}$, the system has **infinitely many solutions**. (Their lines coincide)
  - If $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2}$ but $\dfrac{c_1}{b_1} \neq \dfrac{c_2}{b_2}$, the system has **no solution**. (They have distinct parallel lines)

Exercises

1. Without solving, determine the number of solutions to each of the following systems of linear equations.

   (a) $\begin{cases} 3x - 4y = 5 \\ 2x + 3y = 3 \end{cases}$ (b) $\begin{cases} 6x + 9y = 7 \\ 2x + 3y = 13 \end{cases}$ (c) $\begin{cases} -x + 4y = 7 \\ 2x - 8y = -14 \end{cases}$

2. Applying each of the three methods for solving systems of linear equations, solve each of the following.

   (a) $\begin{cases} -2x - 3y = 5 \\ 2x + 3y = -5 \end{cases}$ (b) $\begin{cases} \dfrac{3}{2}x = 5 - 2y \\ x - 3y = 5 \end{cases}$ (c) $\begin{cases} 0.3x - 0.4y = 1 \\ 0.2x + y = 3 \end{cases}$

3. Solve

   (a) $\begin{cases} 3x - 0.5y = 6 \\ -2x + y = 4 + 2y \end{cases}$ (b) $\begin{cases} \dfrac{2}{x} + \dfrac{3}{y} = -2 \\ \dfrac{4}{x} - \dfrac{5}{y} = 1 \end{cases}$

   Hint: Let $a = \dfrac{1}{x}$ and $b = \dfrac{1}{y}$

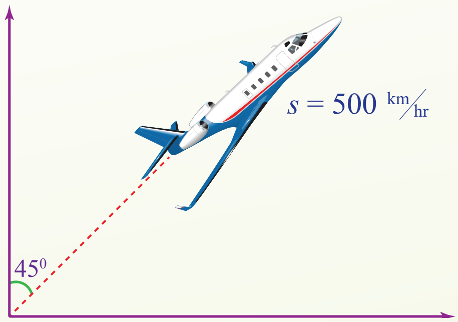
4. There are 32 students in a classroom. The number of girls in the classroom is 4 more than the number of boys. How many girls are in the classroom?
5. The sum of an even number and odd number is 95; and the even number minus the odd number is 17. Find the numbers.
6. In a two digit number, the units digit is three times the tens digit. If 36 is added to the number, the digits interchange their place. Find the number.
7. In a stationary shop, the cost of 3 pencil sharpeners exceeds the price of 2 pens by L$2. Also the total price of 7 pencil sharpeners and 3 pens is L$43. How much is the price of each pencil sharpener and each pen?
8. The total weight of orange and banana that a woman bought is 10 kg. The price of the orange is L$1.20 per kg and the price of the banana is L$0.80 per kg. If the woman paid L$10.40 for the fruits, how many kg of each of the fruits did she buy?

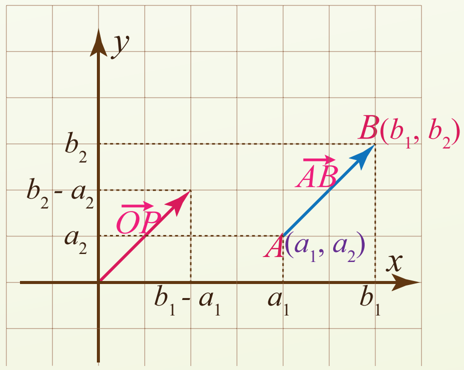
CHAPTER 9

M10CH09

VECTOR IN A PLANE

Chapter Contents





Chapter Outcomes

Learners are able to apply concepts to identify the types of vector quantities, determine the magnitude and direction of vector, perform basic operations (addition, subtraction and scalar multiplication) on vectors.

Objectives: Upon completion of this chapter, learners should be able to:

- discuss the types of vector quantities;
- distinguish between scalar and vector quantities;
- calculate magnitude and direction of vectors;
- add and subtract vectors;
- multiply the vector by a scalar;
- work with position vectors in coordinate plane.

Introduction

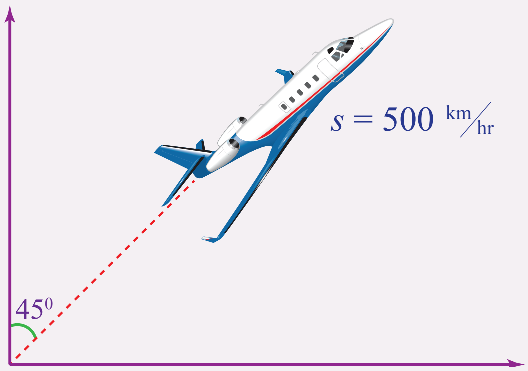
From previous grades, you know about different measurements such as measurement of height, weight, temperature, etc. Such quantities assume real numbers as their measure (with some unit of measurement). For example, the height of a tree is 5 m, the weight of a student is 45 kg, the temperature of a normal person is 36.5°C, etc. However, not all quantities assume only a single real number as their description. There are some quantities, called vectors, whose measures need to involve directions. These will be discussed in this unit.

There are many quantities that require only a number to describe them. For example: What time is it? (9:00 A.M.) How far is Ganta from Monrovia? (201 miles) How fast did you drive the car? (80 km/h) How cold is it? (4°C) What is the area of the rectangle? (6 cm^2) What is the cost of the ticket? (L$10.50) Such quantities as time, distance, speed, temperature, area and amount of money are described by their magnitudes. A quantity which is described by only magnitude is called **scalar**. In fact, the word ***scalar*** is usually used as a synonym for a number.

There are many other quantities that need a direction as well as a magnitude to describe them. For Example: Where is Ganta from Monrovia? (201 miles; Northeast) What force is acting on the car? (1200 lb; down). Such quantities as displacement and force that need a direction as well as magnitude to describe them are called vector quantities (or simply, vectors). The following are more examples of vectors.

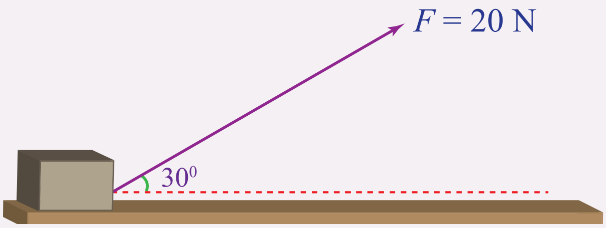
Example 1

An airplane is flying at a speed of 500 km/hr in the direction of N45°E. This is a the velocity of the airplane Velocity is speed and direction of a moving object. It is a vector.



Example 2

A force of 20 N acts to the right at an angle of 30° above the horizontal. Obviously this is also a vector as its magnitude and direction are specified.



The description of some of the quantities given in Activity 1 also involve both magnitude and direction. Try to identify them.

Activity 1

Consider each of the following quantities and identify whether it can be described only by magnitude or whether its description needs direction and magnitude.

(a) Amount of rainfall in mm
(b) Gravity
(c) Volume of a solid figure
(d) Magnetic field
(e) Density
(f) Wind force exerted on tree.
(g) Displacement
(h) Friction force

Now we formally state the definition of a vector.

Definition

Quantities that need both magnitude and direction to describe them are called **vector quantities** (or simply **vectors**).

You may observed in Activity 1 that gravity, magnetic field, wind force, displacement and friction force are examples vectors. See also the following examples.

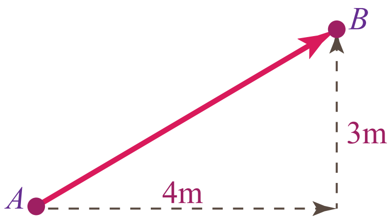
Example 3

The velocity of a car is 80 km/h in the direction of north. This is a vector.

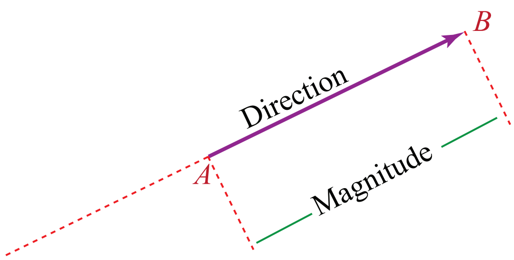
Example 4

Suppose an ant moved 3 m from point A to the East [E] and then 4 m to the North [N] to reach point B. The figure below shows the ant's displacement (from A to B) as a vector.

The direction of move (from point A to B) together with the distance between A and B is called the displacement from A to B. The displacement from point A to B is represented by the directed line segment (arrow) from A to B in the figure above.



A vector is represented by a directed line segment. The direction of the vector is indicated by an arrow pointing from the tail (initial point) to the head (terminal point) as shown in the figure below. If the tail is at point A and the head is at point B, the vector from A to B is written as $\overrightarrow{AB}$.



The length of the directed line segment shows its magnitude and the arrow head points the direction. The vector can also be represented by a single letter of the English alphabet (either small or capital) with arrow above it such as $\vec{V}$ (read as vector v). Sometimes we can also use the small letters in bold to represent vectors.

Example 5

What does each of the vectors in the following figure represent?

Solution

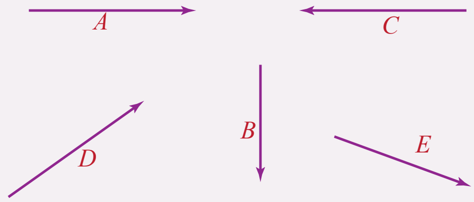
The unit of the magnitude of the vector tells us its type. For instance, N (Newton) is a unit of force and m/sec (meter per second) is a unit of speed.

Therefore, the first vector represents a force of 20N acting at an angle of 30° from the horizontal. (Note that the length of AB is proportional to the magnitude of the force.)

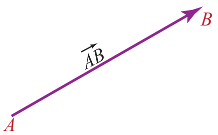
The second vector represents a velocity of 32 m/sec of a moving body in the direction of 60° from the horizontal. (Note that the length of the vector represents the magnitude 32 m/sec of the velocity and is drawn in proportion to it.)

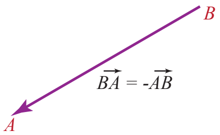
Example 6

The following are examples of vectors. You can determine their magnitudes (lengths) and directions by using ruler and protractor.



The Negative of a vector





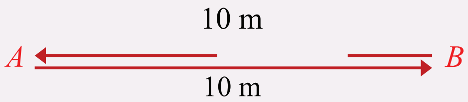
The negative of the vector from A to B is the vector from B to A. It has the same length as $\overrightarrow{AB}$, But the direction is opposite. Thus the negative of the vector and written as $\overrightarrow{BA} = -\overrightarrow{AB}$

The Zero vector

Even though we can say that vectors have length and direction, there is one exception: the ZERO vector. The zero vector **0** is the vector from a point A to the same point A. The magnitude (length) of a zero vector is zero but it does not have a direction.

Example 7

A person moved 10 m to the East from A to B and then 10 m to the West from B to A. Find the resultant displacement.



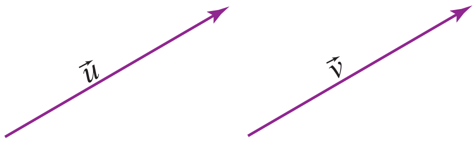
Solution

Here we see that the person ends up at A, hence his displacement is zero. From this we see that if we have $\overrightarrow{AB}$ and $\overrightarrow{BA}$, then the sum of these vectors $\overrightarrow{AB} + \overrightarrow{BA}$ vanishes in the sense that the initial point and the terminal point coincide. Such a vector is zero vector and is denoted by $\vec{0}$ or simply 0. i.e., $\overrightarrow{AB} + \overrightarrow{BA} = 0$. (Recall that $\overrightarrow{BA} = -\overrightarrow{AB}$)

Equality of vectors

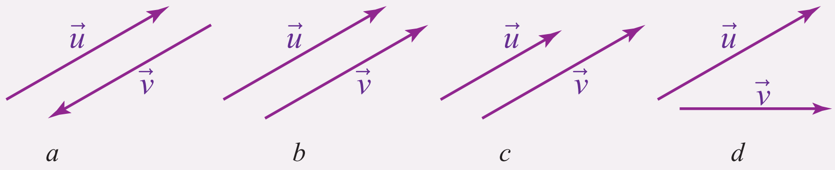
Two vectors are said to be equal, if they have the same length and the same direction.

For example, vectors $\vec{u}$ and $\vec{v}$ are equal vectors, since they have the same length and the same direction.



EXAMPLE 8

Consider each of the following pair of vectors and discuss whether they are equal.



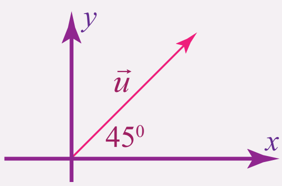
Observe that

1. both vectors have the same length but opposite directions, and hence they are opposite vectors.
2. both vectors have the same direction and equal lengths, and hence they are equal vectors.
3. the pair have the same direction but different lengths, and hence are neither equal nor opposite vectors.
4. the pair have different directions and different lengths and are neither equal nor opposite vectors.

Direction of vectors

The direction of a vector is the angle that is formed by the arrow (that represents the vector) with the horizontal line at its initial point (or with the vertical line in the case of compass directions).

EXAMPLE 9



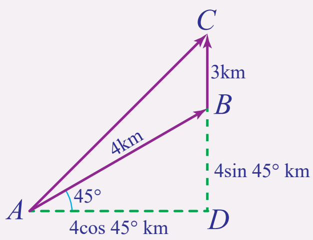
In Figure below, the direction of the vector $\vec{u}$ from the horizontal line at its initial point, as represented below, is 45°. (or N45°E)

EXAMPLE 10

A car Starts at A, travels 4 km north east to B and then 3 km due north to C. Draw a diagram showing the position vectors

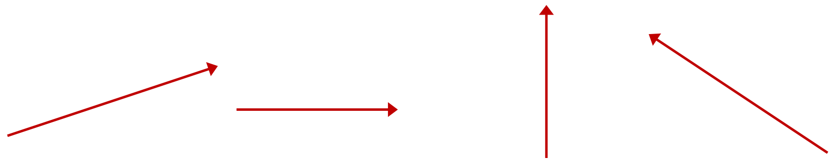
Solution

The point B is 4 km north–east of A and this means that we could also set it by travelling 4 cos 45° km east and 4 sin 45° km north.



Exercises

1. Determine the magnitude and direction of each of the following vectors.

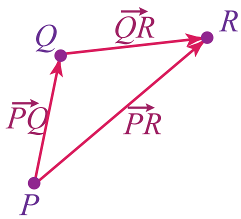


2. Locate each of the following vectors on a coordinate system.
   (a) $\overrightarrow{OP}$ whose length is 3 cm and direction is [N40°E].
   (b) $\overrightarrow{AB}$ whose length is 5 cm and direction is [S45°E].

In this section we will discuss about addition of vectors, subtraction of a vector from another vector and scalar multiplication of a vector (multiplying a vector by a number).

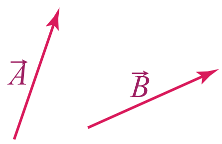
Addition of vectors

Suppose that a particle needs to go from point P to R in the adjacent figure. It can accomplish this by first going to point Q, whose displacement is equal to $\overrightarrow{PQ}$, and then by going to point R, whose displacement is $\overrightarrow{QR}$. The total displacement of the particle is $\overrightarrow{PQ}+\overrightarrow{QR}$. The particle could also make the same displacement by going directly from P to R, whose displacement is $\overrightarrow{PR}$. That is, the displacement $\overrightarrow{PR}$ is the same as $\overrightarrow{PQ}+\overrightarrow{QR}$, i.e., $\overrightarrow{PR}=\overrightarrow{PQ}+\overrightarrow{QR}$. Thus, the vector $\overrightarrow{PR}$ is defined to be the ***sum*** or ***resultant*** of vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$. This called triangular rule of addition.

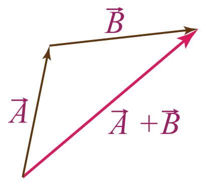


Triangular rule of addition

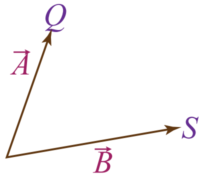
To add two vectors $\vec{A}$ and $\vec{B}$:



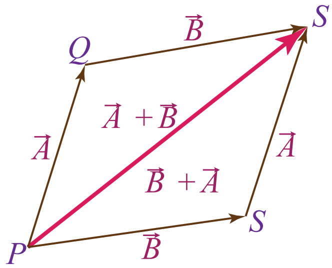
- Place the initial point (tail) of $\vec{B}$ at the terminal (head) of $\vec{A}$,
- The directed line segment from the tail of $\vec{A}$ to the head of $\vec{B}$ is $\vec{A}+\vec{B}$.



Now suppose that the tails (initial points) of two vectors $\vec{A}$ and $\vec{B}$ are at the same point P as shown in the figure below. How do you determine the resultant (sum) of $\vec{A}$ and $\vec{B}$?



Keeping their tails at the same point, let us put the tail of the copy of $\vec{B}$ at the head of $\vec{A}$ (at Q), as shown in the figure below. Also put the tail of the copy of $\vec{A}$ at the head of $\vec{B}$ (at S). These give us the parallelogram $PQTS$ with diagonal PT.

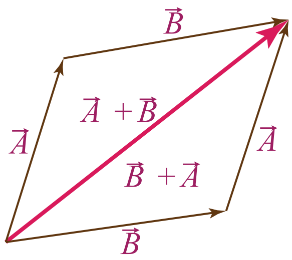


Now by the triangular rule of addition you can see that $\overrightarrow{PT}$ is the resultant (sum) of $\vec{A}$ and $\vec{B}$; i.e., $\vec{A}+\vec{B} = \overrightarrow{PT}$. The same figure shows also that $\vec{B}+\vec{A} = \overrightarrow{PT}$. That is, $\vec{A}+\vec{B} = \overrightarrow{PT} = \vec{B}+\vec{A}$; which means vector addition is commutative.

So, we have established the following result.

Parallelogram rule of addition

Suppose two vectors $\vec{A}$ and $\vec{B}$ have the same initial point. Then, the vectors generate a parallelogram and the directed diagonal from the initial points is the resultant victor $\vec{A}+\vec{B}$.



Moreover, vector addition is commutative; i.e., $\vec{A}+\vec{B} = \vec{B}+\vec{A}$.

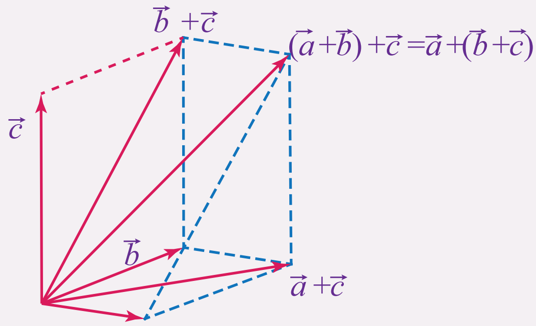
As illustrated in the following example, vector addition is also associative. That is, $(\vec{A}+\vec{B})+\vec{C}=\vec{A}+(\vec{B}+\vec{C})$

Example 11

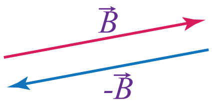
Draw a diagram to show that

$$(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$$

Solution

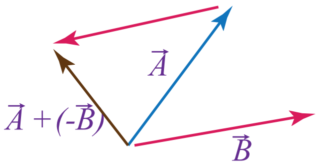


Subtraction of vectors

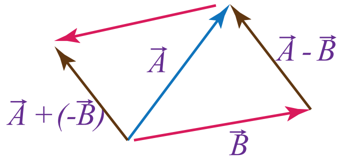


To define subtraction of vectors, recall that if $\vec{B}$ is a vector, $-\vec{B}$ is defined to be the vector with the same magnitude as $\vec{B}$ but **opposite direction**.

By definition, subtracting a vector is adding its negative. That is, $\vec{A}-\vec{B}=\vec{A}+(-\vec{B})$

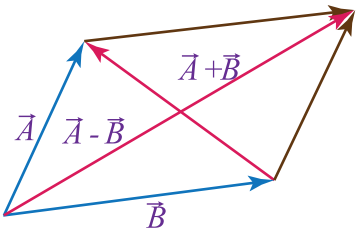


Thus, to show the vector $\vec{A}-\vec{B}$ diagrammatically, put the tail of $-\vec{B}$ at the head of $\vec{A}$ so that, by triangular rule of addition, the vector from the tail of $\vec{A}$ to the head of $-\vec{B}$ is $\vec{A}-\vec{B}$ as in the adjacent figure. In particular, the vector $\vec{A}+(-\vec{B})$ is $\vec{A}-\vec{B}$.



Putting the tails of $\vec{A}$ and $\vec{B}$ at the same point and using the *SSS* postulate of congruency of two triangles, you can show that $\vec{A}+(-\vec{B})$, i.e., $\vec{A}-\vec{B}$ is the vector from the head of $\vec{B}$ to the head of $\vec{A}$

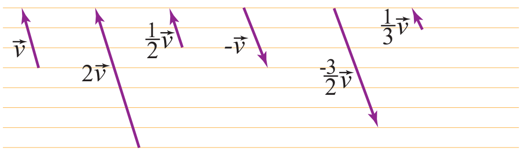
Note that the vectors $\vec{A}+\vec{B}$ and $\vec{A}-\vec{B}$ are both the diagonals of the parallelogram formed by $\vec{A}$ and $\vec{B}$ (see, the figure below).



Scalar multiplication of vectors

Scalar multiplication of a vector is to mean multiplying a vector by a number.

Multiplying a vector by a scalar k gives a vector which is k times as long as the vector. If $k > 0$, then the direction of will be the same as the direction of $\vec{v}$ and if $k < 0$, then the direction of $k\vec{v}$ will be the opposite of the direction of $\vec{v}$. Look at the following examples.



