

# MATHEMATICS

## A Textbook for Grade 11



M11TB

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# Foreword

Liberia, having gone through a period of utmost turmoil till 2003, due to the civil wars, is still reeling under its effect and the added trauma of Ebola in 2014 and effects of the COVID-19 outbreak in 2020. The Liberian government, in the past decade, has made valiant efforts to bring order to the lives of its people. In one such effort, the Ministry of Education (MoE) brought changes to the National Curriculum Framework which are relevant to the present generation, and which would prepare them to meet the challenges of the changing trends of the world. The National Curriculum Framework (NCF) 2018 recommends a change in basic assumptions in the teaching learning process from behaviorist to constructivist approach — moving from hardcore print material to the digital world. Keeping in consideration the sociocultural context and varied experiences of learners as laid down in the Framework, our Teaching Learning Materials are expected to be competent to use multiple methods and techniques like e-learning resources, energized textbooks, and readily available reference material to engage the learners.

As a first initiative, the MoE, through its World Bank-funded Improving Results in Secondary Education (IRISE) project, has adapted textbooks for Grades 10 to 12 in five subjects — English Language and Literature, Mathematics, Biology, Physics and Chemistry.

The National Curriculum Framework, 2018, recommends that children’s learning at school is a reflection of their life outside the school and shows them the path to become a responsible citizen who makes knowledge-based choices. This principle marks a departure from the legacy of teacher centered learning to student centered learning. The syllabi and textbooks developed on the basis of the NCF indicate a serious attempt to implement the idea of Activity Base Learning (ABL). We hope these measures will take us ahead in the direction of building a system of education as outlined in the NCF.

Combined with the efforts by the school principals and teachers this will encourage children to reflect on their own learning and to pursue imaginative activities and questions. With this in mind, perhaps for the first time in our country, we are able to provide separate subject specific textbooks accompanied with guides for teachers for 10–12 grades. Not only have these been developed, adapted and modified to the Liberian context, each of the eight Minimum Learning Competencies (MLCs) have been included in each textbook. So as to reach every high school student, for the first time in the country’s history we have included the digitized form of the textbook accessible by a Quick Response (QR) code given in each book. Not only does it have the digitized textbook, but it provides additional learning materials for use by students, teachers and interested persons. The links to these e-resources and digitized material is being made available on the MoE’s website.

The Textbooks and Teacher Guides have reached the hands of the students after a rigorous quality evaluation by carefully handpicked subject specialists by the MoE, to whom the Ministry expresses gratitude. For the success of this project, I acknowledge the contributions of the IRISE Project Team in the World Bank, and in particular, the Task-Team Leaders; the Project Implementation Team in Liberia headed by its Coordinator Abraham A. Kiazolu II, supported by the Executive Director of the Center of Excellence for Curriculum Development and Textbooks Research, Mrs. Julia K. Sandiman-Gbeyai and her technical working group (TWG), and the International Textbook Consultant and Advisor, Dr Shveta Uppal engaged by the MoE. These notwithstanding would not have been possible without the guidance of the Senior Management Team (SMT) of the Ministry of Education, and in particular, the Deputy Ministers for Instructions, Administration, and Planning, Research and Development, respectively.

Professor Dao Ansu Sonii, Sr.  
Minister of Education  
Republic of Liberia

Monrovia, Republic of Liberia  
January 24, 2023

# Acknowledgments

The development of textbooks contributes to the quality of teaching and learning that go on in the classroom.

The Ministry of Education (MoE) has aligned its Curriculum for Grades 10–12 to the National Curriculum Framework (NCF) of 2018. To ensure the provision of Teaching Learning Materials (TLMs) that support the revised curriculum, the Ministry has sought, reviewed and adapted a new set of textbooks and teacher guides along with digitized contents and e-learning resources for the five core subjects taught at the Senior Secondary education level, namely English Language and Literature, Mathematics, Biology, Chemistry and Physics, through an internationally competitive bidding process from the market supported by the World Bank funded Improving Results in Secondary Education (IRISE) Project.

With profound gratitude and honor, we recognize the Senior Management Team of the Ministry, headed by the Coach, Professor D. Ansu Sonii, Sr., for the strategic decision to make teaching learning materials available and accessible to all in the Liberian Senior Secondary School System, and for providing directions through the process of securing these textbooks and other teaching learning materials for our students and teachers. Our special thanks and appreciation to the World Bank for the financial support towards this policy intervention, and its education task-team including Alonso Sanchez, Oni Lusk-Stover and Binta B. Massaquoi for all their technical inputs offered throughout the process to ensure the kind of quality TLMs the Liberian students deserve are made available for improved learning outcomes.

We would like to specifically recognize the invaluable contributions of the 15 subject experts selected by the MoE from across the various education systems and the West African Examinations Council (WAEC) to evaluate, review and sign off on these teaching learning materials. They didn't just deliver according to our expectations, but also ensured the contextual relevance of the materials

to the Liberian Secondary Education Curriculum and its minimum learning competencies (MLCs). These subject experts include Professor Isaac Saye-Lakpoh Zawolo – *Superintendent* of the Monrovia Consolidated School System (MCSS), Mr. Matthew V.Z. Darblo, Sr. – *Mathematics Instructor* at the University of Liberia (UL), Mr. Charles Tieh Bropleh – *Mathematics Specialist* (MoE), Mrs. Linda Y. Dean – *English Specialist*, Mr. Hassan M. Bangura – *English Language and Literature Expert*, Mr. J. Emmanuel Milton – *English Specialist* (MoE), Mr. Moses K.M. Togbah – *Physics Specialist*, Mr. Prince A. Dossen – *Physics Specialist*, Mr. Benjamin Koryah – *Physics Instructor* at the University of Liberia (UL), Mr. Dominic Dugbe Doe – *Chemistry Specialist*, Mr. Patrick A. Anderson, Sr. – *Director* of the Division of Technical and Vocational Education (MoE), Mr. Kandakai Massaquoi – *Chemistry Specialist*, Ms. Patricia N. Doe – *Head* of Biology Department, African Methodist Episcopal University (AMEU), Mr. Job Carpenter – *Biology Specialist* and Mr. Prince Philip K.A. Aderibigbe – *Biology Specialist*.