Magnitude of a Vector

1. The magnitude (length) of a vector $\vec{A}$ is denoted by $|\vec{A}|$. or $\|\vec{A}\|$.

   Not that $\|\vec{A}\| \geq 0$ (because length is non-negative).

2. If $k \in \mathbb{R}$, $\|k\vec{A}\| = |k|\,\|\vec{A}\|$, where $|k|$ is absolute value of k.
3. A vector $\vec{B}$ is said to be parallel to $\vec{A}$ if $\vec{B} = k\vec{A}$, for some $k \in \mathbb{R}$. In this case, $\vec{A}$ and $\vec{B}$ have the same direction if $k > 0$; and they have opposite directions if $k < 0$.

Definition

- A vector whose magnitude is 1 is called a unit vector.
- If $\vec{v}$ is a non-zero vector, the unit vector in the direction of $\vec{v}$ is given by $\vec{u} = \frac{1}{|\vec{v}|}\vec{v}$,

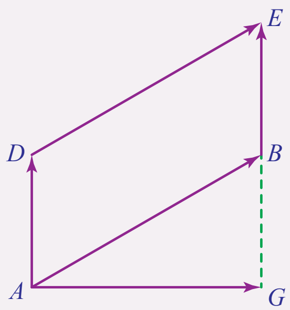
  (Note that the magnitude of $\vec{u}$ is 1 and its direction is the same as the direction of $\vec{v}$).

Example 12

A car starts at A and travels 4 km due north to D, and then 6 km north-east to E.

(a) Are vectors $\overrightarrow{AD}$ and $\overrightarrow{BC}$ equal?

(b) Is it true that $\overrightarrow{AE}$ and $\overrightarrow{AC}$ (in question no.1) are equal?



Solution

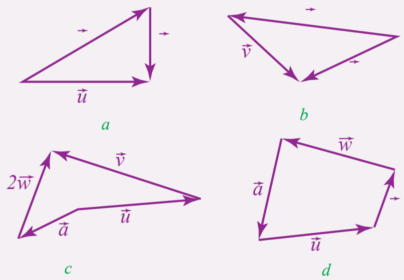
(a) Yes, vectors $\overrightarrow{AD}$ and $\overrightarrow{BC}$ are equal.

(b) Yes, vectors $\overrightarrow{AE}$ and $\overrightarrow{AC}$ (in question no.1) are equal

Therefore, $\angle C = \angle E$

Example 13

Identify the vector in terms of the other vectors in each of the following.



Solution

(a) $\vec{u} - \vec{v} = \vec{a}$

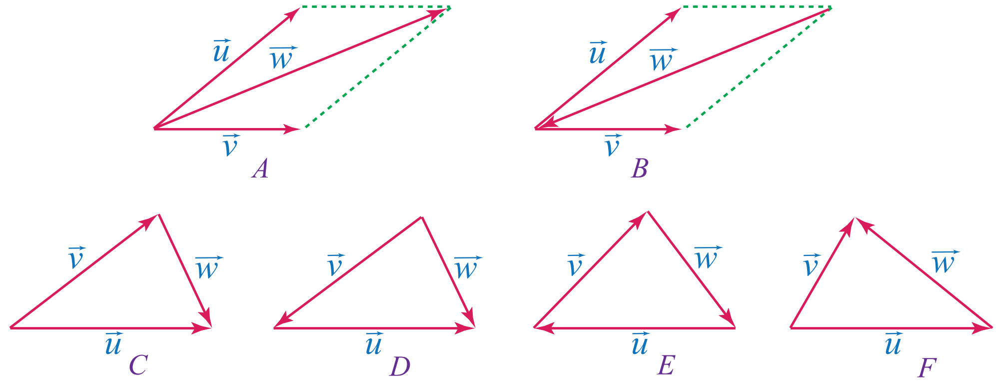
(b) $\vec{a} = \vec{u} - \vec{v}$

(c) $\vec{a} = \vec{u} + \vec{v} - 2\vec{w}$

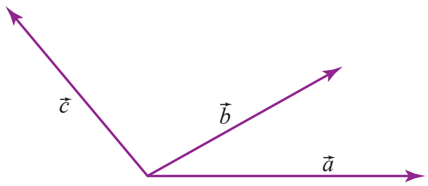
(d) $\vec{a} = -\vec{u} - \vec{v} - \vec{w}$

EXERCISES

1. Identify vector $\vec{w}$ in terms of the vectors $\vec{u}$ and $\vec{v}$

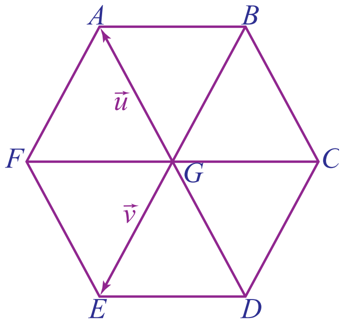


2. Draw a diagram showing $\left(\vec{a}+\vec{b}\right)+\vec{c}=\vec{a}+\left(\vec{b}+\vec{c}\right)$



3. If is a vector of length 2.5 cm, what is the length of the vector $-\frac{7}{2}\vec{v}$

4. *ABCDEF* is a regular hexagon. If $\overrightarrow{GA}=\vec{u}$ and $\overrightarrow{GE}=\vec{v}$, express each of the following in terms of $\vec{u}$ and $\vec{v}$.



(a) $\overrightarrow{AD}$

(b) $\overrightarrow{FE}$

(c) $\overrightarrow{DC}$

(d) $\overrightarrow{AB}$

(e) $\overrightarrow{FC}$

(f) $\overrightarrow{EC}$

(g) $\overrightarrow{BF}$

(h) $\overrightarrow{FD}$

(i) $\overrightarrow{AE}$

5. Let $\vec{v}$ be a non-zero vector and $\vec{u}=\frac{1}{|\vec{v}|}\vec{v}$.

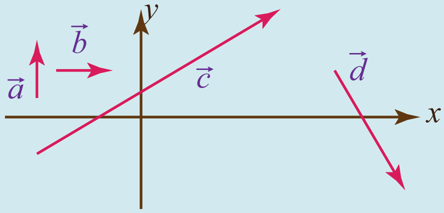
(a) Show that $|\vec{u}|=1$.

(b) Justify that the direction of $\vec{u}$ is the same as the direction of $\vec{v}$.

Up until now, you have used the geometric representation of vectors. Next, we will consider that a vector is given in the *xy*- coordinate plane and discuss expression of a vector using the coordinates of its initial and terminal points.

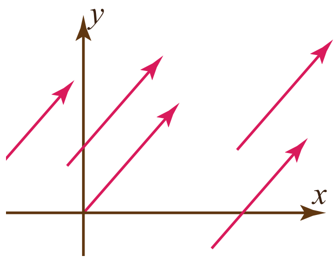
Activity 2

1. Consider the following vectors, which are on the *xy*-coordinate system. Move each vector so that their initial points are at the origin
   (a) How do you differentiate one vector from another?
   (b) The initial point of any of those vectors is (0, 0). How do you express their terminal point?

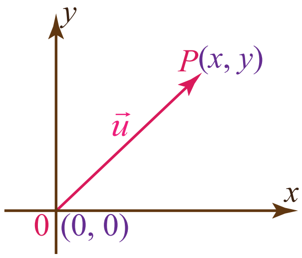


2. If is the vector with initial point A = (1, 2) and the terminal point (3, 4) what will its terminal point be if its initial point is moved to the origin?
3. If is a vector with initial point at origin (0, 0) and terminal point at (2, 5), then how do you expressin terms of the coordinates (2, 0) and (0, 5)?

Notice that the vectors given in the adjacent figure are all equal though they have different initial points. Of these vectors, the one whose initial point is at the origin is called position vector. The position vector represents all those vectors equal to it.



We usually express vectors in component form. This is done by considering the vector with the origin as its initial point and write the coordinates (ordered pair) of its terminal point as a representation of the vector.



For example, in two dimensions, if $\vec{u} = \overrightarrow{OP}$, where O is (0, 0) and P is the point (x, y) then we write $\vec{u}$ = (x, y)

Note

1. Notice that an ordered pair (x, y) represents two notions, namely, a ***point*** in the coordinate plane or a position ***vector***

whose terminal point is (x, y). What it actual stands for is usually clear from the context.

2. It is also common to write a position vector $\vec{u} = (x, y)$ as $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ to distinguish it from coordinate points..

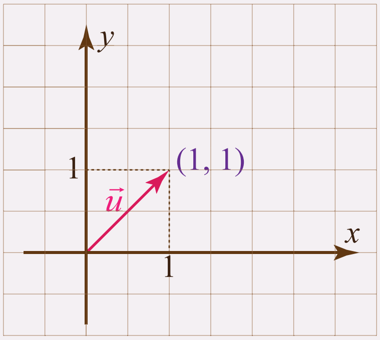
Since the vector $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ has $O(0, 0)$ as its initial point and $P(x, y)$ as its terminal point, its magnitude is $|\vec{u}| = \sqrt{x^2 + y^2}$ which is the length of the line segment from $O(0, 0)$ to $P(x, y)$

Example 14

For the vector given by $\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, its geometric representation is given below. Find the magnitude and direction of the vector.

Solution

From this geometric representation and from the trigonometric identities that you discussed in chapter five, we can determine the direction of the vector.



$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \Rightarrow \tan\theta = \frac{1}{1} = 1.$$

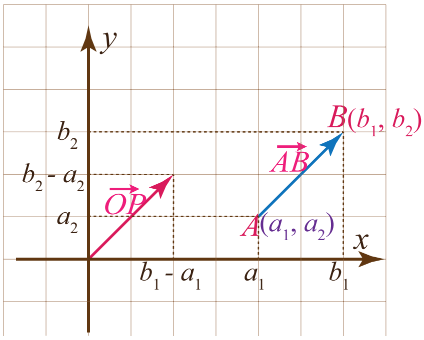
The acute angle whose tangent value is 1 is 45°.

Hence, the direction of the vector is 45°.

The magnitude of the vector is also

$$|\vec{u}| = \sqrt{(1-0)^2 + (1-0)^2} = \sqrt{2}$$

In general, any vectors $\overrightarrow{AB}$, with initial point $A(a_1, a_2)$ and terminal point $B(b_1, b_2)$ is equal to a position vector $\overrightarrow{OP}$ when $\overrightarrow{OP} = \overrightarrow{AB} = \boldsymbol{B} - \boldsymbol{A} =$ $(b_1 - a_1, b_2 - a_2)$ and $\left|\overrightarrow{AB}\right| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}$

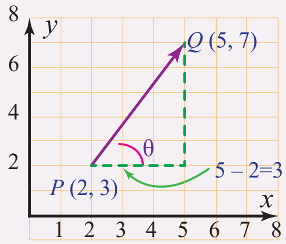


Example 15

Represent the vector whose tail is $P(2, 3)$ and whose head is $Q(5, 7)$ and calculate its magnitude (length).

Solution

Using Pythagorus theorem the magnitude



$$\left|\overrightarrow{PQ}\right| = \sqrt{(5-2)^2 (7-3)^2}$$

$$= \sqrt{3^2 + 4^2}$$

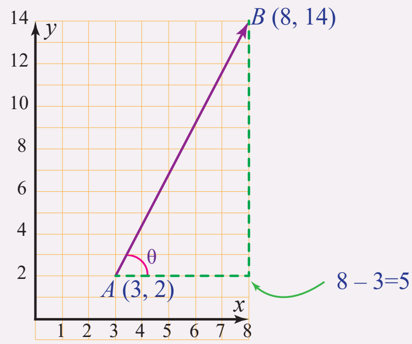
$$= \sqrt{25} = 5$$

In general,

If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are end points of vector $\overrightarrow{PQ}$ or $\vec{v}$ in the plane, the length or magnitude of the vector is given by $\left|\vec{v}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Example 16

Find the length and direction (angle from the horizontal) of the vector , if $\overrightarrow{AB}$ $A(3, 2)$ and $B(8, 14)$.



$$\left|\overrightarrow{AB}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(8-3)^2 + (14-2)^2}$$

$$= \sqrt{5^2 + 12^2}$$

$$= \sqrt{169} = 13$$

$$\tan\theta = \frac{12}{5} = 2.4$$

Therefore, the vector is 13 units long at an angle of 67° to the horizontal.

Example 17

What is the position vector of a vector whose initial point and terminal point are given respectively. Show the corresponding position vector in the x–y coordinate plane.

(a) $A(-5, 1), B(-7, 4)$ (b) $C(7, 1), D(12, 5)$

Solution

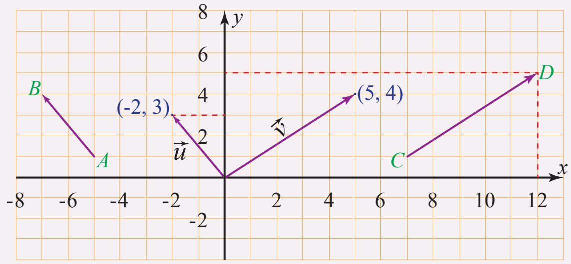
(a) Position vector of $\overrightarrow{AB} = \vec{U} = (-7-(-5), 4-1)$

$$= (-2, 3)$$

(b) Position vector of

$\overrightarrow{CD} = \vec{V} = (12-7, 5-1)$

$= (5, 4)$



EXAMPLE 18

$\overrightarrow{AB}$ is a vector whose initial point A has coordinates (3, 2), and terminal point B. If the position vector of $\overrightarrow{AB}$ is $\vec{v} = (-6, 4)$, what are the coordinates of point B of vector $\overrightarrow{AB}$?

Solution

Let the coordinates of the terminal point B be (x_2, y_2) given the coordinates of A are $(x_1, y_1) = (3, 2)$

Position vector $\vec{v} = (x_2 - x_1, y_2 - y_1) = (-6, 4)$

Therefore, $x_2 - 3 = -6$ and $y_2 - 2 = 4$

$x_2 = -3$ and $y_2 = 6$

Therefore, the coordinates of B are $(-3, 6)$.

Now we consider vector operation algebraically, i.e., when the vectors are position vectors in coordinate plane. To work with vectors algebraically it is better to write the vectors in the coordinate form. So, in the next, we will express a vector $\vec{V}$ in coordinate form as $\vec{V} = (x, y)$.

Vector operations

Let $\vec{A} = (a_1, a_2)$ and $\vec{B} = (b_1, b_2)$ be vectors and $r \in \Re$.

1. **Addition:** $\vec{A} + \vec{B} = (a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$
2. **Subtraction:** $\vec{A} - \vec{B} = (a_1, a_2) - (b_1, b_2) = (a_1 - b_1, a_2 - b_2)$
3. **Scalar multiplication: :** $r\vec{A} = r(a_1, a_2) = (ra_1, ra_2)$

   Note that $\left|r\vec{A}\right| = \sqrt{(ra_1)^2 + (ra_2)^2} = |r|\sqrt{a_1^{\,2} + a_2^{\,2}} = |r|\left|\vec{A}\right|$

Example 19

Given $\vec{u} = (-2, 5)$ and $\vec{v} = (8, -3)$, find

(a) $\vec{u}+\vec{v}$ (c) $\vec{u}-\vec{v}$ (e) $-2\vec{u}$

(b) $\vec{v}-\vec{u}$ (d) $\frac{1}{2}\vec{u}$

Solution

(a) $\vec{u}+\vec{v} = (-2, 5) + (8, -3)$

$= (-2 + 8, 5 + (-3))$

$= (6, 2)$

(b) $\vec{v}-\vec{u} = (8, -3) - (-2, 5) = (8 - (-2), -3 - 5) = (10, -8)$

(c) $\vec{u}-\vec{v} = (-2, 5) - (8, -3)$

$= (-2 - 8, 5 - (-3))$

$= (-10, 8)$

(d) $\frac{1}{2}\vec{u} = \frac{1}{2}(2, 5)$

$= \left(-1, \frac{5}{2}\right)$

(e) $-2\vec{u} = -2(-2, 5)$

$= (-4, -5)$

Example 20

Find the unit vector in the direction of $\vec{v} = (3, 4)$

Solution

Let the unit vector in the direction of $\vec{v}$ be $\vec{u}$

$$\vec{u} = \frac{(3,4)}{|\vec{u}|} = \frac{(3,4)}{\sqrt{3^2+4^2}} = \frac{(3,4)}{\sqrt{25}} = \frac{(3,4)}{5}$$

$$= \frac{1}{5}(3,4) = \left(\frac{3}{5}, \frac{4}{5}\right).$$

Example 21

What is the unit vector in the direction of the position vector $\vec{v} = (0, -9)$

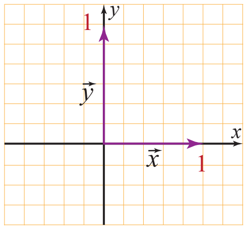
Solution

Let the unit vector in the direction of $\vec{v}$ be $\vec{u}$.

$$w = \frac{\vec{v}}{|\vec{v}|} = \frac{(0,-9)}{\sqrt{0^2+9^2}} = \frac{(0,-9)}{\sqrt{81}} = \frac{1}{9}(0,-9) = (0,-1).$$

The special unit vectors i and j :

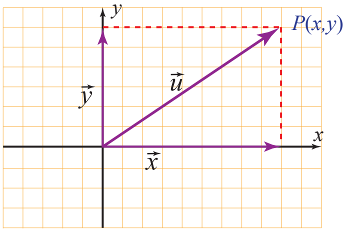
The special unit vectors $\vec{i}=(1,\ 0)$ along the x-axis and $\vec{j}=(0,\ 1)$ along the y-axis respectively, are called **standard unit vectors**.



Any position vector can uniquely be expressed as a linear combination of the unit vectors $\vec{i}$ and $\vec{j}$ in the form $\vec{v}=ai+bj$ where a and b are scalars.

Consider the following position vector $\vec{u}=(x,\ y)$.

Component form of vectors



From the diagram above, you can see that the vector $\vec{u}$ can be expressed as a sum of the vectors $\vec{x}$ and $\vec{y}$ as $\vec{u}=\vec{x}+\vec{y}$ by triangle rule or parallelogram rule for addition of vectors, $\vec{x}=x\vec{i}$ and $\vec{y}=y\vec{j}$ where in which $\vec{i}=\begin{pmatrix}1\\0\end{pmatrix}$ and $\vec{j}=\begin{pmatrix}0\\1\end{pmatrix}$. We say is the component of along the x-axis and is the component of along the y-axis.

$$\vec{u}=\vec{x}+\vec{y}=(x,\ 0)+(0,\ y)=x(1,\ 0)+y(0,\ 1)=x\vec{i}+y\vec{j}$$

EXAMPLE 22

Express each of the following vectors in terms of the unit vectors i and j.

(a) (–2, –5) (b) (–6, 0) (c) (–2, 6)

Solution

(a) $(-2,-5)=-2\vec{i}-5\vec{j}$ (b) $(-6,0)=-6\vec{i}+0\vec{j}$ (c) $(-2,6)=-2\vec{i}+6\vec{j}$

EXAMPLE 23

From the diagram, write the component form of each of the vectors $\vec{u},\vec{v},\vec{w}$ and $\vec{m}$.

Solution

For the vector $\vec{u}$ shown on the diagram, the component in the direction given by the unit vector $\vec{i}$ is 6 and the component in the direction $\vec{j}$ is 5.

Therefore, the position vector in component form is written as $\vec{u}=6\vec{i}+5\vec{j}$ similarly $\vec{v}=5\vec{i}+-4\vec{j}$, $\vec{w}=-5\vec{i}+-6\vec{j}$, $\vec{m}=-5\vec{i}+2\vec{j}$.

Example 24

Express each of the following as a position vector in the coordinate form.

(a) $-2\vec{i}+\vec{j}$ (b) $-6\vec{i}-2\vec{j}$ (c) $0\vec{i}+8\vec{j}$

Solution

(a) $-2\vec{i}+\vec{j}=-2(1,0)+1(0,1)=(-2,0)+(0,1)=(-2,1)$.

(b) $-2\vec{j}=-6(1,0)-2(0,1)=(-6,0)+(0,-2)=(-6,-2)$.

(c) $0\vec{i}+8\vec{j}=0(1,0)+8(0,1)=(0,0)+(0,8)=(0,8)$.

Note

- To add two or more vectors in component form, add the corresponding components.
- To multiply a vector in component form by a scalar, multiply each of the components by the scalar.
- If a vector in component form is, $a\vec{i}+b\vec{j}$ then its magnitude is $\sqrt{a^2+b^2}$. (Pythagoras' theorem).

Example 25

If $\vec{u}=\vec{i}+2\vec{j}$ and $\vec{v}=2\vec{i}+2\vec{j}$, prove that the magnitude of $\vec{u}+\vec{v}$ is 5.

Solution

Let = $\vec{u}+\vec{v}$ = $\vec{w}$

Then $\vec{w}=\left(\vec{i}+2\vec{j}\right)+\left(2\vec{i}+2\vec{j}\right)=\left(\vec{i}+2\vec{i}\right)+\left(2\vec{j}+2\vec{j}\right)$.

Thus $\left|\vec{u}+\vec{v}\right|=\left|\vec{w}\right|=\sqrt{3^2+4^2}=\sqrt{9+16}=\sqrt{25}=5$.

Exercises

1. Draw each of the following vectors in the coordinate plane whose initial point A and terminal point B are given. Then find the magnitude and direction (angle from the horizontal) of each.