The MoE is sincerely grateful to Dr Shveta Uppal, the *International Textbook Consultant* engaged by the IRISE Project to provide technical guidance and quality assurance support to the revising of the Textbooks Management Guidelines (TMG) and the procurement process leading to the provision of textbooks, teacher guides, digital contents and e-learning resources for the Senior Secondary School System in Liberia in accordance with the revised TMG. Heartfelt thanks and appreciations also to the *Executive Director* for the Center of Excellence for Curriculum Development and Textbooks Research, Mrs. Julia K. Sandiman-Gbeyai, and members of her Technical Working Group (TWG) for taking up the responsibility to lead the process of making textbooks and other TLMs available to Liberian students and teachers.

Lastly, we acknowledge the IRISE Project Delivery Team led by Mr. Abraham A. Kiazolu, II – *Project Coordinator*, Mr. Fuseini A. Abu – *International Procurement Specialist* and Mr. Lawrence S. Taylor – *Project Control Specialist* who coordinated the entire process.

We remain grateful to you all!

Hon. Alexander N. Duopu, Sr.,  
*Deputy Minister for Instruction*  
Ministry of Education, Republic of Liberia  
#The Teacher

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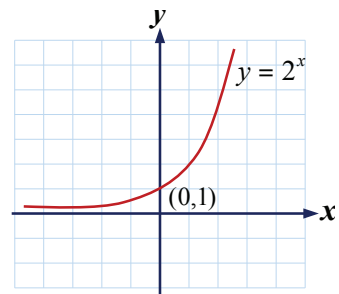
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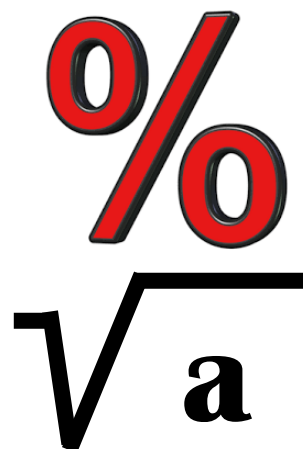


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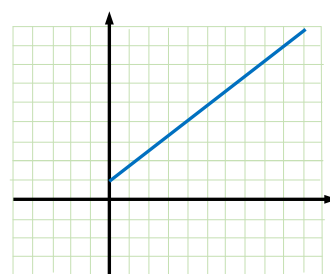
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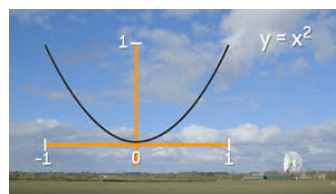
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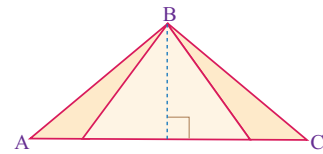
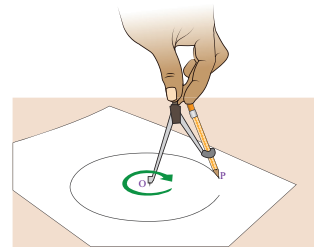
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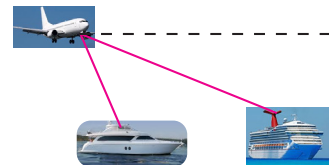
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# CHAPTER

# 1

## MODULAR ARITHMETIC

### Chapter Contents

- 1.1 Polygonal Arithmetic
- 1.2 Modular Arithmetic
- 1.3 Cyclic Variables
- 1.4 Divide using Modular Arithmetic
  - Key Terms
  - Summary
  - Exercises



## **Chapter Outcomes**

By the end of this chapter, you will be able to:

- define and discuss polygonal arithmetic;
- define and discuss modular arithmetic;
- define and discuss cyclic variables;
- divide using modular arithmetic.

## Introduction

In mathematics, modular arithmetic is a system of arithmetic for integers, where numbers “wrap around” when reaching a certain value, called the *modulus*. The modern approach to modular arithmetic was developed by Carl Friedrich Gauss in his book *Disquisitiones Arithmeticae*, published in 1801.

Number patterns have interested mathematicians from the time of the ancient Greeks. Some of this interest centered around the relations between number expressed geometrically and algebraically. This idea is basic to the study of polygonal numbers.

### ACTIVITY 1

1. Define a polygon and give some examples of polygons.
2. What is a regular polygon?
3. Circle is a plane figure. Is it a polygon?
4. Discuss about polygonal numbers?

Polygonal numbers are nonnegative integers represented by geometrical arrangements of equally spaced points that form regular polygons. The most common and basic types of polygonal numbers are triangular, square, pentagonal and hexagonal numbers.

### DEFINITION

A number which corresponds to the number of points on a regular polygon and discrete geometric pattern of equally spaced points is called a *figurate number*.

Polygonal figurate numbers or simply *polygonal numbers* are the numbers which can be arranged as regular polygon such as equilateral triangle, square, pentagon, hexagon, etc. So, polygonal numbers include triangular numbers, square numbers, pentagonal numbers, hexagonal numbers, and so on.

If we represent using “dots” there are special numbers that can form “polygon”. Numbers that can form polygons are called *polygonal numbers*.

The triangular numbers count the number of dots in such a way that when two dots are added to a starting point, an equilateral triangle is formed. By adding three dots to the previous three points, we can obtain a larger equilateral triangle with six

points. A triangle with ten dots can be created by adding four points to the triangle with six points, and so on as shown in Figure 1.

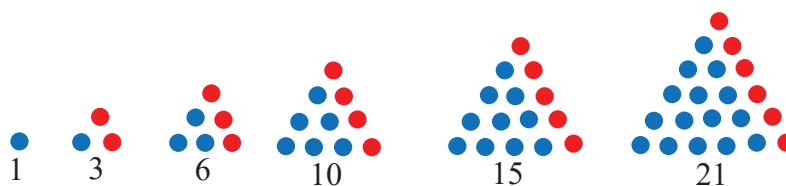


Figure 1.

Continuing with this augmenting arrays of well arranged dots forming an expanded equilateral triangle with respective sum of dots in each such triangle, results in a sequence of numbers 1, 3, 6, 10, 15, 21, 28, ... known as **triangular numbers**.

The square numbers count the number of dots in such a way that when three dots are added to a starting point, a square is formed. By adding five dots to the previous four points, we can obtain a larger square with nine points. A square with sixteen dots can be created by adding seven points to the square with nine points, and so on as shown in figure 2.