   (a) $A=(2, 1)$ and $B=(6, 4)$

   (b) $A=(1, 2)$ and $B=(-2, 6)$

   (c) $A=(-2, -3)$ and $B=(-7, -1)$

   (d) $A=(-2, 3)$ and $B=(-5, -1)$

2. Find the position vector of each of the vectors given in question number 1.
3. Find the magnitude and direction (angle from the horizontal) of each of the following position vectors.

(a) $\vec{u}=(2,\ 2)$	(c) $\vec{w}=(12,\ 4)$	(e) $\vec{n}=(-3,\ 4)$
(b) $\vec{v}=(4,\ 3)$	(d) $\vec{m}=(-5,\ -3)$	(f) $\vec{p}=(4,-3)$

4. The initial point of vector $\overrightarrow{AB}$ is $A(0,\ 1)$. Find the terminal point so that its position vector is $\vec{U}=(4,\ 8)$.
5. The terminal point of vector $\overrightarrow{MN}$ is $M(-2,\ -5)$, find the coordinates of the initial point so that the corresponding position vector is $\vec{U}=(3,\ 10)$.
6. Find the unit vector in the direction of the vector $8\vec{i}-6\vec{j}$.
7. Find the unit vector in the direction of $\vec{v}=5\vec{i}-8\vec{j}$.
8. Find the unit vector in the direction of the vector $\begin{pmatrix}-6\\ 9\end{pmatrix}$.
9. Find the vector of magnitude 4 in the direction of $\vec{v}=4\vec{i}-8\vec{j}$.
10. If $\vec{a}=-2\vec{i}+6\vec{j}$, $\vec{b}=2\vec{i}+5\vec{j}$ and $\vec{c}=2\vec{i}-3\vec{j}$, then calculate:–

(a) $\vec{a}+\vec{b}$	(d) $\vec{a}+\vec{b}+\vec{c}$	(g) $\lvert\vec{a}+\vec{c}\rvert$
(b) $\vec{a}+\vec{c}$	(e) $\vec{a}+2\vec{b}$	(h) $\lvert\vec{a}+\vec{b}\rvert$
(c) $\vec{b}+\vec{c}$	(f) $3\vec{b}-4\vec{a}$	(i) $\lvert 3\vec{a}-\vec{b}\rvert$

Key Terms

- Addition of vectors
- Direction of a vector
- Equality of vectors
- Magnitude(length of) a vector
- Parallelogram law of vector addition
- Parallel vectors
- Position vector
- Zero vector
- Scalar multiplication of a vector
- Scalar quantities
- Subtraction of vectors
- Triangle law of vector addition
- Unit vector
- Vector quantities

Summary

- A quantity which is described by only magnitude is called scalar.
- Quantities that need both magnitude and direction to describe them are called **vector** quantities (or simply vectors).
- A vector is denoted by a directed arrow. Its length is called the magnitude. The direction it points is called the direction of the vector.
- A vector is represented by an arrow ($\overrightarrow{OP}$); the point O is called the initial point and P is called the terminal point. Sometimes, vectors are represented by using letters or a letter with a bar over it such as $\vec{u}$, $\vec{v}$, etc.
- The magnitude of a velocity is the speed; the magnitude of a displacement is distance. Thus, speed and distance are scalar quantities
- A magnitude is always a positive number.
- Vectors can be described geometrically or algebraically: geometrically as a directed arrow and algebraically as a column vector.
- Two vectors are said to be equal if they have the same magnitude and the same direction.
- If two vectors have same or opposite directions then they are parallel. Indeed, two vectors are parallel if one is a scalar multiple of the other.
- For any two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$, $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ (the Triangle law)
- A vector that has no magnitude and direction is called a zero vector or null vector.
- The diagonal of a parallelogram is the sum of the side vectors. This is called the Parallelogram Law.
- Subtraction of vectors $\overrightarrow{AE}$ and $\overrightarrow{AC}$, given as $\overrightarrow{AE}-\overrightarrow{AC}=\overrightarrow{CE}$ is the same as $\overrightarrow{AC}+\overrightarrow{CE}=\overrightarrow{AE}$
- Multiplying a vector by a scalar k either enlarges or shortens the vector. If $|k| > 1$, it enlarges the vector and if $0 < |k| < 1$ it shortens the vector. If $k > 0$, the direction of the vector is unchanged; multiplying a vector by $k< 0$ changes the direction of the vector into the opposite direction.
- If the initial and terminal points of a vector are (x_1, y_1) and (x_2, y_2) then its position vector can be calculated as $P = (x_2 - x_1, y_2 - y_1)$ and is denoted by $P=\begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$.

- Any vector $\vec{A} = (x, y)$ can be written as $\vec{A} = x\vec{i} + y\vec{j}$, where $\vec{i} = (1, 0)$ and $\vec{j} = (0, 1)$.

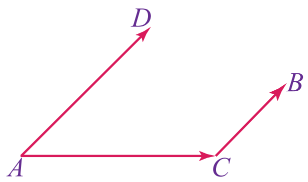
Exercises

1. Describe what is meant by scalar and vector quantities.
2. Write down some examples of scalar and vector quantities.
3. How do you represent a vector geometrically?
4. Sketch a vector whose magnitude is 3 cm in the direction of

   (a) East (b) North 30° East
5. Sketch a vector of length 5 cm whose direction is

   (a) North 45° East (b) West (c) South 20° East
6. From figure below, give a single vector to represent the following

   (a) $\overrightarrow{AC} + \overrightarrow{CB}$ (b) $\overrightarrow{AD} + \overrightarrow{AC}$



7. If $\vec{u} = 20$ m due North and $\vec{v} = 10$ m at 30 degrees E of N, find $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$.
8. When are two vectors parallel?
9. Show, by a diagram, that the sum of two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ is $\overrightarrow{AC}$.
10. If the magnitude of $\overrightarrow{AC}$ is 4 cm, find the magnitude of

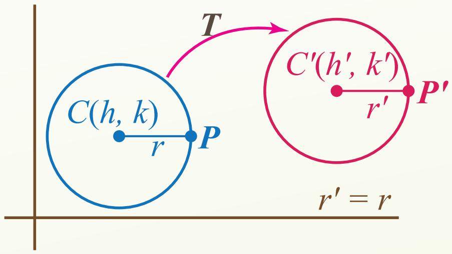
    (a) $3\overrightarrow{AC}$ (b) $\frac{1}{4}\overrightarrow{AC}$ (c) $-\overrightarrow{AC}$
11. If A is the point (4, –2) and B is the point (–3, –6), what is the position vector of B relative to A?
12. The position vector of P (relative to the origin) is $\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$. If the magnitude of $\overrightarrow{OP}$ is 5 units, find the set of all possible values of $\begin{pmatrix} x \\ y \end{pmatrix}$, with $x, y \in \mathbb{Z}$.

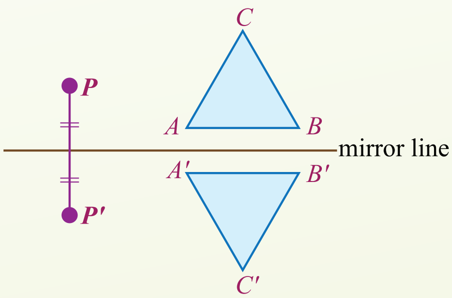
M10CH10

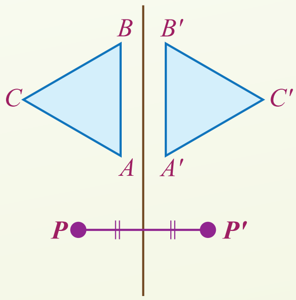
CHAPTER 10

RIGID MOTION

Chapter Contents







Chapter Outcomes

Learners are able to apply concepts to rigid motion by drawing its image using the methods of (i) translation, (ii) reflection ; and determine its symmetry.

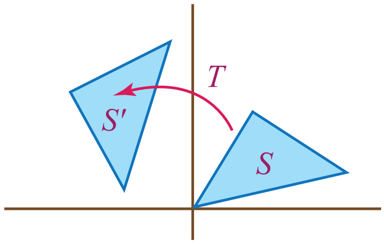
Upon completion of this chapter, you will be able to:

- discuss and draw rigid motion;
- draw and translate images to other position;
- identify and explain reflection of object in the mirror line;
- construct symmetry object.

Introduction

This section deals with some special types of transformation. Transformation is a change of something to some other thing. In mathematics transformation is a function that moves (takes) plane figures from its position to some other location in the plane. There are many versions of transformations, such as, those preserving shape and size, those stretching or shrinking, those preserving orientation (direction) or changing orientation, and any combination of these. In this section, we will consider transformations that changes position and/or orientation of plane figures without changing their shape and size.

A rigid motion is a transformation which moves an object in a plane to a new position while keeping its shape and size constant. A transformation is generally defined by a rule. That is, given an object $\boldsymbol{S}$ (or set of points) in the coordinate plane, a transformation of $\boldsymbol{S}$, denoted by T, is a rule that moves (takes) every point in $\boldsymbol{S}$ to another object (set of points) $\boldsymbol{S'}$ in the plane. In this case we may write,



$$T(\boldsymbol{S}) = \boldsymbol{S'};$$

and say that $\mathbf{S'}$ is the image of $\boldsymbol{S}$ under T.

Also, if $P \in \boldsymbol{S}$ and $P' \in \boldsymbol{S'}$ such that $T(P) = P'$, then P' is said to be the image of P under T. Now, we state the formal definition of a rigid motion.

Definition

Rigid Motion

A transformation T is said to be a rigid motion (rigid transformation) if it preserves distance. That is, for every two points A and B, if $T(A)=A'$ and $T(B) = B'$, then the distance between A' and B' is equal to the distance between A and B ($A'B' = AB$).

Consequently, a rigid motion preserves also angles. For example, an identity transformation is a rigid motion. A transformation is said to be an identity transformation, if the image of every point is itself.

Note

Rigid motion carries any plane figure to a congruent plane figure, i.e., it carries triangles to congruent triangles, rectangles to congruent rectangles, etc.

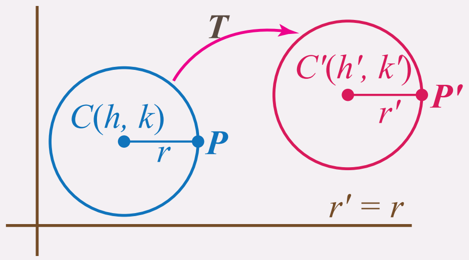
Example 1

Identity transformation is a rigid motion.

Example 2

Given a circle in the coordinate plane with center $C(h, k)$ and radius r, suppose T is a transformation such that $T(C) = C'(h', k')$. If T is a rigid motion, describe the image of the circle. If the image is a circle, what is its radius?

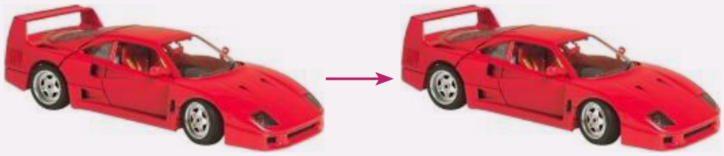
Solution



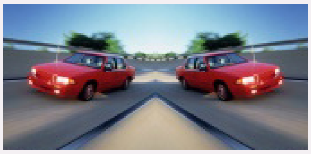
For every point P on the circle, $P' = T(P)$ is a point on the image and $C'P' = CP = r$ since T preserves distance. Therefore, the image of points on the circle are points whose distance from C' is r. Therefore, the image of the circle under T is a circle centered at $C'(h', k')$ with the same radius r.

There are various types of rigid motions. In the next sub-units, we will discuss two types of rigid motions called **translations** and **reflections**.

Translations



Reflections



Exercises

1. In which of the following conditions does the shape or size or both of the object change. Is the change a rigid motion?
   (a) When a rubber is stretched.
   (b) When a commercial jet flies from place to place at a specific time.
   (c) When the earth rotates about its axis.
   (d) When you see your image in a plane mirror.
   (e) When you draw the map of your school compound.
2. Let T be a mapping of the plane onto itself given by $T((x, y)) = (x + 1, -y)$.
   For example, $T((4, 3)) = (4 + 1, -3) = (5, -3)$.
   If $A = (0, 1)$, $B = (-3, 2)$ and $C = (2, 0)$, find the coordinates of the image of A, B and C.

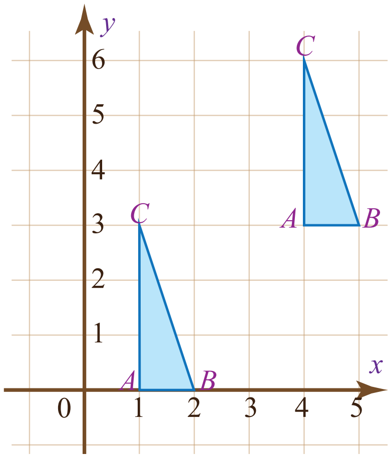
Find the image of ΔABC under T. Is ΔABC congruent to its image? So, is T a rigid motion or not?

3. Suppose T is a mapping of the plane onto itself which sends point P to point P' Let $A = (2, -3)$ and $B = (5, 4)$. Compare the lengths of AB and $A'B'$ and so determine whether T is a rigid motion or not, when

   (a) $T((x, y)) = (x, 0)$
   (b) $T((x, y)) = (x + 1, y - 3)$
   (c) $T((x, y)) = (x, -y)$
   (d) $T((x, y)) = \left(\frac{1}{2}x, 2y\right)$

A translation is a rigid transformation in which every point of a figure is moved along the same direction through the same distance.

For instance, when ΔABC is transformed to $\Delta A'B'C'$, AB and $A'B'$ are parallel to the x-axis, and AC and $A'C'$ are parallel to the y-axis. Moreover, ΔABC and $\Delta A'B'C'$ have the same orientation. i.e., the way they face is the same. This type of transformation is said to be a translation.



DEFINITION

Translation

A transformation T is said to be a **translation** if every point $P = (x, y)$ of a plane is moved the same distance in the direction of a given vector $\vec{u} = (h, k)$. That is,

$$T(P) = P + \vec{u}$$

Or $T(x, y) = (x, y) + (h, k) = (x + h, y + k)$.

In this case, $\vec{u} = (h, k)$ is called a **translation vector**.

Thus, if $\vec{u} = (h, k)$ is translation vector, then the image of a point $P = (x, y)$ is $P' = (x + h, y + k)$. On the other hand, if a point $P = (x, y)$ is translated to $P' = (x', y')$, then the translation vector is

$$\vec{u} = \overrightarrow{PP'} = P' - P = (x' - x, y' - y).$$

Example 3

Let T be a translation that takes the origin to (1, 2). Determine the translation vector and find the images of the following points.

(a) (2, –1) (b) (–3, 5) (c) (1, 2)

Solution

$T((0, 0)) = (1, 2) \Rightarrow \boldsymbol{u} = (1, 2)$ is the translation vector.

$\Rightarrow x \mapsto x + 1$ and $y \mapsto y + 2$. Thus,

(a) $T((2, -1)) = (2 + 1, -1 + 2) = (3, 1)$

(b) $T((-3, 5)) = (-3 + 1, 5 + 2) = (-2, 7)$

(c) $T((1, 2)) = (1 + 1, 2 + 2) = (2, 4)$.

Example 4

Let the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ be translated by the vector $\boldsymbol{u} = (h, k)$. Show that $\left|\overrightarrow{PQ}\right| = \left|\overrightarrow{P'Q'}\right|$.

Solution

Clearly $P' = (x_1 + h, y_1 + k)$ and $Q' = (x_2 + h, y_2 + k)$.

Then, $\left|\overrightarrow{P'Q'}\right| = \sqrt{(x_2 + h - x_1 - h)^2 + (y_2 + k - y_1 - k)^2}$

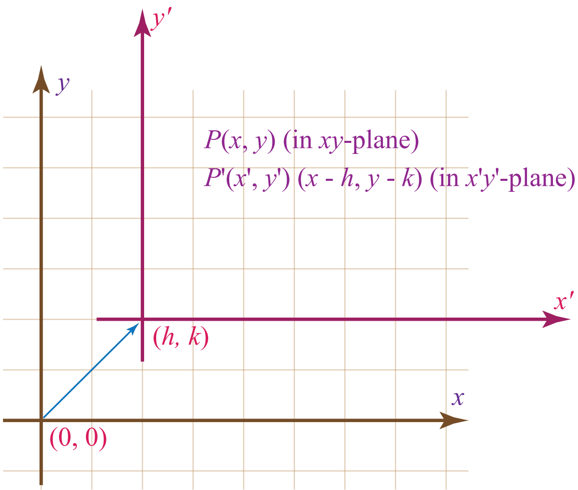
$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \overrightarrow{PQ}.$$

The above example shows that a translation is a rigid motion. You can state a translation formula in terms of coordinates as follows:

(i) If (h, k) is a the translation vector, then

A. The origin is translated to (h, k) i.e., $(0, 0) \to (h, k)$

B. The point $P(x, y)$ is translated to $P'(x + h, y + k)$.



(ii) If the translation vector is $\overrightarrow{AB}$ where $A = (a, b)$ and $B = (c, d)$, then

A. The origin is translated to $(c - a, d - b)$, and

B. The point $P(x, y)$ is translated to $(x + c - a, y + d - b)$.

Example 5

If a translation T takes the origin to $P'(1, 2)$, then $T(x, y) = (x + 1, y + 2)$ and $T(-2, 3) = (-2 + 1, 3 + 2) = (-1, 5)$.

Example 6

If a translation T takes the origin to $(-1, 1)$, then find

(a) The images of the points $P(1, 3)$ and $Q(-3, 6)$.

(b) The image of the triangle with vertices $A(2, -2)$, $B(-3, 2)$ and $C(4, 1)$.

(c) The equation of the image for the parabola whose equation is $y = 2x^2 + 3$.

Solution

(a) The image of the point $P(1, 3)$ is $T(1, 3) = (1 + (-1), 3 + 1) = (0, 4)$.

The image of the point $Q(-3, 6)$ is $T(-3, 6) = (-3-1, 6 +1) = (-4, 7)$.

(b) $T(2, -2) = (2 + (-1), -2 + 1) = (1,-1)$

$T(-3, 2) = (-3 + (-1), 2 + 1) = (-4, 3)$

$T(4, 1) = (4 + (-1), 1 + 1) = (3, 2)$

Thus, $A' = (1, -1)$, $B' = (-4, 3)$ and $C' = (3, 2)$.

The image of ΔABC is $\Delta A'B'C'$.

(c) The image of (x, y) under T is $T(x, y) = (x - 1, y + 1)$.

That is, T changes (maps) x to $x - 1$ and y to $y + 1$.

Thus, the image of $y = 2x^2 + 3$ is the parabola $y + 1 = 2(x - 1)^2 + 3$ or $y = 2x^2 - 4x + 4$.

Example 7

If a translation T takes the point $(-1, 3)$ to the point $(4, 2)$, then find the images of the following lines under the translation T.

(a) $\ell : y = 2x - 3$

(b) $\ell : 5y + x = 1$

Solution

The translation vector is $(h, k) = (4 - (-1), 2 - 3) = (5, -1)$. Thus, the point (x, y) is translated to the point $(x + 5, y - 1)$. A translation maps lines onto parallel lines. Let ℓ' be the image of ℓ under T. Then,

(a) $\ell' : y -(-1) = 2(x- 5) - 3$

$\Rightarrow \ell : y = 2x- 14$

(b) $\ell' : 5(y+ 1) + (x- 5) =1$

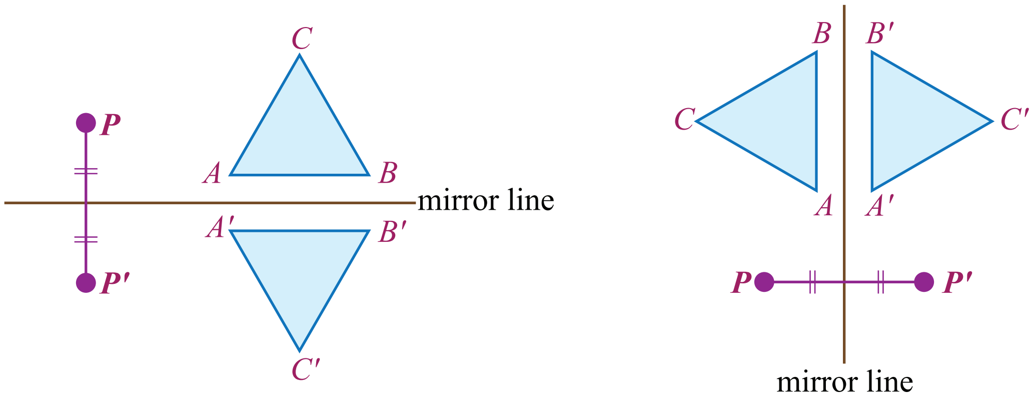
$\Rightarrow \ell' : 5y + x = 1 \Rightarrow \ell' 3= \ell$. Explain!

Exercises

1. If a translation T takes the origin to the point (–3, 2), find the image of the rectangle $ABCD$ with vertices $A(3, 1)$, $B(5, 1)$, $C(5, 4)$ and $D(3, 4)$.
2. Triangle ABC is transformed into triangle $A'B'C'$ by the translation vector (4, 3). If $A = (2, 1)$, $B = (3, 5)$ and $C = (–1, –2)$, find the coordinates of A', B' and C'.
3. Quadrilateral $ABCD$ is transformed into $A'B'C'D'$ by a translation vector (3, –2). If $A = (1, 2)$, $B = (3, 4)$, $C = (7, 4)$ and $D = (2, 5)$, then find A', B', C' and D' and draw the quadrilaterals $ABCD$ and $A'B'C'D'$ on graph paper.
4. What is the image of a circle under a translation?
5. Find the equation of the image of the parabola $y^2 = 3x^2 - 5$ when translated by the vector $\overrightarrow{PQ}$, where $P = (1, –1)$ and $Q = (–4, 3)$.
6. If a translation T takes (2, –5) to (–2, 1), find the image of the line $\ell: 2x - 3y = 7$.

In the previous section, you have seen a translation which moves a given object along a given direction while preserving size and shape of the object. Another rigid motion is a **reflection**.

A reflection fixes a **mirror line** in the plane and flips each point from one side of the mirror line to the point on the other side at the same distance from the line. For instance, the following figure shows the reflection of point P to P' and ΔABC to $\Delta A'B'C'$ using a horizontal and vertical mirror lines.

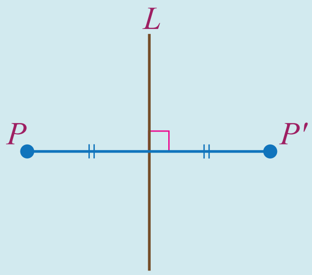


Every reflection has a mirror line, also called line of reflection. It is helpful to think of the mirror line as an actual mirror and the image of the object as its image in the mirror. The formal definition of reflection is as follows.

DEFINITION

Reflection

Let L be a fixed line in the plane. A reflection M about the line L is a transformation of the plane onto itself which carries each point P of the plane into the point P' of the plane such that L is the **perpendicular bisector of** PP'.



The line L is said to be the **line of reflection** or **axis of reflection**.

Note

1. L is a perpendicular bisector of PP' means the following: If $P = (x, y)$ and $P' = (x', y')$, then
   (i) If the slope of L is m, then the slope of PP' is $\frac{-1}{m}$, when $m \neq 0$. (If so, $L \perp PP'$); and
   (ii) The mid-point of PP' is on L ; i.e., $\left(\frac{x+x'}{2}, \frac{y+y'}{2}\right)$ is on L
2. Every point on the line of reflection is its own image.

Notation: The reflection of point P about the line L, is denoted by $M(P)$, i.e. $P' = M(P)$. *(The reflection M(P) may also read as 'mirror image of P'.)*

Reflection has the following properties:

1. A reflection about a line L has the property that, if for two points P and Q in the plane, $P = Q$, then $M(P) = M(Q)$. Hence, reflection is a function from the set of points in the plane into the set of points in the plane.
2. A reflection about a line L maps distinct points to distinct points, i.e., if $P \neq Q$, then $M(P) \neq M(Q)$. Equivalently, it has the property that if, for two points P, Q in the plane, $M(P) = M(Q)$, then $P = Q$. Thus, reflection is a one-to-one mapping.

Theorem

A reflection M is a rigid motion. That is, if $P' = M(P)$ and $Q' = M(Q)$, then $PQ = P'Q'$.

We now consider reflections with respect to the x- and y- axes and the lines $y = mx + b$.

A. Reflection about horizontal and vertical lines

You can easily construct the reflection of a point about x-axes and y-axes. In particular, observe the following:

Suppose $P(x, y)$ is any point on the coordinate plane. Then–

(i) The reflection of P about **x-axis** is $M(x, y) = (x, -y)$, i.e., x unchanged, but $y \mapsto -y$.

(ii) The reflection of P about **y-axis** is $M(x, y) = (-x, y)$, i.e., $x \mapsto -x$, but y unchanged .

EXAMPLE 8

Find the image of the parabola $y = 2x^2 + 1$ under the reflection about

(a) x-axis (b) y-axis

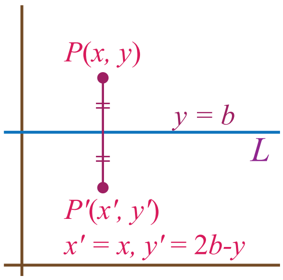
Solution

(a) Under a reflection about x-axis, x is unchanged, but $y \mapsto -y$.
Thus, $y = 2x^2 + 1$ becomes $-y = 2x^2 + 1$ or $y = -2x^2 - 1$.
Therefore, the image of $y = 2x^2 + 1$ under the reflection about x-axis is $y = -2x^2 - 1$.

(b) Under a reflection about y-axis, $x \mapsto -x$ but y unchanged.
Thus, under the reflection about y-axis, the image of $y = 2x^2 + 1$ is $y = 2(-x)^2 + 1 = 2x^2 + 1$ itself.

A reflection about any other horizontal line or vertical line can be constructed in similar manner as the reflection about x-axis or y-axis, respectively. In particular, note the followings.

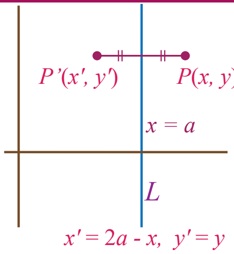
1. If the line of reflection L is horizontal line given by $y = b$, then PP' is vertical (see the adjacent figure). Hence, $x' = x$ and the midpoint of PP' is on L implies $\dfrac{y + y'}{2} = b \Rightarrow y' = 2b - y$.
Thus, the reflection M of any point $P(x, y)$ about the horizontal line $y = b$ is $P'(x, 2b - y)$.
That is, $M(x, y) = (x, 2b - y)$ about the horizontal line $y = b$.



2. Similarly, if the line of reflection L is a **vertical line** given by $x = a$, then PP' is horizontal (see, the adjacent figure). Hence, $y' = y$; and the midpoint of PP' is on L implies $\dfrac{x + x'}{2} = a \Rightarrow x' = 2a - x$.

Thus, the reflection M of any point $P(x, y)$ about a vertical line $x = a$ is $P'(2a - x, y)$.

That is, $M(x, y) = (2a - x, y)$ about the vertical line $x = a$.



EXAMPLE 9

Find the image of (–1, 5) when reflected about the lines

(a) $y = -1$ (b) $x = 1$

Solution

Note that $y = -1$ is a horizontal line ($b = -1$) and $x = 1$ is a vertical line ($a = 1$). Therefore, using the above formula we get the following:

(a) The image of the point (–1, 5) when reflected about the line $y = -1$ is (–1, –7).

(b) The image of the point (–1, 5) when reflected about the line $x = 1$ is (3, 5).

Next we discuss reflection about lines which are not horizontal and not vertical.

B. Reflection in the line *y* = *mx*, where *m* = tan *q*

Let ℓ be a line passing through the origin and making an angle θ with the positive x-axis. Then the slope of ℓ is given by $m = \tan\theta$ and its equation is $y = mx$. *See* (a) below.

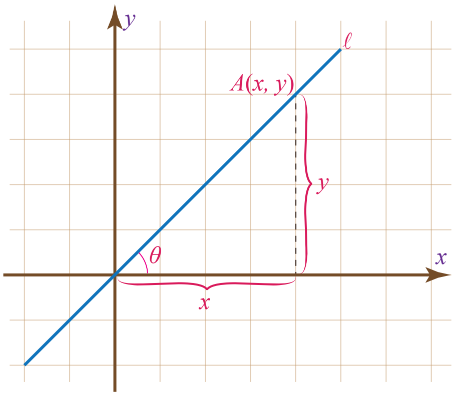


Figure 1.

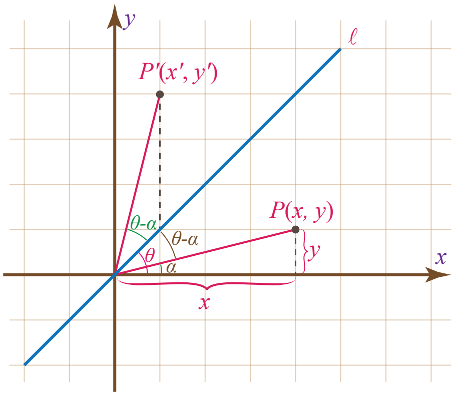


Figure 2.

You will now find the image of a point $P(x, y)$ when it is reflected about this line (*See* Figure 2) above.

Let $P'(x', y')$ be the image of $P(x, y)$.

The coordinates of P are:

$$x = r\cos\alpha \text{ and } y = r\sin\alpha$$

The coordinates of P' are:

$x' = r\cos(2\theta - \alpha)$ and $y' = r\sin(2\theta - \alpha)$

Expanding $\cos(2\theta - \alpha)$ and $\sin(2\theta - \alpha)$,

Now, use the following trigonometric identities which you will learn in a unit of trigonometry:

1. Sine of the sum and the difference
   - $\sin(x + y) = \sin x \cos y + \cos x \sin y$
   - $\sin(x - y) = \sin x \cos y - \cos x \sin y$
2. Cosine of the sum and difference
   - $\cos(x + y) = \cos x \cos y - \sin x \sin y$
   - $\cos(x - y) = \cos x \cos y + \sin x \sin y$

Using these trigonometric identities, you obtain:

$$x' = r\left[\cos\ 2\theta\ \cos\alpha + \sin 2\theta \sin\alpha\right] = x\cos 2\theta + y\sin 2\theta,$$

$$= x\cos 2\theta + y\sin 2\theta, \text{ and}$$

$$y' = r[\sin 2\theta\cos\alpha - \sin\alpha\cos 2\theta] = (r\cos\alpha)\sin 2\theta - (r\sin\alpha)\cos 2\theta,$$

$$= x\sin\ 2\theta - y\cos 2\theta$$

Thus, the coordinates of $P'(x', y')$, the image of the point $P(x, y)$ when reflected about the line $y = mx$ is:

$x' = x\cos 2\theta + y\sin 2\theta$

$y' = x\sin 2\theta - y\cos 2\theta$

where θ is the angle of inclination of the line $\ell: y = mx$.

Based on the value of θ, you will have the following four special cases:

1. When $\theta = 0$, you will have reflection in the x-axis. Thus, (x, y) is mapped to $(x, -y)$ as discussed above.
2. When $\theta = \frac{\pi}{4}$, you will have reflection about the line $y = x$ and hence (x, y) is mapped to (y, x).
3. When $\theta = \frac{\pi}{2}$, you will have reflection in the y-axis and (x, y) is mapped to $(-x, y)$ as discussed above.
4. When $\theta = \frac{3\pi}{4}$, you will have reflection about the line $y = -x$ and (x, y) is mapped to $(-y, -x)$.

Example 10

Find the images of the points (3, 2), (0, 1) and (–5, 7) when reflected about the line $y = mx$, where $m = \tan\theta$ and $\theta = \dfrac{\pi}{4}$

Solution

This is actually a reflection about the line $y = x$. Thus, the images of (3, 2), (0, 1) and (–5, 7) are (2, 3), (1, 0) and (7, –5), respectively.

Example 11

Find the images of the points $P(3, 2)$, $Q(0, 1)$ and $R(-5, 7)$ when reflected about the line $y = \dfrac{1}{\sqrt{3}}x$.

Solution

Since $\tan\theta = \dfrac{1}{\sqrt{3}}$, you have $\theta = \dfrac{\pi}{6}$. Thus, if $P'(x', y')$ is the image of P, then

$$x' = x\cos 2\theta + y\sin 2\theta = 3\cos\left(\frac{\pi}{3}\right) + 2\sin\left(\frac{\pi}{3}\right) = 3\times\frac{1}{2} + 2\times\frac{\sqrt{3}}{2} = \frac{3+2\sqrt{3}}{2}$$

$$y' = x\sin 2\theta - y\cos 2\theta = 3\sin\left(\frac{\pi}{3}\right) - 2\cos\left(\frac{\pi}{3}\right) = 3\left(\frac{\sqrt{3}}{2}\right) - 2\times\left(\frac{1}{2}\right) = \frac{3\sqrt{3}}{2} - 1$$

Hence, the image of P (3, 2) is $P'\left(\dfrac{3+2\sqrt{3}}{2}, \dfrac{3\sqrt{3}}{2} - 1\right)$.

Similarly, you can show that the images of $Q(0, 1)$ and $R(-5, 7)$ are $Q'\left(\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}\right)$ and $R'\left(\dfrac{-5+7\sqrt{3}}{2}, \dfrac{-5\sqrt{3}-7}{2}\right)$, respectively.

Example 12

Find the image of $A = (1,-2)$ after it has been reflected in the line $y = 2x$.

Solution

$y = 2x \Rightarrow y = (\tan\theta)x \Rightarrow \theta = \tan^{-1}(2)$.

But, from trigonometry, you have

$$\sin\theta = \frac{2}{\sqrt{5}} \text{ and } \cos\theta = \frac{1}{\sqrt{5}} \Rightarrow \cos(2\theta) = \cos^2\theta - \sin^2\theta = \frac{1}{5} - \frac{4}{5} = -\frac{3}{5},$$

$$\sin(2\theta) = 2\sin\theta\cos\theta = \frac{4}{5} \Rightarrow x' = -\frac{3}{5}x + \frac{4}{5}y \text{ and } y' = \frac{4}{5}x + \frac{3}{5}y$$

$$\Rightarrow M((1, -2)) = \left(-\frac{11}{5}, -\frac{2}{5}\right).$$

Note

1. If a line ℓ' is perpendicular to the axis of reflection L, then L' is its own image.
2. If ℓ' is a line parallel to the line of reflection L, to find the image of L' when reflected about L, we follow the following steps.

Step a: Choose any point P on ℓ'

Step b: Find the image of P, $M(P) = P'$

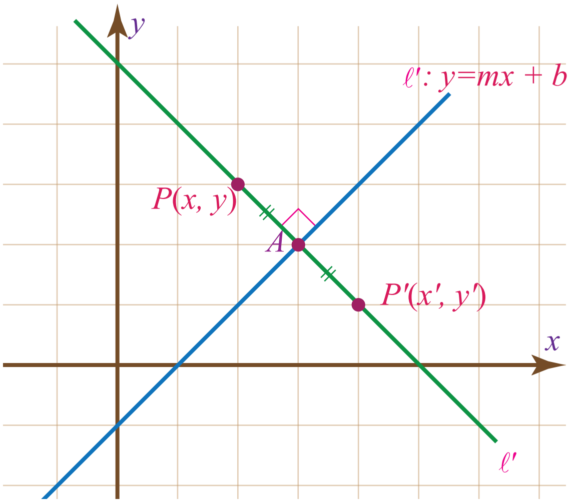
Step c: Find the equation of ℓ', which is the line passing through P' with slope equal to the slope of ℓ.

C. Reflection in the line $y = mx + b$

Let $\ell : y = mx + b$ be the line of reflection, where $m \in \mathbb{R} \backslash \{0\}$.

Let $P(x, y)$ be a point in the plane, not on ℓ.

Let $P'(x', y')$ be the image of $P(x, y)$ when reflected about the line ℓ.



Let ℓ' be the line passing through the points $P(x, y)$ and $P'(x', y')$. Then, ℓ' is perpendicular to ℓ, since ℓ is perpendicular to PP'. Since the slope of ℓ is m, the slope of ℓ' is $-\frac{1}{m}$. Thus, one can determine the equation of the line ℓ'. If A is the point of intersection of ℓ and ℓ', taking A as the midpoint of PP', we can find the coordinates of P'.

Thus, to find the image of a point $P(x, y)$ when reflected about a line ℓ, we follow the following four steps.

Step 1: Find the slope of the line ℓ, say m.

Step 2: Find the equation of the line ℓ', which passes through the point $P(x, y)$ and has slope $-\frac{1}{m}$.

Step 3: Find the point of intersection A of ℓ and ℓ' which serves as the midpoint of PP'.

Step 4: Using A as the mid-point of PP', find the coordinates of P'.

Example 13

Find images of the following lines after reflection in the line $y = 2x - 3$.

(a) $2y + x = 1$ (b) $y = 2x + 1$ (c) $y = 3x + 4$

Solution

(a) The image of ℓ: $2y + x = 1$ is itself. Explain!

(b) ℓ: $y = 2x + 1$ is parallel to the reflecting axis.

Hence $\ell' : y = 2x + b$. We need to determine b.

Let (a, b) be any point on ℓ, say $(0, 1)$, so that its image lies on ℓ'.

By the above reflecting procedure,

$$M((0, 1)) = (a', b') \Rightarrow \frac{b'-1}{a'-0} = -\frac{1}{2} \Rightarrow a' = -2b' + 2.$$

Also, the midpoint of $(0, 1)$ and (a', b') which is $\left(\frac{a'}{2}, \frac{b'+1}{2}\right)$ lies on the reflecting axis

$$\Rightarrow \frac{b'+1}{2} = 2\left(\frac{a'}{2}\right) - 3 \Rightarrow a' = \frac{b'}{2} + \frac{7}{2}.$$

But $a' = -2b' + 2 \Rightarrow 2b' + 2 = \frac{b'}{2} + \frac{7}{2}$

$$\Rightarrow b' = -\frac{3}{5} \Rightarrow a' = -\frac{16}{5} \Rightarrow \left(\frac{16}{5}, -\frac{3}{5}\right) \text{ lies on } \ell'$$

$$\Rightarrow -\frac{3}{5} = 2\left(\frac{16}{5}\right) + b \quad b = -7 \qquad \Rightarrow \ell': y = 2x - 7$$

(c) ℓ: $y = 3x + 4$ and the axis of reflection $y = 2x - 3$ meet at $(-7, -17)$

Next, take a point on ℓ say $(0, 4)$ and find its image (a', b') so that ℓ passes through (a', b'). Perform the technique similar to the problem in b

Thus, $\frac{b'-4}{a'-0} = -\frac{1}{2}$ and $\frac{4+b'}{2} = 2\left(\frac{a'}{2}\right) - 3 \Rightarrow a' = \frac{28}{5}$ and $b' = \frac{6}{5}$

$$\Rightarrow \ell': y = \frac{91}{63}x - \frac{434}{63}$$

Example 14

Find the image of $(-1, 5)$ when reflected about the lines

(a) $y = x + 2$

(b) $y = 2x + 5$

Solution

(a) The slope of $y = x + 2$ is 1.

Let $P'(x', y')$ be the image of P (–1, 5). If ℓ' is the line passing through P and P', then its slope is $\frac{-1}{1} = -1$. Thus, the equation of ℓ' is:

$$\frac{y-5}{x+1} = -1 \Rightarrow \ell' : y = -x + 4$$

The point of intersection of ℓ and ℓ' is (1, 3). Taking (1, 3) as a mid point of $\overline{PP'}$, we get,

$$\frac{-1+x'}{2} = 1 \text{ and } \frac{5+y'}{2} = 3 \Rightarrow -1+x' = 2 \text{ and } 5+y' = 6$$

$$\Rightarrow x' = 3 \text{ and } y' = 1$$

Therefore, the image of P (–1, 5) is P' (3, 1).

(b) The slope of $y = 2x + 5$ is 2. If $P'(x', y')$ is the image of $P(-1, 5)$ and ℓ' is the line through P and P', then its slope is $\frac{-1}{2}$. Thus, the equation of ℓ' is:

$$\frac{y-5}{x+1} = \frac{-1}{2} \Rightarrow \ell' : y = \frac{-1}{2}x + \frac{9}{2}$$

The point of intersection of ℓ and ℓ' is $A\left(\frac{-1}{5}, \frac{23}{5}\right)$. Taking A as the midpoint of PP', find the coordinates of P' as:

$$\frac{-1+x'}{2} = \frac{-1}{5} \text{ and } \frac{5+y'}{2} = \frac{23}{5} \Rightarrow -5 + 5x' = -2 \text{ and } 25 + 5y' = 46$$

$$\Rightarrow 5x' = 3 \text{ and } 5y' = 46 - 25 = 21 \Rightarrow x' = \frac{3}{5} \text{ and } y' = \frac{21}{5}$$

Hence, the image of P (–1, 5) is $P'\left(\frac{3}{5}, \frac{21}{5}\right)$.

Example 15

Find the image of the line $\ell' : y = -3x - 7$ after a reflection about the line $\ell : y = -3x + 1$.

Solution

Pick a point P on ℓ', say P (1, –10).

To find the image of the point P (1, –10) when reflected about the line $y = -3x + 1$, proceed as follows:

Since slope of ℓ is –3, the slope of the perpendicular line is $\frac{1}{3}$. Thus, the equation of the line through (1, –10) with slope $\frac{1}{3}$ is: $\frac{y+10}{x-1}=\frac{1}{3}$

$$\Rightarrow y=\frac{1}{3}x-\frac{31}{3}.$$

The point of intersection of $y=-3x+1$ and $y=\frac{1}{3}x-\frac{31}{3}$ is $A\left(\frac{34}{10},\frac{-92}{10}\right)$.

Taking A as a midpoint of PP', find the coordinates of the image $P'(x', y')$ of P, i.e.,

$$\frac{1+x'}{2}=\frac{34}{10} \text{ and } \frac{-10+y'}{2}=\frac{-92}{10}$$

$$\Rightarrow 10+10x'=68 \quad \text{and} \quad -100+10y'=-184$$

$$\Rightarrow x'=\frac{58}{10} \qquad \text{and} \qquad y'=\frac{-84}{10}$$

Therefore, the image of $P(1,-10)$ is $P'\left(\frac{58}{10},\frac{-84}{10}\right)$.

Now, you need to find the equation the line passing through P' with slope –3, i.e.,

$$\frac{y+\frac{84}{10}}{x-\frac{58}{10}}=-3 \Rightarrow \frac{10y+84}{10x-58}=-3$$

$$\Rightarrow 10y+84=-30x+174$$

$$\Rightarrow 10y=-30x+174-84$$

$$\Rightarrow 10y=-30x+90$$

$$\Rightarrow y=-3x+9.$$

Hence, the image of the line $\ell': y=-3x-7$ when reflected about the line $y=-3x+1$ is $y=-3x+9$

Exercises

1. The vertices of triangle ABC are A (2, 1), B (3, –2) and C (5, –3). Give the coordinates of the vertices after:
   (a) a reflection in the x-axis
   (c) a reflection in the y-axis
   (b) a reflection in the line $x + y = 0$
   (d) a reflection in the line $y = x$.
2. Find the image of the point (–4, 3) after a reflection about the line $\ell: y=x-2$
3. If the image of the point (–1, 2) under reflection is (1, 0), find the line of reflection.