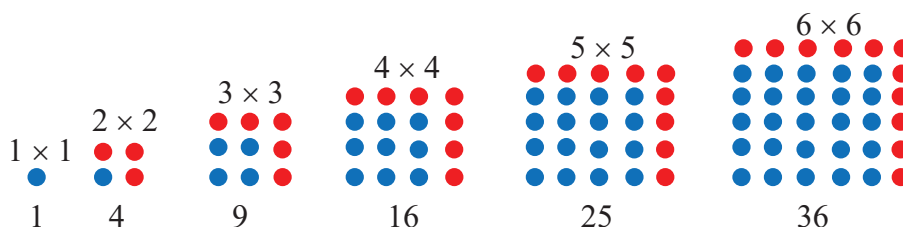


Figure 2.

Similarly, as shown in figure 2, as the numbers three, five, seven, nine, etc. are added to a starting dot, forming a square array as the figure grows outwardly, the sequence of numbers 1, 4, 9, 16, 25, 36, 49, ... are known as **square numbers**.

The pentagonal numbers count the number of dots in such a way that when four dots are added to a starting point, a pentagon is formed. By adding seven dots to the previous five points, we can obtain a larger pentagon with twelve points. A pentagon with twenty-two dots can be created by adding ten points to the pentagon with twelve points, and so on as shown in figure 3.

As shown in figure 3, as the numbers four, seven, ten, thirteen, etc. are added to a starting dot, forming a pentagon array as the figure grows outwardly, the sequence of numbers 1, 5, 12, 22, 35, 51, 70, ... are known as **pentagonal numbers**.

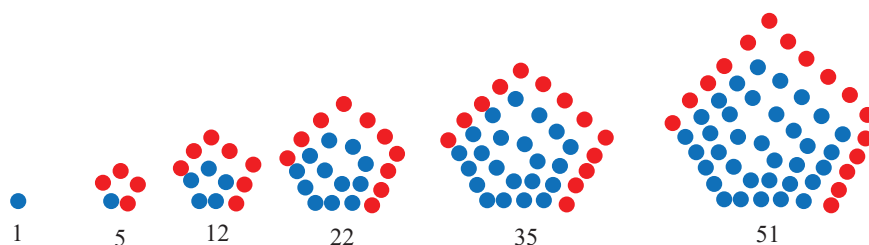


Figure 3.

We have seen that polygonal numbers can be defined geometrically by counting the number of dots in a regular  $k$ -gon as shown in Figure 3. For  $k \geq 3$ , a  $k$ -gonal number is the total number of dots inside one of the regular  $k$ -gon in the figure.

Let  $P_n(k)$  denote the  $n^{\text{th}}$  term of the  $k^{\text{th}}$  polygonal number. For example,  $P_n(3)$ ,  $P_n(4)$ ,  $P_n(5)$  and  $P_n(6)$  represents the  $n^{\text{th}}$  triangular, square, pentagonal and hexagonal numbers, respectively. For convenience, we set  $P_0(k) = 0$ ,  $P_1(k) = 1$ .

Suppose that the first  $n$  regular  $k$ -gons have already been constructed, so that each edge of the outermost  $k$ -gon now has length  $n$ -dots. To construct the  $(n + 1)^{\text{st}}$   $k$ -gon, first we extend the two edges attached to the base point by one dot, and then add edges parallel to the previous edges, so that each (outer) edge now has  $(n + 1)$  dots. The total number of dots we are adding to the figure is  $1 + (k - 2)n$ . Thus, we have the relation,

$$P_{n+1} = P_n + (k - 2)n + 1$$

The  $n^{\text{th}}$  term of  $k$ -gonal number is given by the formula:

$$P_n(k) = \frac{n}{2}[2 + (n - 1)(k - 2)] \quad \text{or} \quad P_n(k) = \frac{n}{2}[(k - 2)n^2 - (k - 4)n]$$

Triangular, square, pentagonal and hexagonal number can be summarized by the table shown below.

$k$ -gon	Name	Formula: $P_n(k) = \frac{n}{2}[2 + (n - 1)(k - 2)]$	Numbers
3	Triangular numbers	$P_n(3) = 1 + 2 + 3 \dots + n = \frac{n(n+1)}{2}$	0, 1, 3, 6, 10, 15, 21, 28, ...
4	Square numbers	$P_n(4) = 1 + 3 + 5 \dots + (2n - 1) = n^2$	0, 1, 4, 9, 16, 25, 36, 49, ...
5	Pentagonal numbers	$P_n(5) = 1 + 4 + 7 + \dots + (3n - 2) = \frac{n(3n-1)}{2}$	0, 1, 5, 12, 22, 35, 51, 70, ...
6	Hexagonal numbers	$P_n(6) = 1 + 5 + 9 + \dots + (4n - 3) = n(2n - 1)$	0, 1, 6, 15, 28, 45, 66, ...

**EXAMPLE 1**

Use the  $n^{\text{th}}$  term formula for polygonal numbers to find each of the following.

- The 27<sup>th</sup> term of triangular number.
- The 12<sup>th</sup> term of pentagonal number.

**Solution**

- (a) The  $n^{\text{th}}$  triangular number is given by the formula:

$$T_n = \frac{n(n+1)}{2}$$

$$T_{27} = \frac{27(27+1)}{2} = \frac{27 \times 28}{2} = 27 \times 14 = 378.$$

Therefore, the 27<sup>th</sup> triangular number is 378.

- (b) The  $n^{\text{th}}$  pentagonal number is given by the formula:

$$T_n = \frac{n(3n-1)}{2}$$

$$T_{12} = \frac{12(3 \times 12 - 1)}{2} = \frac{12 \times 35}{2} = 6 \times 35 = 210.$$

Therefore, the 12<sup>th</sup> pentagonal number is 210.

**Exercises**

- What type of polygonal numbers does the following sequence numbers illustrate?
  - 1, 3, 6, 10, 15, . . .
  - 0, 1, 7, 18, 34, . . .
- Find the value of  $n$  such that a regular  $n$ -gon has interior angle measure of  $140^\circ$ .
- Use the  $n^{\text{th}}$  term formula for polygonal numbers to find the following.
  - the 19<sup>th</sup> term of triangular numbers.
  - the 21<sup>st</sup> term of square numbers.
  - the 39<sup>th</sup> term of pentagonal numbers.
  - the 12<sup>th</sup> term of hexagonal numbers.
- Find the 8<sup>th</sup> heptagonal number.
- What is the next three numbers in this sequence: 1, 5, 12, 22, 35, \_\_\_\_\_, \_\_\_\_\_, \_\_\_\_\_.
- Derive the formula to find the  $n^{\text{th}}$  heptagonal number.

Modular Arithmetic is a special type of arithmetic that involves only integers. In this sub-topic, we will learn about polygonal arithmetic and the basics of modular arithmetic while presenting cyclic variables and solving problems of division using modular arithmetic.

### ACTIVITY 2

1. What is the concept of modular arithmetic?
2. What time will it be 19 hours after 9 o'clock?
3. What time will it be 9 hours before 4 o'clock?
4. Is there any difference between 12-hours clock and 24-hours clock?
5. It is now December, what month was it 19 months ago?
6. What does  $a \equiv b \pmod{n}$  mean?

Modular arithmetic is sometimes known as clock arithmetic as one of the most familiar uses of modular arithmetic is in the 12-hour clock which has the time period divided into two equal halves. The best way to introduce modular arithmetic is to think of the face of a clock.

The numbers go from 1 to 12 but when we get to “13 o'clock”, it actually becomes 1 o'clock again. So, 14 becomes 2, 15 becomes 3, 16 becomes 4 and so on.

This can be keep going, so when you get to “27 o'clock”, we are actually back round to where 3 o'clock on the clock face (and also where 15 o'clock was too).

So, in this clock world, we only care where we are in relation to the numbers 1 to 12. In this world, 1, 13, 25, 37, 49, . . . are all thought of as the same thing, as are 2, 14, 26, 38, 50, . . . and so on.

What we are saying is “ $13 = 1 + \text{some multiple of } 12$ ”, “ $50 = 2 + \text{some multiple of } 12$ ”, or, alternatively, “the remainder when we divide 13 by 12 is 1” and “the remainder when we divide 14 by 12 is 2”. Thus, we can say that 0 coincides with 12 and as the hour hand moves to the right, 1 coincides with 13, 2 with 14 and so on.

The hour hand rotates clockwise which corresponds with numbers increasing when moving to the right on a number line. However, 12 is equivalent to 0 on this circle, which can be written as follows:

$$12 \equiv 0 \pmod{12},$$



Figure 4.

This can be read as 12 is *congruent to 0 modulo 12*. The usual “=” sign is reserved for the straight number line; we use “ $\equiv$ ” on the circle instead. The symbol “**mod 12**” tells us that the circle is divided into 12 equal parts, so that 12 coincides with 0, 13 with 1, etc. We can write this mathematically as:

$$12 \equiv 0(\text{mod } 12), 13 \equiv 1(\text{mod } 12), 14 \equiv 2(\text{mod } 12), \dots, 22 \equiv 10(\text{mod } 12), 23 \equiv 11(\text{mod } 12).$$

**EXAMPLE 2**

It is 5 o'clock now; what time will it be in 100 hours?

**Solution**

$100 = 12 \times 8 + 4$ . So, 4 hours after 5 o'clock is 9 o'clock.

**EXAMPLE 3**

It is now December; what month was it 27 months ago?

**Solution**

Jan	Feb	Mar	Apr	May	June	July	Aug	Sep	Oct	Nov	Dec
0	1	2	3	4	5	6	7	8	9	10	11

Since we have numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 and 11 we will do our arithmetic in modulo 12. It is now December (Month 11) hence to find 27 months ago, we can write

$$27 \equiv 12 \times 2 + 3$$

Therefore, the answer is the 3<sup>rd</sup> month before December which is September.

Modular Arithmetic is a form of arithmetic dealing with the **remainders** after integers are divided by a fixed “*modulus*”  $m$ . Basically, it is a kind of integer arithmetic that reduces all numbers to ones that belongs to a fixed set  $\{0, 1, 2, 3, \dots, n - 1\}$ .

In general, if we are working in mod  $m$ , where  $m$  is any whole number, we write  $a \equiv b \text{ mod } m$ , if  $a$  leaves a remainder of  $b$  when we divide it by  $m$ .

**EXAMPLE 4**

- (a)  $27 \equiv 3 \text{ mod } 4$
- (b)  $24 \equiv 0 \text{ mod } 6$
- (c)  $15 \equiv 1 \text{ mod } 7$
- (d)  $121 \equiv 1 \text{ mod } 12$

**EXAMPLE 5**

Evaluate the following:

- (a)  $11 \pmod{4}$       (c)  $39 \pmod{3}$       (e)  $2 \pmod{3}$       (g)  $5 \pmod{2}$   
 (b)  $65 \pmod{7}$       (d)  $96 \pmod{13}$       (f)  $1 \pmod{3}$       (h)  $31 \pmod{7}$

**Solution**

- (a)  $11 \pmod{4} \equiv 3$  because  $4 \times 2 + 3 = 11$       (e)  $2 \pmod{3} \equiv 2$  because  $3 \times 0 + 2 = 2$   
 (b)  $65 \pmod{7} \equiv 2$  because  $7 \times 9 + 2 = 65$       (f)  $1 \pmod{3} \equiv 1$  because  $3 \times 0 + 1 = 1$   
 (c)  $39 \pmod{3} \equiv 0$  because  $3 \times 13 + 0 = 39$       (g)  $5 \pmod{2} \equiv 1$  because  $2 \times (3) + 1 = 5$   
 (d)  $96 \pmod{13} \equiv 5$  because  $13 \times 7 + 5 = 96$       (h)  $31 \pmod{7} \equiv 4$  because  $7 \times (5) + 4 = 31$

We can say two integers,  $a$  and  $b$ , are congruent **mod**  $m$ , where  $m$  is a natural number, if both numbers divided by  $m$  produce the same remainder.

**EXAMPLE 6**

$52 \div 9$  has remainder 7 and  $79 \div 8$  also has remainder 7. This similarity helps us to define a new number relationship called **congruence**.