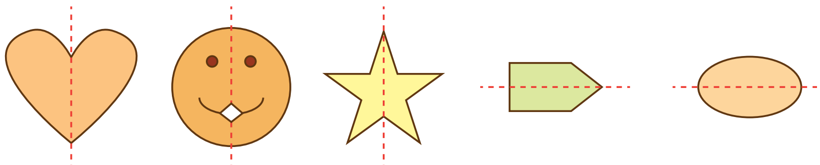
4. Find out some of the figures which are their own images in reflection about the line $y = x$.
5. Find the image of the line $\ell: y = x + 4$ after it has been reflected about the line $L: y = x - 3$.
6. Find the image of the line $\ell: y = 2x + 1$ after it has been reflected about the line $L: y = 3x + 2$.
7. If T is a translation that sends (0, 0) to (3, –2) and M is a reflection that maps (0, 0) to (2, 4), find
   (a) $T(M(1, 3))$ (b) $M(T(1, 3))$
8. In a reflection, the image of the line $y - 2x = 3$ is the line $2y - x = 9$. Find the axis of reflection.

You may have often heard of the term 'symmetry' in day to day life. Symmetry is a balanced and proportionate similarity found in two halves of an object; that is, one-half is the mirror image of the other half. In other word, a shape is said to be **symmetrical** if it can be divided into two identical parts. A shape which is not symmetric is referred to as **asymmetrical**.

Line of symmetry

An imaginary line along which a symmetrical shape can be divided to obtain identical halves is called a **line of symmetry** (or **axis of symmetry**). Every symmetrical shape has a line of symmetry. Conversely, if a shape has a line of symmetry, then it is symmetrical. For instance, a figure (or a shape) has a line of symmetry, if it can be folded so that one half of the figure coincides with the other half.

Symmetrical objects are found all around us, in nature, architecture, and art. The following figure shows some examples of symmetrical shapes. The broken line through the shape is its line of symmetry.



In geometry, a plane figure is symmetrical if it can be divided into two congruent parts. That is, the part of the figure that lies in one side of the line of symmetry is **congruent** to the part in the other side. If so, the shape is symmetric.

EXAMPLE 16

Let ΔABC be an isosceles triangle such that $AB = AC$. Show that ΔABC is symmetric and determine its line of symmetry.

Solution

We need to determine a line that divides ΔABC into two parts such that the parts are congruent. To do this, construct a line AD from vertex A to side BC such that

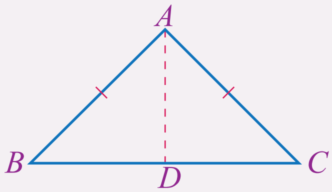
AD is a bisector of BC; i.e., $DB \equiv DC$.

So, the line containing AD divides ΔABC into two triangles, ΔADB and ΔADC, and we have to show that these two triangles are congruent. To show this:

(i) $AB = AC$ (given)

(ii) $DB = DC$ (by construction)

(iii) $AD = AD$ (Common)



Therefore, by SSS, ΔADB is congruent to ΔADC.

Hence, ΔABC is symmetric and its line of symmetry is through angle A bisecting side BC.

EXAMPLE 17

Show that a rectangle $ABCD$ is symmetric.

Solution

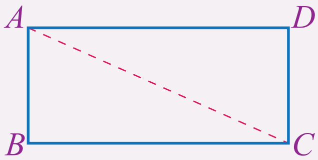
We need to determine a line that divides $ABCD$ into two congruent parts. To do this, let us take a diagonal, say AC, so that the rectangle is divided into two parts: ΔABC and ΔCDA. Now, to show these are congruent:

Now, to show ΔABC is congruent ΔCDA:

(i) $AB = CD$ (*Opposite sides of the rectangle*)

(ii) $BC = DA$ (*Opposite sides of the rectangle*)

(iii) $AC = CA$ (*Common*)

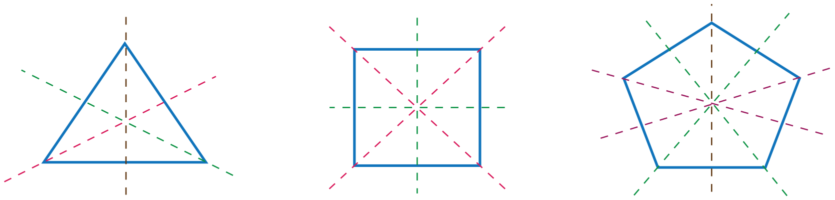


Therefore, by SSS, ΔABC is congruent to ΔCDA.

Hence, the rectangle is symmetric and its line of symmetry is along its diagonal.

Note: A symmetric figure can have more than one lines of symmetry. In such cases, the lines of symmetry always intersect at one point.

For example, equilateral triangles, squares, and regular pentagons have as many lines of symmetry as their sides (See, the figure below).



Equilateral triangle Square Regular pentagon

EXAMPLE 18

Show that the parabola given by $y = 3(x–5)^2 + 1$ is symmetric.

Solution

Using the concept of reflection, we need to show that the image of the parabola is the parabola itself under the reflection about the vertical line $x = 5$. From the previous section, you know that $M(x, y) = (2a–x, y)$ under a reflection about a vertical line $x = a$.

Therefore, for the reflection about $x = 5$, we get $M(x, y) = (2(5)–x, y) = (10 –x, y)$.

That is, $x \mapsto 10 - x$, but y unchanged.

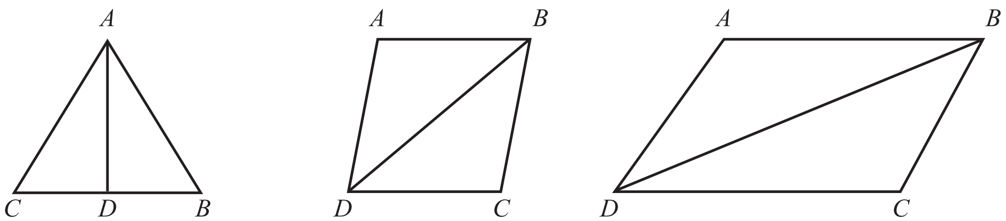
Thus, using this change in $y = 3(x – 5)^2 + 1$ we get

$$\begin{aligned} y &= 3((10 - x) - 5)^2 + 1 \\ &= 3(10 - x - 5)^2 + 1 \\ &= 3(5 - x)^2 + 1 \\ &= 3(x - 5)^2 + 1, \text{ which is just the original equation.} \end{aligned}$$

That is, the reflection of the parabola $y = 3(x – 5)^2 + 1$ about $x = 5$ is the parabola itself. Therefore, the parabola $y = 3(x – 5)^2 + 1$ is symmetric and its line of symmetry is $x = 5$.

EXERCISES

1. How many lines of symmetry does each of the following shapes have?
   - (a) A rectangle
   - (b) An equilateral trapezium
   - (c) A rhombus
   - (d) A regular hexagon
2. Using the notion of congruency of shapes, show that the following figures are symmetric. Determine also a line of symmetry of the shape.



(a) An equilateral triangle
(b) A rhombus
(c) A parallelogram

3. Using the notion of reflection, show the followings.
   (a) A parabola given by $y = a(x - h)^2 + k$ is symmetric with line of symmetry $x = h$.
   (b) The graph of $y = 3x^4$ is symmetric with line of symmetry $x = 0$ (y-axis).

KEY TERMS

- Axis of reflection
- Axis of symmetry
- Identity transformation
- Line of reflection
- Line of symmetry
- Reflection
- Rigid motion
- Symmetry
- Transformation
- Translation

SUMMARY

- **Rigid motion**
  (i) Transformation can be classified as rigid motion and non-rigid motion.
  (ii) Rigid motion is a motion that preserves distance. Otherwise it is non-rigid.
  (iii) Identity transformation is a transformation that image of every point is itself.
- **Translation**
  (i) Translation is a transformation in which every point of a figure is moved along the same direction through the same distance.
  (ii) Translation vector: If point P is translated to P', the vector PP' is said to be the translation vector.
  (iii) If $\boldsymbol{u} = (h, k)$ is a translation vector, then $T(x, y)= (x + h, y + k)$.
- **Reflection**
  A reflection M about a fixed line L is a transformation of the plane onto itself which maps each point P of the plane into the point P' of the plane such that L is the perpendicular bisector of PP'.
  (i) Point of reflection in the x-axis, $M(x, y) = (x, -y)$.
  (ii) Point of reflection in the y-axis, $M(x, y) = (-x, y)$.

(iii) Point of reflection in the line $y = x$, $M(x, y) = (y, x)$.

(iv) Point of reflection in the line $y = -x$, $M(x, y) = (-y, -x)$.

(v) Reflection in the line $y = mx$, $M(x, y) = (x', y')$

$x' = x \cos 2\theta + y \sin 2\theta$ $\qquad$ $y' = x \sin 2\theta - y \cos 2\theta$; $m = \tan \theta$

- **Symmetry**
  - A shape is said to be **symmetrical** if it can be divided into two identical pieces. The line that divides a symmetrical shape into two identical parts called a line of symmetry or axis of symmetry.
  - In symmetrical figure, its line of symmetry divides the figure into two congruent parts
  - In a symmetrical figure, its halves are the mirror image of one another. That is, the image of the figure is the figure itself under reflection about the line of symmetry. That is, the line of symmetry is the line of reflection.

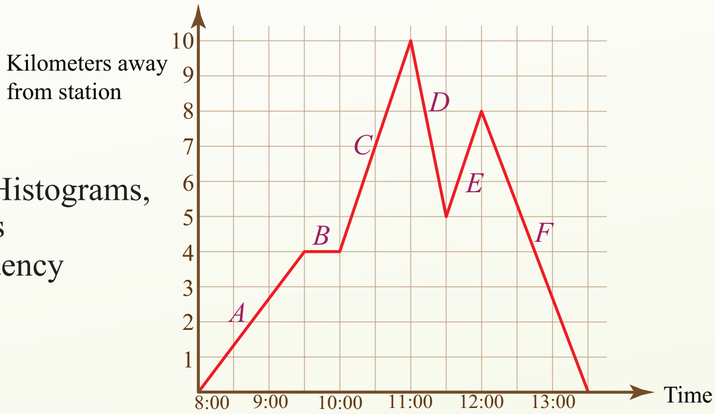
Exercises

1. If a translation T carries the point (7, –12) to (9,–10) find the images of the following lines and circles.
   (a) $y = 2x - 5$
   (b) $y = 4x^2 - 3$
   (c) $2y - 5x = 4$
   (d) $x^2 + y^2 = 3$
   (e) $x + y = 10$
   (f) $x^2 + y^2 - 2x + 5y = 0$
2. In a reflection the image of the point P (3, 10) is P' (7, 2). Find the equation of the line of reflection.
3. Find the image of the followings under the reflection about the line L: $y = 2x + 1$.
   (a) the point $P(3, 2)$
   (b) the circle $(x - 3)^2 + (y - 2) = 6$.
4. Find the image of the line ℓ: $y = 2x + 1$ under the reflection about the line L: $y = 2x$.
5. Find the image of $\ell:\ y = \frac{2}{3}x + 2$ under the reflection about the line L: $y = 2x - 3$.
6. $x^2 + y^2 = r^2$ is symmetric about any line passing through the origin. (Note: the equation of a line passing through origin is $y = mx$, $m \in \mathbb{R}$.)

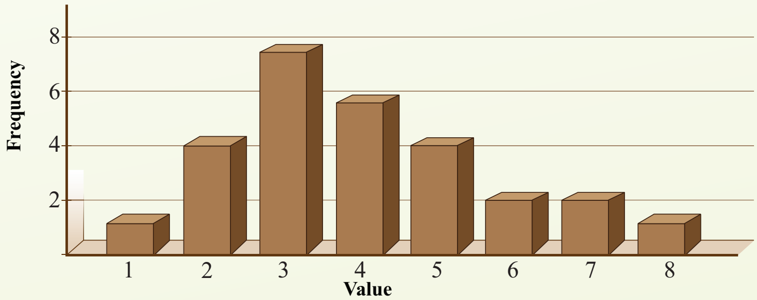
M10CH11

CHAPTER 11

STATISTICS, RATIOS, RATES AND PERCENTAGES



Chapter Contents



Chapter Outcomes

Learners are able to apply concepts of statistics, ratio, rates and percentages to solve problems on these topics. The students will be able to analyze and interpret or explain practical data and solve problems relating to scale drawing , ratio, rates, and percent.

Upon completion of this chapter, learners should be able to:

- define and discuss statistics;
- construct frequency table;
- discuss graphical display;
- discuss and define averages;
- define and discuss ratio and rates;
- demonstrate the use of rate in problem solving;
- define and analyze scales and scale drawing;
- define and discuss travel graphs and conversion graphs;
- identify and discuss percentages;
- solve problems using percentages.

Introduction

This unit deals with various topics: statistics, ratios and rates, scale and scale drawing, travel graphs and conversion graphs, percentage and using percentages in problem solving.

Statistics plays an important role in all our lives. For example, citizens are counted in the population census taken by the government for various reasons. Business people conduct surveys to find out what products are most frequently wanted by consumers; and so on.

Meaning of Statistics

There are many definitions of the term statistics given by different scholars. However, for the purpose of this unit, we will confine ourselves to the following:

Definition

Statistics is the science of *collecting*, *organizing*, *presenting*, *analyzing* and *interpreting data* (quantitative information) in order to draw conclusions.

Basically, statistics is a procedural process performing five logical steps on numerical data. These are:

Collection of data → **Organization of data** → **Presentation of data** → **Analysis of data** → **Interpretation of data**

1. Collection of data: This is the process of obtaining measurements or counts. For example, measuring the heights of students in your class, or counting the number of persons admitted to a certain hospital are examples of data collection.
2. Organization of data: The second step of statistics refers to the organization of data. Collected data has to be organized in a suitable form to understand the information gathered. The collected data must be edited, classified and tabulated.
3. Presentation of data: The main purpose of data presentation is to facilitate statistical analysis. This can be done by illustrating the data using graphs and diagrams like bar graph, histograms, pie-charts, pictograms, frequency polygons, etc.

4. Analysis of data: In order to meet the desired purpose of investigation, data has to be analyzed. The purpose of analyzing data is to get information useful for decision making.
5. Interpretation of data: Based on analyzed data, conclusions have to be drawn. This step usually involves decision making about a large collection of objects (the population) based on information gathered from a small collection of similar objects (the sample).

The following examples show us how statistics plays a major role in decision making in different sectors.

Example 1

Information gathered about the incidence or prevalence of diseases, such as Covid and Ebola, in a community provides useful information on changing trends in health status and mortality.

Example 2

Statistics is used to study existing conditions and the prevalence rate of HIV/AIDS in order to design new programs or to study the merits of different methods adopted to control HIV/AIDS. It assists in determining the effectiveness of new medication and the importance of counseling.

Example 3

Demographic data about population size, its distribution by age and sex and the rate of population growth, etc., all help policy makers in determining future needs such as food, clothing, housing, education, health facilities, etc.

Example 4

Recording annual temperatures in a country provides the community with timely warning of environmental hazards.

Example 5

Statistical data collected on customer services provides feedback that can help to reform policies and systems.

Definition of basic terms or concepts

Population and Sample: In its ordinary usage, population refers to the number of people living in an area or country. In statistics, however, population refers to the complete collection of individuals, objects or measurements that have a common characteristic.

Gaining access to an entire group (or population) is often difficult, expensive and sometimes destructive. Therefore, instead of examining the entire group, a researcher examines a small part of the group, called a sample.

Data: Data can be defined as a record of quantitative or qualitative values of a **variate** (or **variable**). Variable or variate, represented by a letter, say x or V, is an attribute who can have different values in measurements or observations. This depends on the type of the study for which the measurement is undertaken. For instance, if the measurements are the height of students, the variable x represents height. In this case, x_1, x_2, x_3, ..., x_n represent the height of the first student, height of the second student, ..., height of the n-th student, respectively.

Source of data: Data sources are classified as primary and secondary. Data is said to be primary, if it is obtained first hand for the particular purpose on which one is currently working. Primary data is original data, obtained personally by the researcher from primary sources by observation, interview or direct measurement.

Data which has been collected previously(for similar or different purpose) is known as secondary data**.** Secondary data refers to that data which is not originated by the researcher himself/herself, but which he/she obtains from someone else's records.

Raw data is data that has not processed for use. This is a record which is not yet organized in a desired form.

Data can be classified as qualitative or quantitative. A data is qualitative if it is about some characteristic whose values are not numbers, such as the eye color of peoples, gender, religion or place of birth. On the other hand, a data is **quantitative** if it is numerical values such as height, weight, age or scores in tests. Here, in this unit, we will deal only with quantitative data.

Exercises

1. Describe in your own words the difference between the following pair of statistical terms. Explain also using illustrative examples.
   (a) population, sample.
   (b) primary data, secondary data
   (c) qualitative data, quantitative data
2. Mention at least four uses of statistics.
3. What is a raw data?
4. Collect data on age, height, and first semester mathematics exam score of the students in your class.

11.2 Frequency Distributions, Histograms, Stem-plots and Pie Charts

As mentioned in the previous subsection, statistics involves the collection of data and analyzing it. For the data to be effectively analyzed, they should be organized or arranged in a compact and orderly form. This procedure is commonly known as tabulation.

Methods for organizing raw data include the drawing of a table called the table of **frequency distribution**. Frequency of a variate (data value) is the number of times it appears in the data.

Definition

A frequency distribution is a tabular or graphical representation of a data showing the frequency associated with each variate (data value).

Example 6

Suppose there are 12 students whose scores in a mathematics test is as follows:

13, 11, 12, 14, 13, 12, 11, 12, 13, 12, 15, 14

Organize the data in tabular form. What are the variates? Give the frequency of each variate.

Solution

The data given above is raw data. We may arrange the scores in ascending order, determine the frequency of each score (number of times it appears), and tabulate the result in the form given below. .

Score (V)	11	12	13	14	15
Number of students (f)	2	4	3	2	1

The table given above is the frequency distribution table. for the scores of the students. The scores are the variate and the number of students getting a particular score is the frequency of the variate.

Histograms

A histogram is a visual (graphical) display of data using bars (rectangles) of different heights. Each rectangle represents a variate (data value) and the height of the rectangle corresponds to the frequency of the variate.

Definition

A histogram is a graphical representation of a frequency distribution in which the variate (V) is plotted on the x-axis (horizontal axis) and the frequency (f) is plotted on the y-axis (vertical axis).

Main parts of a histogram:

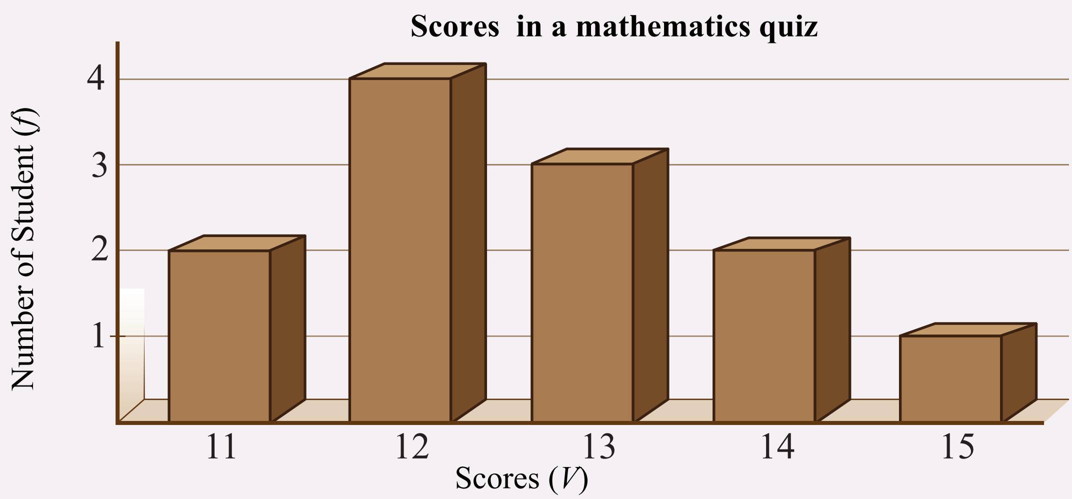
1. **The title**: the title describes the information included in the histogram.
2. ***X*-axis**: The x-axis (horizontal) shows the variates (data values/items). These can also be intervals that show the scale of values which the measurements fall under.
3. ***Y*-axis:** The y-axis (vertical) shows the frequencies of the variates. (or values in the intervals)
4. **The bars:** The height of a bar (rectangle) shows the frequency of the variate it represents. The width of each bar should be the same. Usually, there is no gap between the bars.

Example 7

Let us draw a histogram representation of the frequency distribution given in the above example. i.e.,:

Score (V)	11	12	13	14	15
Number of students (f)	2	4	3	2	1

The following is the histogram that displays this frequency distribution.



Grouped frequency distribution

Usually a large number of data are collected in an ungrouped form and then grouped in some meaningful order. For example, a row data of 40 marks (scores) of students in a test are listed in the following table.