**DEFINITION**

Two integers  $a$  and  $b$  are congruent modulo  $m$  if they differ by an integer multiple of  $m$ , that is,  $b - a \equiv km$  for some  $k \in \mathbb{Z}$ . This equivalence is written  $a \equiv b \pmod{m}$ . We read “ $a \equiv b \pmod{m}$ ” as “ $a$  is **congruent** to  $b$  modulo  $m$ ”, which means  $\frac{m}{a} - b$ .

**EXAMPLE 7**

- (a)  $8 \equiv 3 \pmod{5}$  because  $\frac{5}{8} - 3$       (c)  $13 \equiv -1 \pmod{7}$  because  $\frac{7}{13} + 1$   
 (b)  $20 \equiv 4 \pmod{8}$  because  $\frac{8}{20} - 4$       (d)  $1 \equiv -5 \pmod{2}$  because  $\frac{2}{1} + 5$

**Note**

Let  $n$  be a positive integer. For all  $a, b \in \mathbb{Z}$

1.  $a \equiv a \pmod{m}$
2. If  $a \equiv b \pmod{m}$ , then  $b \equiv a \pmod{m}$ .
3. If  $a \equiv b \pmod{m}$  and  $b \equiv c \pmod{m}$ , then  $a \equiv c \pmod{m}$ .

**EXAMPLE 8**

Find the possible values of  $m$  that satisfy each congruence.

(a)  $15 \pmod{m} \equiv 3$

(c)  $17 \equiv 7 \pmod{m}$

(b)  $60 \pmod{m} \equiv 0$

(d)  $15 \equiv 4 \pmod{m}$

**Solution**

(a) If  $15 \pmod{m} \equiv 3$  has a value of 3, then the remainder is 3 when  $15 - 3 = 12$  divided by  $m$ , which occurs when  $m$  is a factor of 12, where  $m \neq 1$ . Thus, we need to find all factors of 12. That is,  $m = \{2, 3, 4, 6, 12\}$ .

(b) If  $60 \pmod{m}$  has a value of 0, then the remainder is 0 when 60 divided by  $m$ , which occurs when  $m$  is a factor of 60, where  $m \neq 1$ . Thus, we need to find all factors of 60. That is,

$$m = \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$$

(c) By definition of congruence,  $17 - 7 = 10$  must be divisible by  $m$ . So, the possible set of values of  $m$  are the positive divisors of 10 are  $m = \{2, 5, 10\}$  remember by definition  $m \neq 1$ .

(d)  $15 - 4 = 11$  must be divisible by  $m$ . So,  $m = \{11\}$ .

**EXAMPLE 9**

If today is Thursday, what day of the week will it be in 100 days' time?

**Solution**

Sun	Mon	Tue	Wed	Thurs	Fri	Sat
0	1	2	3	4	5	6

Since we have numbers 0, 1, 2, 3, 4, 5 and 6 we will do our arithmetic in modulo 7. Today is Thursday (day 4) hence to find 100 days later, we can write

$$4 + 100 = 104 \equiv 6 \pmod{7}$$

Therefore, the answer is the 6<sup>th</sup> day which is Saturday.

**Note**

If the number is less than the modulo, we leave it as it is. That is, if  $a < m$ , then

$$a \equiv a \pmod{m}$$

**EXAMPLE 10**

(a)  $2 \equiv 2 \pmod{3}$

(c)  $8 \equiv 8 \pmod{12}$

(b)  $5 \equiv 5 \pmod{7}$

(d)  $13 \equiv 13 \pmod{15}$

The modulus of a negative number is obtained by adding multiple of the modulo to the number until we get the first positive number.

For example,  $-13 \bmod 6 = -13 + 6 = -7$ .

This is not a positive number, so we continue adding.  $-7 + 6 = -1$ .

This is still not a positive number, so we continue adding.  $-1 + 6 = 5$ . This is positive.

So,  $-13 \bmod 6 = 5$ .

In other words,  $-13 \bmod 6 = 5$  because  $6 \times (-3) + 5 = 13$ .

### EXAMPLE 11

Evaluate the following:

(a)  $-1 \pmod{3}$

(c)  $-12 \pmod{5}$

(b)  $-5 \pmod{2}$

(d)  $-31 \pmod{7}$

#### Solution

(a)  $-1 \pmod{3} \equiv 2$  because  $3 \times (-1) + 2 = 1$

(b)  $-5 \pmod{2} \equiv 1$  because  $2 \times (-3) + 1 = -5$

(c)  $-12 \pmod{5} \equiv 3$  because  $5 \times (-3) + 3 = -12$

(d)  $-31 \pmod{7} \equiv 4$  because  $7 \times (-5) + 4 = -31$ .

## Operation on modulo of numbers

In this section, we will study the operation of *addition*, *subtraction* and *multiplication* of numbers on a given modulo.

The sum of two or more numbers in a given modulo is found by first adding the numbers before converting to the given modulo. That is, if  $a \equiv b \pmod{m}$  and  $c \equiv d \pmod{m}$ , then  $a + c \equiv b + d \pmod{m}$ .

### EXAMPLE 12

(a)  $9 + 8 \pmod{5} = 17 \pmod{5} = 2 \pmod{5} = 2$ .

(b)  $4 + 5 \pmod{10} = 9 \pmod{10} = 9$ .

(c)  $-10 + 4 \pmod{5} = -6 \pmod{5} = 4 \pmod{5} = 4$ .

(d)  $9 + 12 \pmod{13} = 21 \pmod{13} = 8 \pmod{13} = 8$ .

The difference of two numbers in a given modulo is found by first subtracting the numbers before converting to the given modulo. That is, if  $a \equiv b \pmod{m}$  and  $c \equiv d \pmod{m}$ , then  $a - c \equiv b - d \pmod{m}$ .

**EXAMPLE 13**

(a)  $19 - 7 \pmod{5} = 12 \pmod{5} = 2 \pmod{5} = 2.$

(b)  $2 - 5 \pmod{4} = -3 \pmod{4} = 1 \pmod{4} = 1.$

(c)  $-6 - 3 \pmod{7} = -9 \pmod{7} = 5 \pmod{7} = 5$

The product of two or more numbers in a given modulo is found by first multiplying the numbers before converting to the given modulo. That is, if  $a \equiv b \pmod{m}$  and  $c \equiv d \pmod{m}$ , then  $a \times c \equiv b \times d \pmod{m}$ .

**EXAMPLE 14**

(a)  $7 \times 9 \pmod{5} = 63 \pmod{5} = 3 \pmod{5} = 3.$

(b)  $8 \times 12 \pmod{13} = 96 \pmod{13} = 5 \pmod{13} = 5.$

### Addition ( $\oplus$ ) and multiplication ( $\otimes$ ) of tables in a given modulo

The signs ( $\oplus$ ) and ( $\otimes$ ) are used to differentiate from ordinary addition and multiplication respectively. For example, in modulo 5, we can evaluate  $7 \oplus 12 = 19 = 4 \pmod{5} = 4$  and  $7 \otimes 12 = 84 = 4 \pmod{5} = 4$ .

We use the set of numbers  $\{0, 1, 2, 3, 4\}$  to draw the table for arithmetic modulo 5. For modulo 8, we use the set of numbers  $\{0, 1, 2, 3, 4, 5, 6, 7\}$  to draw the table.

**EXAMPLE 15**

Draw addition ( $\oplus$ ) and multiplication ( $\otimes$ ) tables for modulo 6.

$\oplus$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$\otimes$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

## Properties of modulo operator

In this section, we will see four basic properties of modulo and verify using example.

**Property 1:** If the value of  $A$  is increased by a multiple of  $B$  (say,  $kB$ , where  $k$  is any integer), then

$$A \bmod B = (A + kB) \bmod B; \text{ where } k \text{ is any integer.}$$

### EXAMPLE 16

If  $A = 18$  and  $B = 5$ , then verify that  $A \bmod B = (A + kB) \bmod B$  for  $k = 6$ .

#### Solution

Given If  $A = 18$  and  $B = 5$ , we have

$$A \bmod B = 18 \bmod 5 = 3$$

$$\text{Also, } (A + kB) \bmod B = (18 + 6 \times 5) \bmod 5 = 58 \bmod 5 = 3 \text{ because } 58 = 5 \cdot 11 + 3.$$

$$\text{Thus, } 18 \bmod 5 = 258 \bmod 5 = 3$$

Therefore,  $A \bmod B = (A + kB) \bmod B$  is verified.

**Property 2:** Addition property of modulo arithmetic: If  $A$ ,  $B$  and  $C$  are positive integers, then

$$(A + B) \bmod C = (A \bmod C + B \bmod C) \bmod C$$

### EXAMPLE 17

Verify that  $(17 + 8) \bmod 3 = (17 \bmod 3 + 8 \bmod 3) \bmod 3$

#### Solution

$$\text{L.H.S: } (17 + 8) \bmod 3 = 25 \bmod 3 = 1.$$

$$\text{R.H.S: } (17 \bmod 3 + 8 \bmod 3) \bmod 3 = (2 + 5) \bmod 3 = 7 \bmod 3 = 1.$$

L.H.S = R.H.S. Hence verified.

**Property 3:** Subtraction property of modulo arithmetic: If  $A$ ,  $B$  and  $C$  are positive integers, then

$$(A - B) \bmod C = (A \bmod C - B \bmod C) \bmod C$$

### EXAMPLE 18

Verify that  $(19 - 6) \bmod 5 = (19 \bmod 5 - 6 \bmod 5) \bmod 5$ .

#### Solution

$$\text{L.H.S: } (19 - 6) \bmod 5 = 13 \bmod 5 = 3.$$

$$\text{R.H.S: } (19 \bmod 5 - 6 \bmod 5) \bmod 5 = (4 - 1) \bmod 5 = 3 \bmod 5 = 3 \text{ as } 5 \times 0 + 3 = 3.$$

L.H.S = R.H.S. Hence verified.

**Property 4:** Multiplication property of modulo arithmetic is if  $A$ ,  $B$  and  $C$  are positive integers, then

$$(A \times B) \bmod C = (A \bmod C \times B \bmod C) \bmod C.$$

### EXAMPLE 19

Verify that  $(20 \times 11) \bmod 8 = (20 \bmod 8 \times 11 \bmod 8) \bmod 8$ .

#### Solution

L.H.S:  $(20 \times 11) \bmod 8 = 220 \bmod 8 = 4$ .

R.H.S:  $(20 \bmod 8 \times 11 \bmod 8) \bmod 8 = (4 \times 3) \bmod 8 = 12 \bmod 8 = 4$ .

L.H.S = R.H.S. Hence verified.

### Exercise

- If it is 3 o'clock now, what will it be in:
 

(a) 34 hours	(d) 86 hours
(b) 40 hours	(e) 103 hours
(c) 57 hours	(f) 122 hours
- If the time now is 5 o'clock, what time was it before:
 

(a) 22 hours	(d) 42 hours
(b) 29 hours	(e) 56 hours
(c) 18 hours	(f) 75 hours
- If it is 9 o'clock now, what will it be in:
 

(a) 3 hours	(d) 91 hours
(b) 22 hours	(e) 913 hours
(c) 51 hours	(f) 122 hours
- Evaluate the following:
 

(a) $-2 \pmod{3}$	(c) $-14 \pmod{5}$
(b) $-7 \pmod{2}$	(d) $-43 \pmod{7}$
- If it is now April, what month was it 41 months ago?
- Find the possible values of  $m$  that satisfy each congruence.
 

(a) $13 \pmod{m} \equiv 2$	(c) $19 \equiv 7 \pmod{m}$
(b) $40 \pmod{m} \equiv 0$	(d) $17 \equiv 6 \pmod{m}$