Table 1

40 Marks (out of 100) to the nearest integer							
93	71	57	60	91	79	63	69
82	83	65	78	65	92	72	72
75	58	75	75	70	69	75	74
65	81	65	52	74	77	70	80
80	87	97	81	55	66	62	75

In this haphazard arrangement not much information can be readily be obtained. So, we arrange the marks in **intervals**. The number of values in each interval is called the **frequency of the interval**. The frequency of each interval is determined by tallying (counting) every values that lie in the interval.

Example 8

Make grouped frequency distribution for the marks listed in Table 1 above with each intervals to include five marks, such as 50 − 54, 55 − 59, ..., 95 − 99.

Solution

We determine the frequency (number of marks) in each interval by tallying as shown in Table 2 below.

Table 2

Interval	Tally	Frequency
50 – 54	|	1
55 – 59	///	3
60 – 64	///	3
65 – 69	~~////~~ //	7
70 – 74	~~////~~ //	7
75 – 79	~~////~~ ///	8
80 – 84	~~////~~ /	6
85 – 89	/	1
90 – 94	///	3
95 – 99	/	1
	Total	**40**

Once data values are grouped into intervals, each interval is represented by the middle number in the interval. For example, the interval 70–74 is represented by the number 72. The middle number in an interval is called the **class mark** of the interval.

The purpose of grouping a data is to present it in a compact manner. The suitability of the chosen intervals for the desired compactness depends on what we want to do with the data. For instance, the grouped frequency distribution given in Table 2 is not compact enough if the purpose of grouping the scores is to decide letter grades. In particular, if these marks are converted to latter grades, where A is 90–100, B is 80–89, C is 70–79, D is 60–69 and F is 59 and below, we obtain a more compact frequency distribution for the same data set as shown below in Table 3.

Table 3

Letter Grade	Interval	Frequency
A	90 – 100	4
B	80– 89	7
C	70 –79	15
D	60 – 69	10
F	50 – 59	4
	Total	**40**

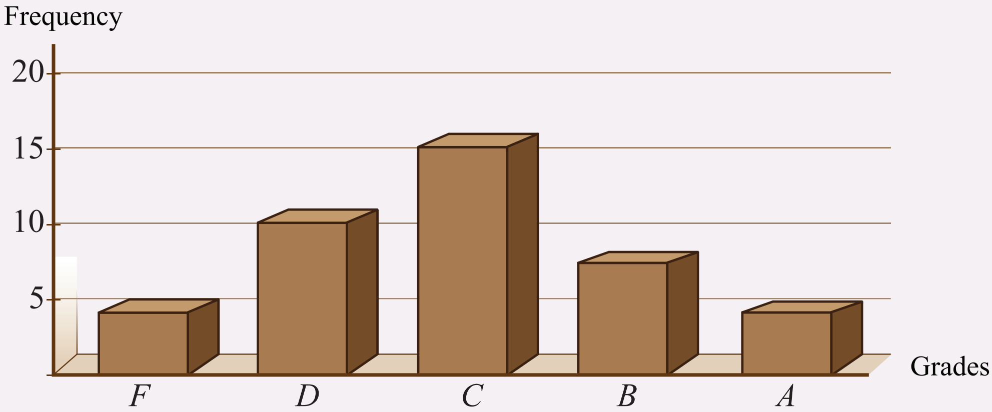
Histogram is also used to represent a grouped frequency distribution. In this case, a bar (rectangle) is constructed on each interval whose height is the frequency of that interval. All others are the same as for the case of simple frequency distribution.

Example 9

Draw histogram for the grouped frequency distribution given in Table 3 above.

Solution

The histogram is shown below. You can represent each of the intervals by its middle number. Here, however, we indicated each interval by the corresponding letter grade since the frequency distribution is constructed for the letter grades.



Stem and Leaf Plot

A stem and leaf plot is a way to plot data where each data value is split into two parts, called **stem** and **leaf**. Here, the stem of a number is the digit at the largest place-value of the number and leaf is the digit at the smallest place-value. For instance, for 25, stem is 2 and leaf is 5. We plot(write) 25 as 2|5. Hence, in a stem plot, say, 2|357 represents 23, 25 and 27. Space between leaves is possible but it is not a requirement.

Example 10

The following is a stem and leaf plot of a data set:

2	1 3
3	0 0 2
4	3 6 6 7 7 7
5	4 7
6	0 1 4 4 7
7	2 3 6 8
8	4

In this plot, the first line shows: 21, 23

Second line: 30, 30, 32

Third line: 43, 46, 46, 47, 47, 47;

and so on. The smallest value in this data set is 21 and the highest value is 84.

In general, you may follow the following steps to make a stem and leaf plot of a data set.

1. Arrange the data values in ascending order.
2. The largest place-value digit of a data element is placed at the stem (in the left column). The stems are listed down in ascending order. The leaves of each stem are listed in ascending order in the line (row) containing the stem.
3. Draw a vertical line between the stems and their leaves.

Example 11

Make a stem and leaf plot from the following data set, which represents students scores in a test: 64, 82, 79, 79, 96, 81, 97, 80, 81, 80, 84, 87, 98,

75, 60, 88, 82, 78, 81, 86, 80, 52, 84, 78, 83, 82.

Solution

Arranging the scores in ascending order and following the steps given above, we get:

5	2
6	0 4
7	5 8 8 9 9
8	0 0 0 1 1 1 2 2 2 3 4 4 6 7 8
9	6 7 8

Note: Like histogram, a stem and leaf plot is a visual representation of a data and allows us to compare data values or to overview their distribution. But the advantage of the stem and leaf plot is that we can see all of the values (histogram usually shows just totals). A disadvantage of stem and leaf plots is that they are useful only for small data sets.

Pie Charts

A pie chart is a pictorial representation of the frequency distribution of data on a **circular region**. The circle (pie) is partitioned into sectors so that each sector represents an individual item or a class of the data. The central angle (or area) of a sector is proportional to the relative frequency of the individual it represents. That is, if the frequency of a given individual is f then the angle of the sector representing the individual is

$$\theta = \frac{f}{n} \times 360°,$$

Where n is the total number of the individuals in the data.

Example 12

Draw a pie chart for the following frequency distribution of blood groups in a sample.

Blood groups	**A**	**AB**	**B**	**O**
frequencies (f)	172	43	387	258

Solution

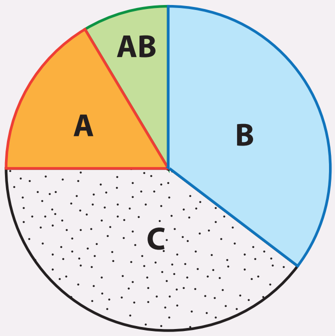
The total number (total frequencies) of the blood groups in the sample is $n = 860$. Hence, the central angles for each of the sectors representing the four blood groups are:

for **A** group: $\theta = \frac{172}{860} \times 360° = 72°$, for **AB** group: $\theta = \frac{43}{860} \times 360° = 18°$,

for **B** group: $\theta = \frac{387}{860} \times 360° = 162°$, and

for **O** group: $\theta = \frac{258}{860} \times 360° = 108°$.

Now, using a protractor, a circle is partitioned into sectors of 72°, 18°, 162° and 108° to represent the frequencies of A, AB, B and O groups, respectively, as shown below.



A pie chart of frequency distribuon of blood groups

EXERCISES

1. Give two reasons why raw data should be summarized into a frequency distribution.
2. What is the difference between a frequency distribution table and a histogram?
3. Suppose the following data represents the number of persons who took counseling on HIV/AIDS on 40 consecutive days:

   Table 1

10	5	10	3	4	5	12	9	11	13
10	9	6	10	8	7	3	7	9	10
4	6	8	6	7	6	4	4	11	8
10	9	5	8	8	7	8	8	6	12

   (a) Construct a frequency distribution table from the data.
   (b) Construct a histogram.
   (c) On what percent of days did more than 10 people take counseling?
4. Maket the stem and leaf plot of the 40 scores given in Table 1 above.
5. Here are the number of sacks (each weighs 100 kg) of fertilizer distributed to 50 farmers.

   Table 2

20	24	22	19	20	13	18	24	10	15
31	20	33	19	20	42	14	32	27	18
18	43	14	18	40	35	14	22	47	20
15	14	15	46	21	16	24	26	18	24
10	12	15	36	14	21	20	38	15	41

   (a) Construct a grouped frequency distribution with interval 10–14, 15–19, ..., etc.
   (b) Construct a histogram of the grouped frequency distribution.
   (c) Construct the stem and leaf plot of the data set.

Quantitative variables contained in raw data or in frequency tables can also be summarized by means of a few numerical values. A key element of this summary is called the measures of central tendency or measures of average. The three commonly used measures of central tendency are the arithmetic mean (or the mean), the median and the mode.

1. The arithmetic mean

DEFINITION

The arithmetic mean (or the mean) of a variable is the sum of all the data values, divided by the total frequency (number of observations). That is,

If $x_1, x_2, x_{3, \ldots,} x_n$ are n observations of a variable, then the mean, $\bar{x}$, is given by

$$\text{Mean}: \bar{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n} = \frac{\text{sum of values}}{\text{total number of values}}.$$

When used in everyday language the word "average" often stands for the arithmetic mean.

EXAMPLE 13

Find the mean of the following data:

7, 21, 2, 17, 3, 13, 7, 4, 9, 7, 9

Solution

$$\bar{x} = \frac{7+21+2+17+3+13+7+4+9+7+9}{11} = \frac{99}{11} = 9.$$

Note: The mean of a data can also be calculated from its frequency distribution. So, if the values $x_1, x_2, x_3, \ldots, x_n$ occur $f_1, f_2, f_3, \ldots, f_n$ times, respectively, then the mean ($\bar{x}$) is given by

$$\text{Mean: } \bar{x} = \frac{x_1 f_1 + x_2 f_2 + \ldots + x_n f_n}{f_1 + f_2 + \ldots + f_n}$$

EXAMPLE 14

The following table shows the age of 14 students in a certain class:

Age in years (x)	12	13	16	18
Number of students (f)	3	4	2	5

Compute the mean age of the students.

Solution

$$\bar{x} = \frac{12\times3+13\times4+16\times2+18\times5}{3+4+2+5} = \frac{36+52+32+90}{14} = \frac{210}{14} = 15 \text{ years}.$$

2. The median

A second measure of central tendency of a quantitative data is the median.

DEFINITION

The median is the value that lies in the middle of the data when it is arranged in ascending or descending order. So, half the data is below the median and half the data is above the median.

EXAMPLE 15

Find the median of each of the following:

(a) 6, 7, 9, 7, 11, 13, 15

(b) 27, 23, 36, 38, 27, 40, 45, 39

Solution

(a) First arrange the data in ascending order as 6, 7, 7, 9, 11, 13, 15

There are seven values (an odd number of values) and the middle value is the 4^{th} element of the list which is 9. Therefore, 9 is the median of the data.

(b) First, arrange the data in ascending order as 23, 27, 27, 36, 38, 39, 40, 45

There are eight values (an even number). The two middle values are the 4^{th} and 5^{th} elements of the list which are 36 and 38. The median is the average of the two middle values. That is, the median is half the sum of 36 and 38. So, the median = $\frac{36+38}{2}=37$.

EXAMPLE 16

Find the median of the following distribution

x	1	2	3	4	5
f	2	3	2	4	2

Solution

There are 13 data values. So, the median is the 7^{th} piece of data, which is 3.

Note that the median of a set of data with values arranged in ascending or descending order is:

(i) the middle value of the list if there is an odd number of values.

(ii) average of the two middle values if there is an even number of values.

3. The mode

A third measure of central tendency is the mode.

Definition

A data value which occurs most frequently in a data set is called a **mode**.

Example 17

Find the mode(s) of each of the following data sets, if any.

(a) 9, 8, 7, 10, 8, 7, 6, 8

(b) 10, 11, 13, 10, 14, 12, 18, 17

(c) 4, 6, 12, 5, 10, 7

Solution

(a) 8 is the mode because it occurred three times (most frequent in the data set).

(b) The values 10 and 12 both occur twice, while the others occur only once. It has two modes and the data is a bimodal.

(c) It has no mode because each value occurs only once.

The above example shows that a set of data can have no mode, one mode (unimodal), two modes (bimodal) or more than two modes (multimodal). A data has no mode if the frequencies of every data values are equal.

Example 18

Find the mean, median and mode of the following distribution of temperatures in a certain town for one month.

Temperature in °C(*V*)	20	21	23	24	26	28
Number of days (*f*)	2	4	5	9	3	7

Solution

Mean: $\bar{x} = \dfrac{(20\times 2)+(21\times 4)+(23\times 5)+(24\times 9)+(26\times 3)+(28\times 7)}{2+4+5+9+3+7}$

$= \dfrac{40+84+115+216+78+196}{30} = \dfrac{729}{30} = 24.3$

Therefore, the mean is 24.3°C.

The number of observations is an even number which is 30. So, the median is half the sum (average) of the 15^{th} and 16^{th} values.

i.e., median $= \dfrac{15^{th} \text{ value} + 16^{th} \text{ value}}{2} = \dfrac{24+24}{2} = 24$.

Therefore, the median is 24°C.

The value with highest frequency is the number 24. Therefore, the mode is 24°C.

Exercises

1. Find the mean, mode and median of the following data.

   11, 9, 14, 12, 11, 14, 10, 21, 18, 14, 15, 18

2. Given below is a frequency distribution of values V.

   (a) Find the mean, mode and median of the following distribution.

   (b) How many of the values are non-negative?

V	–2	–3	0	1	2	3
f	3	2	3	6	5	1

3. Calculate the mean, median and mode of the following data;

Value	10	15	20	25	30	35	40
Frequency	15	10	50	4	10	8	3

4. The following raw data represents the number of HIV/AIDS patients waiting for counseling at 8:00 am on 40 consecutive Saturdays at a certain hospital.

11	6	5	8	11	6	3	7	4	6
5	4	13	14	9	11	13	8	10	9
10	9	6	5	10	7	8	7	8	3
8	7	8	9	6	10	11	8	8	4

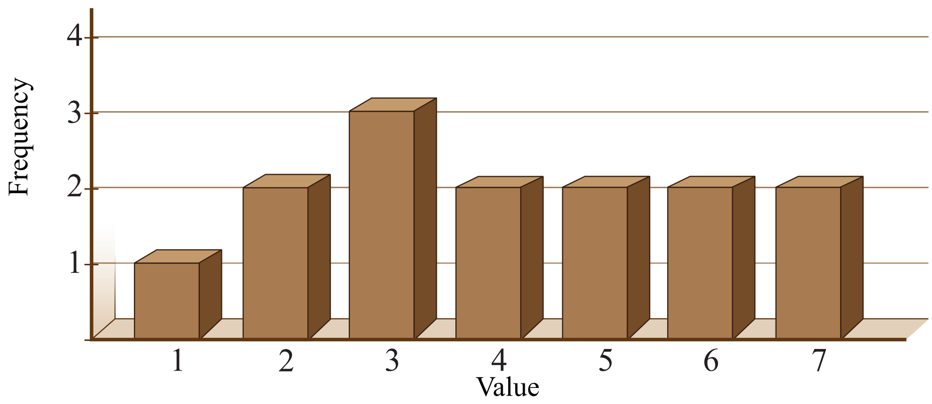
   (a) Draw a frequency distribution table.

   (b) Calculate the mean, median and modal number of HIV/AIDS patients.

   (c) Draw a histogram.

   (d) Make a steam and leaf plot.

5. Find the mean, median and mode of the data represented by the histogram below.



A box and whisker plot is a visual(graphical) tool used to display the variance or dispersion of a data using the three quartiles and two extremes (lowest and highest values) of the data.

The **quartiles** of a data, denoted by, $\boldsymbol{Q_1}$**,** $\boldsymbol{Q_2}$ and $\boldsymbol{Q_3}$**,** are values that divides the data into four equal parts. In particular, arranging the values of the data in increasing order,

$\boldsymbol{Q_2}$ is the median of the entire data;

$\boldsymbol{Q_1}$ is the median of half of the data values below the median; and

$\boldsymbol{Q_3}$ is the median of half of the data values above the median.

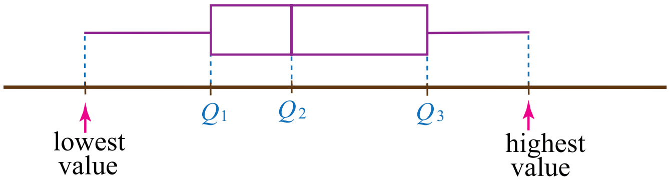
$\boldsymbol{Q_1}$ is called the **lower quartile** and Q_3 is called the **upper quartile**. The difference between the highest value (*HV*) and the lowest value (*LV*) of the data is called **range**; and the difference between upper and lower quartiles, is called the **interquartile range** (IQR) of the data. That is,

range = $\boldsymbol{HV - LV}$, and $\boldsymbol{IQR = Q_3 - Q_1}$,

The three quartiles divide the data values into four equal parts (each containing one-fourth of the values). However, the length of the parts may not be equal. In a longer part, the data values are relatively scattered wider apart than in the shorter part. In other words, a longer part indicates higher variability of the data values in that part. A box and whisker plot displays graphically the relative length of each of the four parts. This can effectively help us to see the variability of the data values over each of the four parts.

To draw a box and whisker plot

1. Arrange the data values in ascending order (from lowest to highest value).
2. Identify the quartiles, Q_1, Q_2, Q_3, of the values and label them on the number line.
3. Draw a box (rectangle) whose length is equal to the interquartile range above the line such that the left side of the rectangle is on Q_1 and the right side of the rectangle is on Q_3. Then, divide the rectangle into two parts with a line segment at Q_2 (the median).
4. Draw a whisker (horizontal line segment) from the lowest value of the data to the box and another whisker from the box to the highest value of the data. (See the following diagram)



Illustrave diagram for box-and-whisker

We illustrate this using the following examples.

Example 19

Draw a box and whisker plot for the following information:

Range = 75, Lowest value (LV) = 5, Median = 30, Upper quartile = 58;

Interquartile range (IR) = 43;

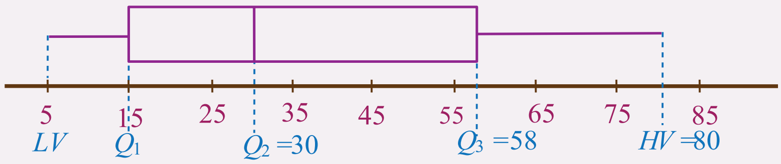
Solution

To draw the box and whisker diagram, we need five values: LV, Q_1, Q_2(median), Q_3, and HV (highest value). $LV = 5$, $Q_2 = 30$ and $Q_3 = 58$ are given. We need to compute Q_1 and HV from the given information ($IR = 43$ and Range = 75). In particular,

$IR = Q_3 - Q_1 \Rightarrow Q_1 = Q_3 - IR = 58 - 43 = 15$; and

Range $= HV - LV \Rightarrow HV = LV +$ Range $= 5 + 75 = 80$.

Once we have the five values, we can draw the box and whisker plot as shown below.



Note that the length of the right hand side box is longer than the left side box. This indicates that the data values that are between the median and upper quartile have larger variability (dispersion) than those values between the lower quartile and the median. The whiskers also show that the data values above the upper quartile have larger variability than those below the lower quartile.

Example 20

A utility company in a city is researching the cost of household electricity and water bills. They have analyzed the data from hundreds of households and found that the median cost is L\$66 and the interquartile range is L\$21. The range is L\$27.50, the cheapest price was L\$47.50 and the upper quartile was L\$74. Represent this information with a box-and-whisker plot.

Solution

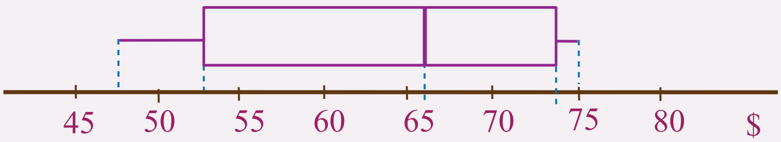
Given: $Q_2 = 66$, $IR = 21$, Range = 27.50, $LV = 47.50$, $Q_3 = 74$.

We need to find the highest value (HV) and the lower quartile (Q_1). To compute these:

$HV = LV$ + Range = 47.50 + 27.50 = 75.

$Q_1 = Q_3 - IR = 74 - 21 = 53$.

Once we have all of the values we need we can draw the box and whisker plot as follows:



Now, using this box and whisker plot, you can discuss the variability of the data values in the four part.

Exercises

1. Make a box and whisker plot for a data with the following information:
   Median = 15.75, upper quartile =20.25, highest value = 24.5;
   range = 16 and interquartile range = 9.
2. Draw a box-and-whisker plot for the following data and explain their distribution:
   (a) Ages of the members of a basketball team: 24, 30, 30, 22, 25, 22, 18, 25, 28, 30, 25.
   (b) Test score: 34, 18, 27, 52, 59 , 61, 54, 78, 68, 87, 91, 93, 100, 82, 85
3. Make a box and whisker plot of the following data and discuss its variability.

Value	10	15	20	25	30	35	40
Frequency	15	10	50	4	10	8	3

4. The following raw data represents the number of HIV/AIDS patients waiting for counseling at 8:00 am on 40 consecutive Saturdays at a certain hospital.

11	6	5	8	11	6	3	7	4	6
5	4	13	14	9	11	13	8	10	9
10	9	6	5	10	7	8	7	8	3
8	7	8	9	6	10	11	8	8	4

Draw a box and whisker plot of this data and discuss its distributions (variability).

In this section, we will discuss two related concepts, ratios and rates, that are used to compare two quantities. Things are not always of the same size. Thus, there is always a need to compare quantities using division to see how much bigger a quantity is when compared to another.

A ratio is used to compare two like quantities or numbers with the same units. On the other hand, rates are used to compare two quantities of different units. We will discuss both starting with ratios.