7. If today is Saturday, what day of the week will it be in 125 days' time?
8. Simplify the following.
- |                    |                     |
|--------------------|---------------------|
| (a) $23 \pmod{3}$  | (g) $68 \pmod{7}$   |
| (b) $17 \pmod{5}$  | (h) $96 \pmod{8}$   |
| (c) $66 \pmod{7}$  | (i) $104 \pmod{11}$ |
| (d) $57 \pmod{9}$  | (j) $91 \pmod{13}$  |
| (e) $63 \pmod{8}$  | (k) $239 \pmod{13}$ |
| (f) $86 \pmod{12}$ | (l) $391 \pmod{10}$ |
9. Find the sum of the following in the modulo given.
- |                        |                        |
|------------------------|------------------------|
| (a) $29 + 19 \pmod{6}$ | (c) $45 - 25 \pmod{7}$ |
| (b) $49 + 52 \pmod{8}$ | (d) $29 - 16 \pmod{8}$ |
10. Evaluate each of the following in the modulo given.
- |                            |                              |
|----------------------------|------------------------------|
| (a) $8 - 2 + 13 \pmod{11}$ | (c) $19 - 7 - 5 \pmod{3}$    |
| (b) $15 - 3 + 7 \pmod{4}$  | (d) $88 + 25 - 34 \pmod{12}$ |
11. Find the product of the following in the modulo given.
- |                              |                                      |
|------------------------------|--------------------------------------|
| (a) $9 \times 11 \pmod{6}$   | (e) $8 \times 2 \times 13 \pmod{11}$ |
| (b) $17 \times 9 \pmod{7}$   | (f) $15 \times 3 \times 7 \pmod{5}$  |
| (c) $25 \times 8 \pmod{9}$   | (g) $9 \times 7 \times 5 \pmod{12}$  |
| (d) $29 \times 16 \pmod{13}$ | (h) $8 \times 17 \times 3 \pmod{10}$ |
12. If  $A = 19$  and  $B = 12$ , then verify that  $A \pmod{B} = (A + kB) \pmod{B}$  for  $k = 7$ .
13. If  $A = 21$ ,  $B = 12$  and  $C = 7$ , then verify each of the following.
- |   |
|---|
| (a) $(A + B) \pmod{C} = (A \pmod{C} + B \pmod{C}) \pmod{C}$           |
| (b) $(A - B) \pmod{C} = (A \pmod{C} - B \pmod{C}) \pmod{C}$           |
| (c) $(A \times B) \pmod{C} = (A \pmod{C} \times B \pmod{C}) \pmod{C}$ |
14. Draw an addition  $\oplus$  table for arithmetic modulo 5.
- Using your table,
- (i) Evaluate,
- |  |   |
|--|---|
| (a) $(2 \oplus 4) \oplus (3 \oplus 4)$ | (c) $2 \oplus (3 \oplus 4)$                       |
| (b) $(2 \oplus 3) \oplus 4$            | (d) $[(1 \oplus 2) \oplus (3 \oplus 4)] \oplus 4$ |
- (ii) Find the truth set of:
- |                      |                      |
|----------------------|----------------------|
| (a) $3 \oplus n = 2$ | (b) $n \oplus 3 = 4$ |
|----------------------|----------------------|

15. Draw a multiplication  $\otimes$  table for arithmetic modulo 7.

Using your table,

(i) Evaluate,

(a)  $(3 \otimes 5) \otimes (4 \otimes 5)$

(c)  $4 \otimes (5 \otimes 6)$

(b)  $(2 \otimes 5) \otimes 6$

(d)  $[(2 \otimes 3) \otimes (4 \otimes 5)] \otimes 6$

(ii) Find the truth set of:

(a)  $6 \otimes n = 0$

(c)  $n \otimes 5 = 2$

(b)  $n \otimes n = 1$

(d)  $4 \otimes n = 3$

Modular arithmetic is a branch of mathematics used to predict the outcomes of *cyclic events* such as days of the week, market days, months of the year, time etc.

### ACTIVITY 3

- The shorter hand of a clock points to 4 on a clock face. What number does it point to after 27 hours?
- Find the following numbers in their simplest form in modulo 4.
 

(a) 12	(b) 19	(c) 93	(d) 103
--------	--------	--------	---------
- L is equivalence classes modulo 3.

Modular Arithmetic is a special base arithmetic in which only the *remainder* is reported. That is, it can be considered as *remainder arithmetic*. For example, considering the number of days of the week on a circular dial or circular clock face as shown below.

If we take Sunday as the starting point (0), Monday (1) and so on clockwise around the dial, we can find any day we want.

### EXAMPLE 20

Starting at Sunday, what day will be in 19 days?

#### Solution

When we start on Sunday, we will return to Sunday every *seven* days. So, 19 days from Sunday will be  $2(7) + 5$ ; this is **Friday**.

We could also solve this problem by using the idea of remainder. That is, dividing 19 by 7 then we consider the remainder.

$$19 \div 7 = \frac{19}{7} = 2 \text{ remainder } 5.$$

Now the remainder 5 shows there are five days after Sunday and that is **Friday**.

From the above example, the number of days in the cycle is called its *modulus* and each day is called a *cyclic variable*. This type of circular arithmetic is called **Modular Arithmetic**.

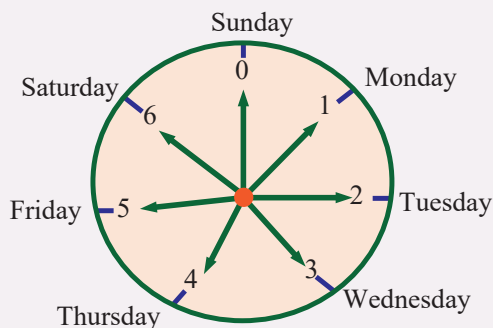


Figure 5.

A familiar use of modular arithmetic is in the 12-hour clock, in which the day is divided into two 12-hour periods. For example, 5 hours after 11 o'clock, it is 4 o'clock. This is quite natural despite the fact that  $5 + 11$  is 16 not simply 4, we identify time that are 12 hours apart as the same time of the day. That is,  $16 = 12 + 4$ . The 12-hours in the cycle is called its *modulus* and each hour is called a *cyclic variable*.

The modular arithmetic base 7 is written as “*mod 7*”,

base 12 is written as “*mod 12*” and so on. In general, in any “*mod n*” arithmetic, the remainder will form a set with elements  $\{0, 1, 2 \dots n - 1\}$

#### EXAMPLE 21

- If  $n = 7$ ,  $\text{mod } 7 = \{0, 1, 2, 3, 4, 5, 6\}$ .
- If  $n = 12$ ,  $\text{mod } 12 = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ .

#### EXAMPLE 22

It is 3 o'clock now; what time will it be in 17 hours?

**Solution**

$17 = 12 \times 1 + 5$ . So, 5 hours after 3 o'clock is 8 o'clock.

#### EXAMPLE 23

Today is Thursday, what day of the week was it 10 days ago?

**Solution**

$10 = 7 \times 1 + 3$ . So, 3 days. Therefore, the 3<sup>rd</sup> day before Thursday is Monday.