Ratios

DEFINITION

A **ratio** is the quotient of two numbers (or quantities of the same unit).

The ratio of a to b is written as $a : b$ or $\frac{a}{b}$ (where, $b \neq 0$).

(The ratio $a : b$ is read as 'a to b'.)

Note

1. A ratio has no unit because it is a quotient of quantities of the same unit.
2. Since a ratio is a fraction, all rules for fractions apply to ratios. For instance, $ka : kb = a : b$ since $\frac{ka}{kb} = \frac{a}{b}$. We usually express a ratio in its lowest term.

EXAMPLE 21

There are 12 red apples and 4 green apples in a basket. What is the ratio of red apples to green apples in the basket? Interpret the result.

Solution

Since red apples =12 and green apples 4, the ratio of red apples to green apples is $12 : 4 = \frac{12}{4} = 3$.

This means there are 3 times more red apples than green apples in the basket.

EXAMPLE 22

Suppose that 8 boys and 16 girls are in a classroom.

(a) The ratio of the boys to the girls is $8 : 16 = \frac{8}{16} = \frac{1}{2}$ (or 1: 2). It means that there are half as many boys as girls in the classroom.

(b) However, the ratio of the girls to the boys is $16 : 8 = \frac{16}{8} = 2$. It means that there are two times as many girls as boys in the classroom.

As this point you may have noticed that the order is important when stating a ratio. That is, when we say 'the ratio of *a* to *b*', then *a* is numerator and *b* is denominator.

Example 23

In a classroom the ratio of males to females is 2 to 3. If 20 students are males, how many students are females?

Solution

Let f be the number of females in the class room. Then, the given ratio of males to females is $20 : f = 2 : 3$

$$\Rightarrow \frac{20}{f} = \frac{2}{3} \Rightarrow 2f = 3\times 20 \Rightarrow f = \frac{3\times 20}{2} = 3\times 10 = 30.$$

Therefore, the number of female students in the classroom is 30.

Rates

Similar to a ratio, a rate is also the quotient of two quantities. Indeed, a rate is a ratio that compares two different quantities that have different units. For example, if we say John types 50 words in a minute, then his rate of typing is 50 words per minute. Similarly, if Peter types 80 words in two minutes, then his rate is $\frac{80 \text{ words}}{2 \text{ minutes}} = \frac{40}{1} = \frac{\text{words}}{\text{minutes}}$ which is 40 words per minute.

The word "per" gives a clue that we are dealing with a rate. The word "per" can be further replaced by the symbol "/" in problems. For instance, km per hour can be written as km/hour.

In general, we can write the formula for rate as the ratio between two quantities with different units. That is,

$$\text{Rate} = \frac{\text{Quantity 1}}{\text{Quantity 2}}$$

Both quantity1 and quantity 2 are stated with their units of measurement, say unit1 and unit2, respectively. Hence, the simplest form of this ratio (quotient) is the rate with "unit 1 / unit 2" being the unit for the value of the rate found.

Some common examples of rates: Distance per unit time (called speed), interest per year (for a saving account), birth rate of a country (number of births per year), purchased quantity per cost, number of heart beats per minute, etc., are examples of rates.

Example 24

Ben rode his bike for 2 hours and traveled 30 km. Find the speed at which he rode.

Solution

The speed $= \dfrac{\text{Distrance travelled}}{\text{travel time}} = \dfrac{30 \text{ km}}{2 \text{ hours}} = 15$ km/hour .

Example 25

A bat beats its wings 180 times in 10 seconds. Write the rate as a fraction in lowest terms.

Solution

The rate is $\dfrac{180 \text{ beats}}{10 \text{ seconds}} = 18$ beats/second

So, the rate is 18 beats per second.

Example 26

A mountain climber is 3200 meters from the peak. He climbs 50 meters per hour for 8 hours per day. How many days will it be before he reaches the peak?

Solution

The first job is to figure out the rate per day.

(0 meters/1 hour) × (8 hours/ day) = 50(8) hour/day = 400 meter/day.

Hence, he is climbing at a rate of 400 meters per day.

Now divide 3200 by the daily rate to find the number of days it will take him to reach the top.

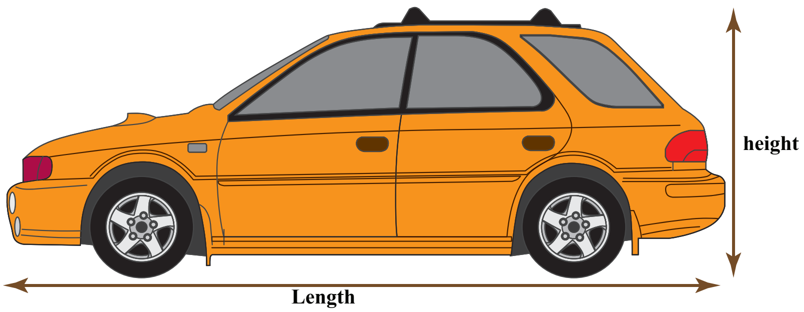
That is, $\dfrac{3200 \text{ meter}}{400 \text{ meter/day}} = 8$ days.

Exercises

1. If 5: 9 = 2: x, then find x
2. If the ratio $A : B = 2 : 3$ and $B : C = 6 : 5$, then find the ratio $A : C$
3. In a classroom, the ratio of male students to females is 2 to 3. If there are 50 students in the classroom, how many students are females?
4. On a trip of 40 travelers, if there were 10 more women than men, what is the ratio of the women to the men ?
5. If the ratio of m to n is 2 to 3 and $\dfrac{2n+5}{m} = 4$, then find m.

6. The ratio of students to teachers in certain school is 39**:** 2. The number of students in the school is 819. If the number of female teachers is 25, what is the number of male teachers in the school ?
7. Convert the speed of 54 km/hour to meter/second.
8. A tiger in a zoo has consumed 39 kg of meat in 6 days. If it continues to eat at the same rate, in how many more days will its total consumption be 91 kg?
9. On a trip, a man drove for 2 hours 80 km per hour and for 3 hours 60 km per hour. What was his average speed for the whole trip?
10. A man completed a journey in 5 hours. He traveled the first half of the journey at the rate of 9 km/hr and the second half at 6 km/hr. How long was the total journey?
11. If one worker can pack 3 boxes every 5 minutes and another worker can pack 4 boxes every 7 minutes, how many minutes will it take these two workers, working together, to pack 82 boxes?

Scale drawing, also known as drawing to scale, is to copy an object by reducing or enlarging its size without changing its shape. To draw a figure similar to a large object, like the drawing of a van below, we usually need to reduce the actual size of the object.



In real-life, the length of this van may measure 6 meters (600 cm). However, you may draw the van on a sheet of paper with a length of 24 cm. This means, the actual length is reduced by a factor of $\frac{600}{24} = 25$. In other words, 24 cm length on the drawing is $\frac{1}{25}$ of the real length of the van. We write this situation as a ratio 1**:**25 (1 to 25). This ratio, $\frac{1}{25}$, is called the **scale factor** of the drawing. If the drawing is to the scale, the reduction on the height should be also with the same factor. That is, if the actual height of the van is, say 200 cm, then the corresponding height of the drawing should be $\frac{1}{25}\times 200 = 8$ cm.

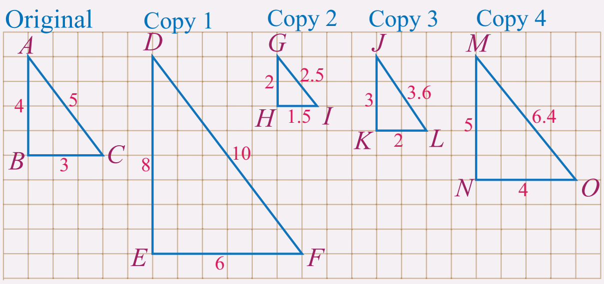
For any scaled drawing, a scale is shown as a ratio, such as 1:25. A drawing at a scale of 1:25 means 1 unit of length on the drawing represents 25 units of length on the real object. This means also that the drawing is 25 times smaller than the real size of the object.

In general, when a copy (or drawing) is described as a scaled drawing at a scale of $a : b$, it means each sides of the drawing is $\frac{a}{b}$ times as long as the corresponding sides in the original object. We call this number (ratio) $\frac{a}{b}$ the **scale factor** of the drawing. The size of the scale factor affects the size of the copy. A scale factor greater than 1 makes a copy which is larger than the original. A scale factor less than 1 makes a copy which is smaller than the original.

Notice that, when we state a scale $a : b$ $\left(\text{or } \frac{a}{b}\right)$, the first number (the numerator) always refers to the length of the copy and the second number (the denominator) refers to the real length of the original object.

Example 27

An original triangle *ABCD* and its four copies are given in the figure below. For each copy, tell whether it is a scaled copy of the original triangle. If so, what is the scale factor?



Solution

The given original triangle is ΔABC. For each copy (drawing), we need to check whether the ratio of each side to the corresponding side of the original are equal.

(i) The ratios of the sides of ΔDEF to the corresponding sides of ΔABC is $\frac{8}{4}=\frac{6}{3}=\frac{10}{5}=2$

Hence, Copy 2 is a scaled copy of the original triangle. The scale factor is 2, i.e, each side in Copy 1 is twice as long as the corresponding side in the original triangle.

(ii) The ratios of the sides of ΔGHI to the corresponding sides of ΔABC is $\frac{2}{4}=\frac{1.5}{3}=\frac{2.5}{5}=\frac{1}{2}$.

Hence, Copy 1 is a scaled copy of the original triangle with the ratio of 1:2; i.e., the scale factor is $\frac{1}{2}$. That is, each side in ΔGHI is half of the corresponding side in ΔABC.

(iii) Copy 3, ΔJKL is not a scaled copy of ΔGHI because $\frac{KJ}{AB} \neq \frac{KL}{BC}$ as $\frac{3}{4} \neq \frac{2}{3}$.

(iv) Copy 4, ΔMNO is also not a scaled copy of ΔGHI (You can check the ratio of the sides).

Note: Scale can be written without units (e.g., 1 to 400) or with units (e.g., 1 cm to 5 km). When a scale does not have units, the same unit is used for distances on the scale drawing and actual distances. For a scale with units, say, "1 cm to 5 km", 1 centimeter on the drawing represents 5 actual kilometers on the ground.

Example 28

A man drew a floor plan of a hall using the scale 1 cm to 2.5 meters. If the man's drawing is 4 cm wide and 6 cm long, what are the dimensions of the actual hall?

Solution

Given: the width of the drawing is 4 cm and its length is 6cm .

Let W be the width and L be the length of the actual hall.

The given scale is 1 *cm* to 2.5 *m*. Consequently,

$$\frac{4\text{ cm}}{W}=\frac{1\text{ cm}}{2.5\text{ m}} \Rightarrow W=\frac{4\text{ cm}\times 2.5\text{ m}}{1\text{ cm}}=4\times 2.5\, m=10\, m;\text{ and}$$

$$\frac{6\text{ cm}}{L}=\frac{1\text{ cm}}{2.5\text{ m}} \Rightarrow L=\frac{6\text{ cm}\times 2.5\text{ m}}{1\text{ cm}}=15\, m.$$

Therefore, the hall is 10 meters wide and 15 meters long.

Exercises

1. The scale drawing of a tree is 1:1100. If the height of the tree on paper is 2 cm, what is the height of the tree in real life?

2. Suppose that the length of a vehicle is drawn to scale with the scale of 1:20. If the length of the drawing of the vehicle on paper is 12 inches, how long is the vehicle in real life?
3. A container is a rectangular box with length 9 meters, width 6 meters and height 4 meters.

   Determine whether each of the drawings of the container with the following dimensions is a scaled drawing of the container. If so, what is the scale factor?

   (a) length 15 cm, width 10 cm, and height 7 cm.

   (b) length 14 cm, width 10.5 cm, and height 7 cm.

   (c) length 13.5 cm, width 9 cm, and height 6 cm.
4. Hanna draw a floor plan of her classroom using the scale 1 inch to 6 feet

   (i) Hanna's drawing is 4 inches wide and 5.5 inches long. What are the dimensions of the actual classroom?

   (ii) A table in the classroom is 3 feet wide and 6 feet long. What size should it be on the scale drawing?

   (iii) Hanna wants to make a larger scale drawing of the same classroom. Which of these scales could she use? (Note: 1 foot = 12 inches)

   (a) 1 to 50 (b) 1 to 72 (c) 1 to 100

In this section, you will learn how to read or use conversion graphs and travel graphs

A. Conversion graphs

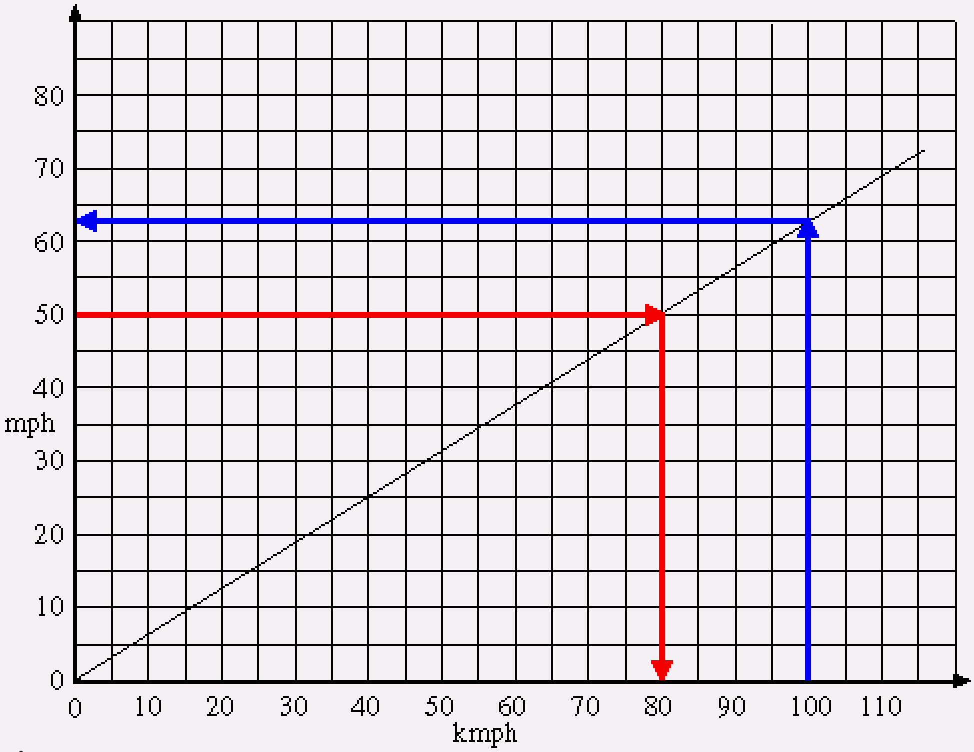
Conversion graphs are straight line graphs that show a relationship between two units and can be used to convert from one unit to the other. This could be changing between miles and kilometres, dollars to other currency, etc.

Example 29

Look at the conversion graph below (the straight line from the origin towards the upper right corner). The vertical axis shows miles per hour (mph) and the horizontal axis shows kilometres per hour (kmph). The line on the graph can be used to convert speeds from mph to kmph and vice versa. For instance, convert

(a) 50 mph to kmph

(b) 100 kmph to mph



Solution

(a) To convert 50 mph into kmph: Start by finding 50 on the mph axis (the vertical axis). When you have found it, draw a horizontal line from this point to the conversion line; and then, from the conversion line draw a vertical line down to the kmph axis as shown on the graph above. This brings you to the value 80 on the kmph axis.

This means, 50 mph = 80 kmph.

(b) To convert 100 kmph into mph, starting at the point of 100 on kmp axis, draw a vertical line to the conversion line; and then, from the conversion line draw a horizontal line to the mph axis. This brings you to the point that corresponds to 63 on the mph axis.

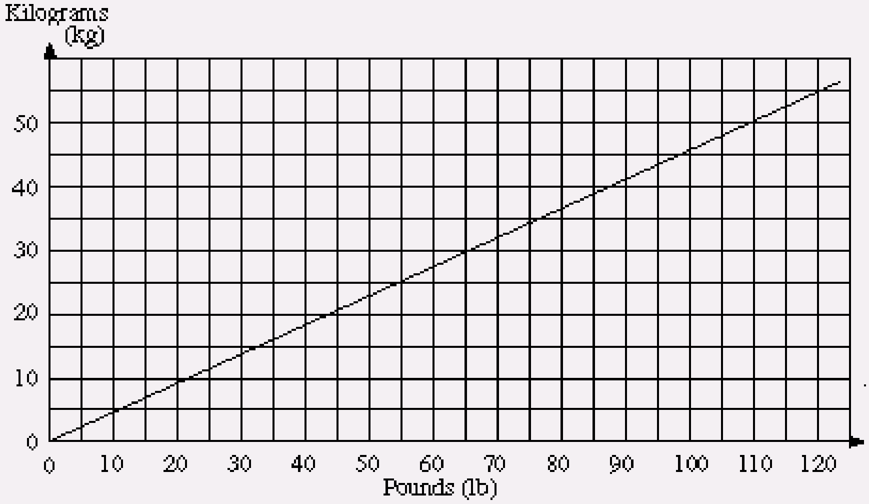
This means, 100 kmph = 63 mph.

Example 30

The graph below can be used for converting between weight in kilograms (kg) and weight in pounds (lb). Use the graph to convert

(a) 30 kg to lb

(b) 110 lb to kg



Solution

Notice that the vertical axis is for kg and the horizontal axis is for lb. To get the desired conversion, we proceed exactly as in the previous example.

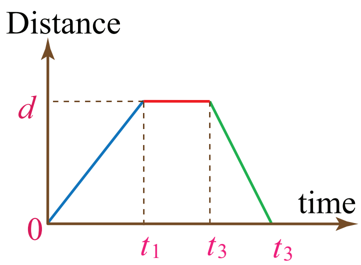
(a) To convert 30 kg to lb: Starting from 30 on the kg axis, draw a horizontal line to the conversion line. When you meet the conversion line, draw from there vertically down to the lb axis. This brings you to the point that corresponds to 65 on the lb axis. Therefore, 30 kg = 65 lb.

(b) To convert 110 lb into kg: Starting from 110 on lb axis, draw a vertical line to the conversion line; and then, from the conversion line draw a horizontal line to the kg axis. This brings you to 50 on the kg axis. This means, 110 lb =50 kg.

B. Travel graphs

A travel graph shows the distance travelled away from a starting point against time. This allows us to identify the speed of the traveler, points at which the traveler were stationary (not moving), and when the traveler is returning to the starting point.

A travel graph is a distance-time graph; i.e., the vertical axis shows the distance and the horizontal axis shows the time elapsed. Here, it is important to identify the following types of movement.



- The horizontal line indicates a stationary period as the distance is not changing,. For instance, the adjacent graph shows that the object is at rest (stationary) between time t_1 and t_2.

- A straight line segment shows that the object is moving with a constant speed. The speed is given by the slope of the line segment, i.e.,

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}.$$

- When the traveler is moving away from the starting point, the slope is positive. Negative slope indicates that the object is moving back toward the starting point. For instance, the above graph shows that an object is moved away a distance of d units within time t_1. It is remained stationary during the period from t_1 to t_2. It is then moved back toward the point of start during the time interval from t_2 to t_3. At time t_3 its displacement from the starting point is 0. That is, it is back to the starting point by the time t_3.

Example 31

The travel graph given below shows a man's journey. Look at each part of the graph and answer the following questions.

(a) What is happening in the section of the journey marked A?

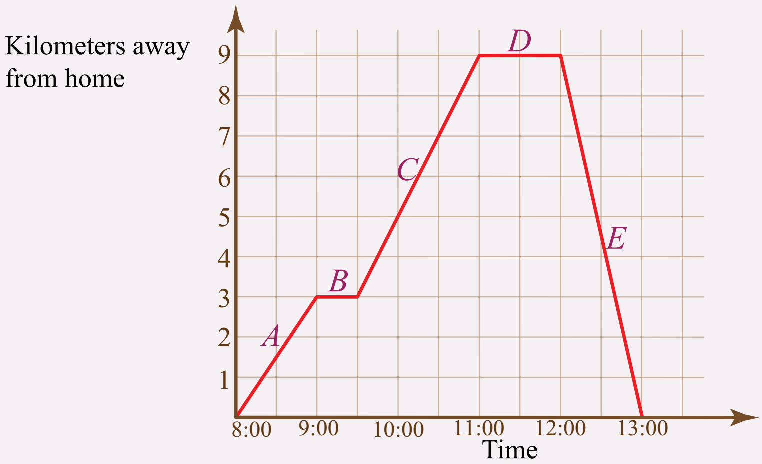
(b) What can you say about section B of the journey?

(c) At what speed is the person travelling in section C?

(d) Describe what is happening at section D.

(e) At what speed is the person travelling in the last part of the journey, section E?

(f) What distance is travelled overall?



Solution

(a) The first part of the journey starts at 8:00 and finishes at 9:00, lasting an hour. Miles away from home begins with 0 (at home) and finishes at 3 miles. Hence,

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}} = \frac{3 \text{ km}}{1 \text{ hour}} = 3 \text{ km per hour on the journey marked } A.$$

(b) B indicates that he stopped for 30 minutes (0.5 hours) between 9:00 and 9:30.

(c) The next section of movement (C) is from 9:30 to 11:00, which is 1 hour 30 minutes, i.e., 1.5 hours. This section goes from 3 km to 9 km on the graph indicating a distance of $9 - 3 = 6$ km. Therefore, the speed in this section is

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}} = \frac{6 \text{ km}}{1.5 \text{ hour}} = 4 \text{ km/hour.}$$

(d) The part of the graph marked *D* shows that he stopped for one hour (from 11:00 to 12:00)

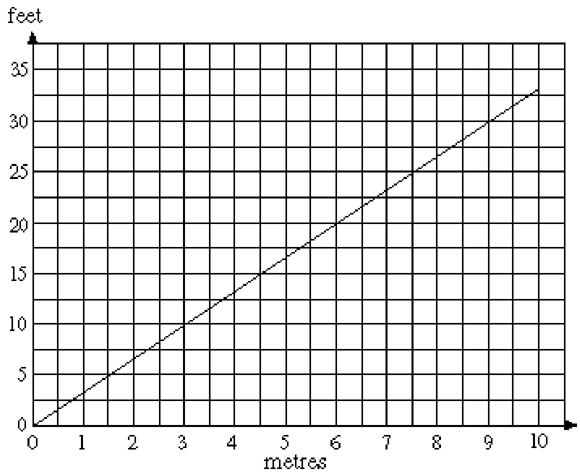
(e) Section E indicates that,

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}} = \frac{(0-9) \text{ km}}{(13-12) \text{ hour}} = -9 \text{ miles/hour.}$$ This means, he travelled at the speed of 9 km/hour back toward home.

(f) From the graph you can see that he travelled 9 km away and 9 km back. Hence, overall he travelled 18 km.

Exercises

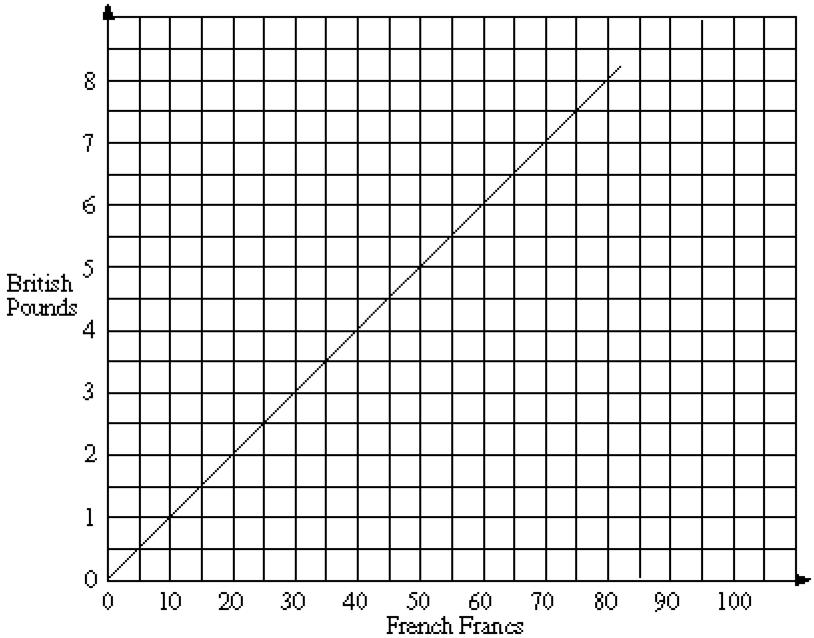
1. The graph below can be used for converting between length in metres (m) and length in feet (ft). Use the graph to convert each of the following.



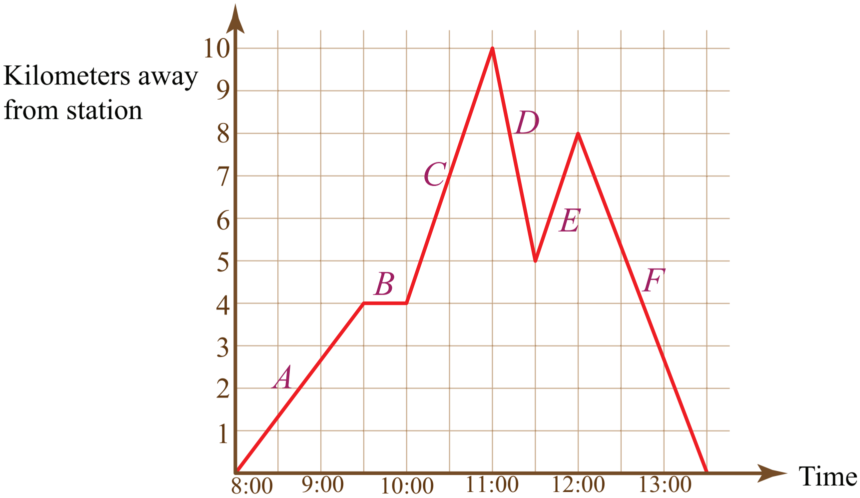
   (a) 1.5 m to ft
   (b) 5 m to ft
   (c) 9 m to ft
   (d) 15 ft to m
   (e) 12.5 ft to m
   (f) 20 ft to m

2. The graph below can be used to convert between British Pounds and French Francs (which was used before EURO). Use the graph to convert each of the following.

   (a) 25 Francs to Pounds
   (b) 50 Franks to Pounds
   (c) 2.50 Pounds to Francs
   (d) 6 Pounds to Francs.



3. A cyclist leaves home at 19:00 and cycles at a constant speed for 10 miles, reaching his destination at 20:15. He stays there for half an hour and returns home cycling at a constant speed at 21:30. Draw a travel graph to represent this journey.
4. The travel graph shown below shows the journey of a bus from station. Describe the journey, including the speed, in each of the sections marked A to F. Find also the overall distance travelled.



The term "percentage" was adapted from the Latin word "per centum", which means "by the hundred". Percentages are fractions with 100 as the denominator.

DEFINITION

Percentage is defined as a given part or amount in every hundred. It is a **fraction with 100 as the denominator** and is represented by the symbol '%'.

That is, $n\%$, read as n percent, means $\frac{n}{100}$.

For example, $25\% = \frac{25}{100} = \frac{1}{4}$, $50\% = \frac{50}{100} = \frac{1}{2}$, etc.

Similarly, $100\% = \frac{100}{100} = 1$ which means the whole. The fact that 100 % = 1 is used to express fractions as a percentage. That is, $\frac{a}{b} = \frac{a}{b} \times 1 = \frac{a}{b} \times 100\%$. Thus, we can express fractions as percentages by multiplying the given fraction by 100 and attach the symbol % to the result.

EXAMPLE 32

(a) In expressing $\frac{1}{5}$ as a percentage, we get $\frac{1}{5} \times 100\% = 20\%$.

Similarly, express $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$ and $\frac{1}{8}$ as percentages.

(b) In expressing 0.3 as a percentage, we get $0.3 \times 100\% = 30\%$.

Similarly, express 0.2, 0.125, and 1.5 as percentages.

EXAMPLE 33

In a contest for a seat in parliament, Mr. Musa has got 750 votes in a village. If there were 1250 voters in the village, what percentage of the votes did Mr. Musa get?

Solution

His votes are 750 out of 1250. In fraction, this is 750/1250. We know how to change this fraction to percentage. That is, $\frac{750}{1250} \times 100\% = 60\%$.

Hence, he got 60% votes from the village.

We usually need to know a percentage of certain quantity. Calculating percentage of a certain quantity means to find the share of a whole, in terms of 100. By the definition, $n\%$ of a quantity Q is equal to $\frac{n}{100} \times Q$.

Example 34

(a) 40 % of 60 is $\frac{40}{100}\times 60 = 24$.

(b) 12.5% of 64 is $\frac{12.5}{100}\times 64 = 8$.

(c) 150% of 70 is $\frac{150}{100}\times 70 = 105$.

Percentage increases and Percentage decreases

There are various real life problems that involve percentages. One of the common usages of percentage is to analyze the relative change in values. Relative change in values is described by either **percentage increase** (if original value is increased) or **percentage decrease** (if original value is decreased). These are given as follows:

$$\text{Percentage increase} = \frac{\text{New value} - \text{Original value}}{\text{Original value}} \times 100\%$$

$$\text{Percentage decrease} = \frac{\text{Original value} - \text{New value}}{\text{Original value}} \times 100\%$$

Example 35

There were 15 fishes in a tank. After a year the fish have bred and there are now 20 fishes in the tank. By what percentage has the number of the fishes increased?

Solution

$$\text{Percentage increase} = \frac{20-15}{15}\times 100 = \frac{5}{15}\times 100 = 3.3\%.$$

Example 36

If the price of an item is increased from L$60 to L$75, what is the percentage increase in the price?

Solution

$$\text{Percentage increase} = \frac{\text{New price} - \text{Original price}}{\text{Original price}}\times 100 = \frac{75-60}{60}\times 100 = \frac{15}{60}\times 100 = 25\%$$

Therefore, the price is increased by 25%.

Example 37

The price of an item was L$20. The item is now sold at L$18. What is the percentage decrease in the price?

Solution

$$\text{Percentage decrease} = \frac{\text{Original value} - \text{New value}}{\text{Original value}} \times 100 = \frac{20-18}{20} \times 100$$

$$= \frac{2}{20} \times 100 = 10\%.$$

Exercises

1. Find the following percentages .
   (a) 15 % of 450
   (b) 45.5 % of 320
   (c) 120 % of 65
   (d) 12.5% of $\frac{6}{5}$ of
2. Express each of the following fractions or decimals as percentage.
   (a) $\frac{2}{5}$
   (b) $\frac{5}{6}$
   (c) $\frac{5}{2}$
   (d) 0.27
   (e) 0.344
   (f) 3.2
3. What percentage of 60 cm is 15 cm?
4. Three friends *K*, *L* and *M* received votes for their contest for an office. *K* secured 45% of the votes, *L* had 33% of the votes and *M* had the rest of the votes. If *M* secured 1,430 votes, calculate
   (a) The total number of votes cast;
   (b) How many more votes *K* received than *L*?
5. The ratio of girls to boys in a classroom is 1 to 2. What is the percentage of the girls in the classroom?
6. The area of a rectangular plot is 1120 square meters. If its width is 30% less than its length, what is the perimeter of the plot (in meter)?
7. In the morning there were 45 chocolates in a shop. At noon there were only 12 left. Determine the percentage of chocolates sold.

8. An item is sold for L$120 after 20% discount from its original price. How much was the original price of the item?
9. A businessman paid L$2000 to rent an office for one year. The rent is then increased each year by 20% of the previous year. How much will to pay for the first three years he rents the office?
10. Type A and Type B alloys contain 20% and 30% copper, respectively. How many Kilo grams of each alloy should be mixed to obtain 10 kg of an alloy containing 24% copper?

Key Terms

- Arithmetic mean
- Average
- Box and whisker plot
- Central tendency
- Class mark
- Conversion graph
- Data
- Data sources
- Data tabulation
- Frequency
- Frequency distribution
- Grouped data
- Grouped frequency distribution
- Histogram
- Interquartile range
- Lower quartile
- Median
- Mode
- Percentage
- Percentage decrease
- Percentage increase
- Population
- Primary data
- Qualitative data
- Quantitative data
- Quartiles
- Range
- Rates
- Ratios
- Raw data
- Sample
- Scale factor
- Scale drawing
- Secondary data
- Statistics
- Stem and leaf plot
- Travel graph
- Upper quartile
- Variability

SUMMARY

- Statistics is the science of collecting, organizing, presenting, analysing and interpreting data in order to draw conclusions.
- A population is the complete collection of individuals, objects or measurements that have a characteristic in common.
- A small part (or a subset) of a population is called a sample.
- If the categories of a classification are based on some attribute or characteristics whose values are not numbers, then it is called qualitative classification.
- If the characteristic of interest is numerical, then it is called quantitative classification.
- Data is said to be primary, if it is obtained first-hand for the particular purpose on which one is currently working.
- Data that has been previously collected for a similar or different purpose is called secondary data.
- A statistical table is a systematic presentation of data in columns and rows.
- The quantity that we measure from observation is called a variate (or variable).
- A frequency distribution is a distribution showing the number of observations associated with each data value.
- A histogram is a pictorial representation of a frequency distribution in which the variables (V) are plotted on the x-axis and the frequency of occurrence is plotted on the y-axis.
- If $x_1, x_2, x_3, \ldots x_n$ are the n observations of a variable then the mean ($\bar{x}$) is given by

  $$\bar{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}.$$
- The median of a variable is the value that lies in the middle of the data when arranged in ascending or descending order.
- The mode of a variable is the most frequent observation of the variable that occurs in the data set.
- The range R of a set of numerical data is the difference between the maximum and minimum values.

  Range = highest value – lowest value

- The quartiles of a data, denoted by, Q_1, Q_2 and Q_3, are values that divides the data into four equal parts. **Inrequartile range** is the difference between the upper quartile (Q_3) and the lower quartile (Q_1).
- A stem and leaf plot is a way to plot data where each data value is split into two parts, called stem and leaf. The stem of a number is the digit at the highest place-value. The leaf of the number is the digit at the lowest place value.
- A box and whisker plot is a visual(graphical) tool used to display the variance or dispersion of a data using the three quartiles and two extremes (lowest and highest values) of the data.
- A ratio is the quotient of two like quantities or numbers with the same units. A ratio a to b is written as $a : b$. It is the fraction $\frac{a}{b}$.
- A rate is the quotient of two quantities of different units.
- Scale drawing is to draw or copy an object by reducing or enlarging its size without changing its shape.
- Conversion graphs are straight line graphs that show a relationship between two units and can be used to convert from one unit to the other.
- A travel graph shows the distance travelled away from a starting point against time. This allows us to identify the speed of the traveler, points at which the traveler were stationary (not moving), and when the traveler is returning to the starting point.
- Percentage is defined as a given part or amount in every hundred. It is a fraction with 100 as the denominator and is represented by the symbol '%'. That is, $n\%$ means $\frac{n}{100}$.

Exercises

1. What is meant by a frequency distribution of a data?
2. The marks of 30 students in a mathematics test are given below:

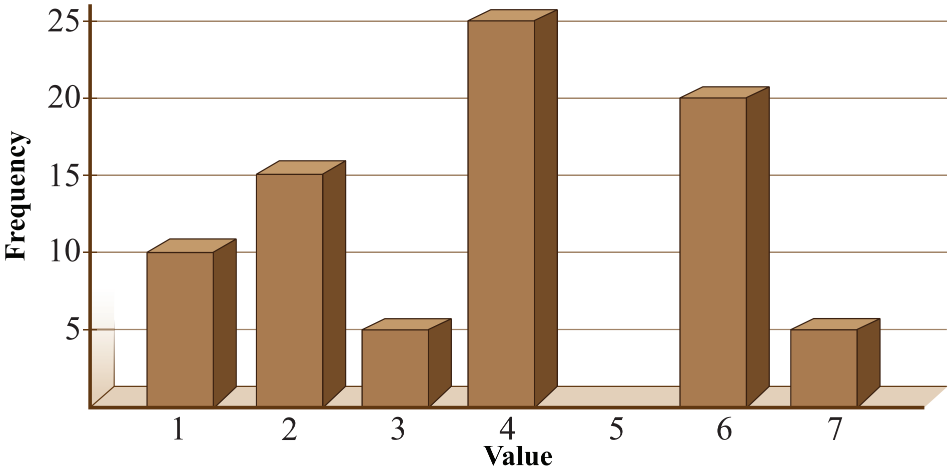
3	5	4	6	8	12	14	5	6	5
8	5	9	10	9	10	12	10	12	10
12	13	10	15	14	15	14	15	14	14

(a) Construct a frequency distribution table.

(b) Draw a histogram to represent the data.

(c) What percent of the students have scored less than 15?

3. Refer to the following histogram to answer the questions that follow.



(a) Prepare a frequency distribution table.

(b) What is the highest variable?

(c) What is the highest frequency?

(d) How many variates occur 5 times?

(e) Which variates have the minimum frequency?

4. Find the mean, median, mode,range, variance and standard deviation of the population function whose distribution is given in the table below.

V	2	3	4	5	6
f	2	4	1	2	3

5. The table below shows the number of students who scored marks 3, 4 or 5 in a math test.

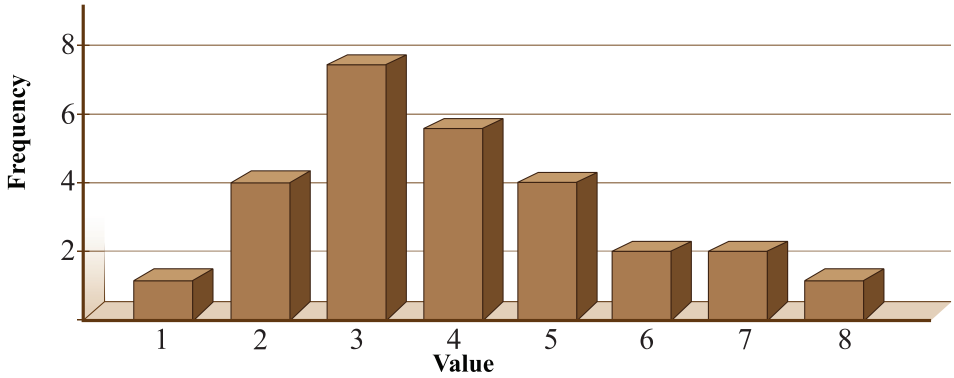
Mark	3	4	5
Number of Students	3	x	4

If the mean mark is 4.1, how many students scored 4?

6. In a study, 20% of a group of 80 men and 15% of a group of 120 women were found to be diabetic. What is the mean percentage of diabetics for both the groups combined?

7. The median of $x - 4$, x, $2x$ and $2x + 12$ is 9, where x is a positive integer. Find the value of x.

8. Find the mean, median, mode, range from the histogram given below.



9. In a class of boys and girls, the mean weight of 8 boys is 55 kg and the mean weight of a group of girls is 48 kg. The mean weight of all the children is 50.8kg. How many girls are there?

10. From the members of a farmers' association 50 farmers cultivated wheat. An agricultural expert wants to study the farmers' yield in terms of quintals they harvested per hectare and found the following

50	45	45	50	46	48	55	48	52	54
51	52	45	55	46	50	55	54	49	51
48	46	51	52	47	45	49	54	46	48
53	52	48	46	55	47	51	47	50	53
47	53	48	45	54	48	50	46	52	54

(a) Prepare a frequency distribution that represents the data.

(b) Draw a histogram.

(c) Make the stem and leaf plot of the data.

(d) Find the mode of the data.

(e) What percentage of the farmers have produced more than 52 quintals per hectare.

(f) Construct the box and whisker plot of the data and discuss its variability.

11. The ratio of men to women on a trip is 3 to 4. What is the percentage of men on the trip?

12. In a classroom, the ratio of girls to boys is 5 to 3. If the class has a total of 40 students, how many more girls are there than boys?

13. Among the people on a conference, 55% can speak French, 40% can speak Arabic and 15% can speak both French and Arabic. What percentage of the people can speak neither French nor Arabic?

14. The following table shows the age distribution of members of a youth club:

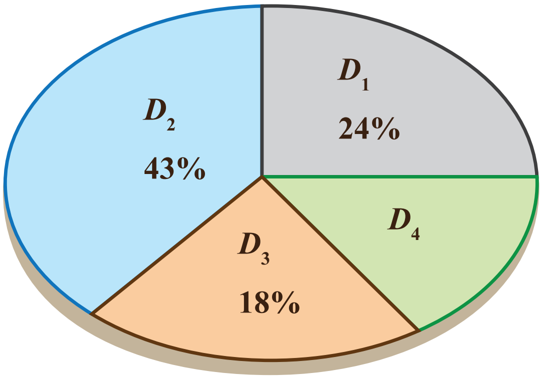
Age (year)	18	19	20	21	22
Frequency	4	9	7	11	9

(a) What is the median age of the members of the club (in years) ?

(b) What is the mean age of the members (in years)?

(c) What is the percentage of the members whose age is equal to the mode?

15. The following chart shows the percentage of students enrolled in four departments D_1, D_2, D_3 and D_4 of a college.



(a) What percentage of the students have enrolled in D_4?

(b) If the total number of students in the college is 1600, what is the number of students in D_3?

(c) What is the ratio of the number of students in D_1 to those in D_3?

(d) If the number of students in D_1 is 120 more than those in D_3, what is the total number of students in the college?

What is Bullying?

Any unwanted written, verbal, graphic, or physical act by an individual or group toward another person(s) that causes harm or distress.

Types of Bullying

- Physical
- Verbal
- Social
- Emotional
- Cyber

Signs of Bullying

- Headaches
- Depression
- Loss of friends
- School absenteeism
- Academic problems

What You Can Do

PREVENT

- Be a role model for positive communication, healthy relationships, and self-care.
- Reinforce acts of kindness, respect, and inclusion.
- Set policies and rules about bullying.

RECOGNIZE

- Know the definition of bullying and its many forms.
- Talk with and actively listen to the youth who confide in you.
- Watch for warning signs of bullying.

INTERVENE

- If you witness bullying behavior
- Respond quickly and consistently to send the message that it is not acceptable.
- Separate the students involved.
- Meet any immediate medical or mental health needs.
- Stay calm and model respectful behavior.

Source: Teacher's Diary on *Cyber-Crime Awareness* by UNODC, Cybercrime and MoE, Republic of Liberia

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MINISTRY OF EDUCATION

TEXTBOOK MANAGEMENT

Student Textbook Assignment Form

School : ____

District : ______________________ Country: ______________________

Teacher : ______________________ Subject: ______________________

Grade : ____

No.	Name of the Student	Name of the Textbook	Barcode Number	Condition of the TB when received	Date Received	Date Returned	Condition of TB when returned	Signature of Student
1								
2								
3								
4								
5								
6								
7								
8								
9								
10								
11								
12								
13								
14								
15								