**Exercise**

- If it is 7 o'clock now, what will it be in:
  - 13 hours?
  - 19 hours?
  - 27 hours?
  - 7 hours?
- If the time now is 11 o'clock, what time was it before:
  - 13 hours?
  - 23 hours?
  - 12 hours?
  - 3 hours?
- If it is 3 o'clock now, then
  - in 33 hours, what time will the clock show?
  - what time did the clock show 19 hours ago?
- If today is Saturday, then
  - What day of the week will it be 76 days from now?
  - What day of the week was it 19 days ago?
- If your birthday was on a Tuesday last year, then
  - on what day will your birthday fall this year?
  - on what day of the week were you born?

In modular arithmetic, the numbers we are dealing with are just integers and **the operations used are addition, subtraction, multiplication and division.**

**ACTIVITY 4**

- State Euclid's division algorithm.
- Define coprime numbers.
- Divide each of the following and express the result in terms of " $a = bq + r$ ."
  - $27 \div 12$
  - $59 \div 7$
  - $127 \div 29$
  - $230 \div 25$
- Simplify the following.
  - $21 \pmod{7}$
  - $19 \pmod{6}$
  - $87 \pmod{12}$
  - $107 \pmod{35}$

We say  $a|b$  " $a$  divides  $b$ " if there exists an integer  $c$  such that  $b = ac$ .

**EXAMPLE 24**(a)  $7 \mid 21$  because  $7 \times 3 = 21$ (b)  $6 \nmid 40$  because 40 is not divisible by 6.**Euclid's Division Algorithm**

Let  $a$  be any integer and  $d$  be a positive integer. Then, there exists unique  $q$  and  $r$  ( $0 \leq r < d$ ) such that

$$a = dq + r.$$

$d$  is called the **divisor**,  $a$  is called the **dividend**,  $q$  is called the **quotient** and  $r$  is called the **remainder**.

**EXAMPLE 25**

What are the quotient and remainder:

(a) When 411 is divided by 19

(b) When 790 is divided by 27

**Solution**(a)  $411 = 19 \times 21 + 12.$ 

Thus, the quotient,  $q$  is 21 and the remainder,  $r$  is 12.

(b)  $790 = 27 \times 29 + 7.$ 

Thus, the quotient,  $q$  is 29 and the remainder,  $r$  is 7.

**Modular Arithmetic**

Let  $a$  be any integer and  $d$  be a positive integer. If  $a = dq + r$ , then  $a \bmod d = r$ .

**EXAMPLE 26**(a)  $64 = 12 \times 5 + 4$  so,  $64 \bmod 12 = 4.$ (b)  $216 = 19 \times 11 + 7$  so,  $216 \bmod 19 = 7.$ 

If  $b$  divides  $a$  as integers, then one might guess that we could use the usual division.

Let us consider the relation:

$$7 \equiv 1 \pmod{3}.$$

If we divide both sides by 7, we get  $1 = \frac{1}{7} \pmod{3}$  which is incorrect because  $\frac{1}{7}$  is not an integer and the congruence relation holds true for integers. Thus, the usual division does not work on modular arithmetic.

If after division all numbers are integers, then the congruence relation holds?

Let us consider the relation:

$$10 \equiv 4 \pmod{6}.$$

If we divide both sides by 2, we get  $5 \equiv 2 \pmod{6}$ . Even though all numbers in this relation are integers, it is still incorrect relation. However, this does not mean that we can never do division using modular arithmetic. Consider the relation:

$$14 \equiv 2 \pmod{3}.$$

If we divide both sides by 2, we get  $7 \equiv 1 \pmod{3}$  and this relation is correct.

Thus, what can we conclude about division of integers using modular arithmetic? Let us just compare the two different cases  $14 \equiv 2 \pmod{3}$  and  $10 \equiv 4 \pmod{6}$  together. The most important difference is the modulus used. In the relation  $14 \equiv 2 \pmod{3}$  that works uses *odd* modulus while the relation  $10 \equiv 4 \pmod{6}$  that doesn't work uses *even* modulus.

We can generalize to this statement: when dividing both sides by 2 on, we need the modulus to be odd. In fact, we can further generalize division on modular arithmetic by the following two rules.

**Rule 1:** If  $ka$  is congruent to  $kb$  modulo  $m$ , then we can divide both sides by  $k$ , provided that  $k$  and  $m$  are coprime. That is, if  $ka \equiv kb \pmod{m}$ , then  $a \equiv b \pmod{m}$  whenever  $k$  and  $m$  are coprime

The first rule of division relies on the fact that  $k$  and  $m$  are coprime so what if  $k$  and  $m$  are not coprime?

Let us consider the relation  $10 \equiv 4 \pmod{6}$ . This relation can be expressed as:

$$5 \times 2 \equiv 2 \times 2 \pmod{3 \times 2}.$$

So, 2 and 6 are not coprime because they have common factor, 2. This is the greatest common divisor (GCD) of 2 and 6. Thus, dividing both sides of the relation by 2, we obtain the correct relation

$$5 \equiv 2 \pmod{3}.$$

When we generalize this, we get the second rule for division on modular arithmetic.

**Rule 2:** If  $ka$  is congruent to  $kb$  modulo  $km$ , then we can divide both sides by  $k$ , provided that  $k$  and  $m$  are not coprime. That is, if  $ka \equiv kb \pmod{km}$ , then  $a \equiv b \pmod{m}$  whenever  $k$  and  $m$  are not coprime.

**EXAMPLE 27**

Simplify the following.

(a)  $\frac{3}{4}(\text{mod } 5)$

(c)  $\frac{5}{3}(\text{mod } 7)$

(b)  $90 \div 10 (\text{mod } 4)$

(d)  $7 \div 2 (\text{mod } 5)$

**Solution**

Let the unknown be  $x$ . Then,

(a)  $\frac{3}{4} = x(\text{mod } 5)$

$$3 = 4x (\text{mod } 5)$$

$$4x = 3 (\text{mod } 5)$$

Now, let us find the sum of a multiple of 5 and 3 which is divisible by 4. That is,

$$4x = 3 + 5 (\text{mod } 5)$$

$$4x = 8 (\text{mod } 5) \dots\dots\text{dividing both side by 4 gives}$$

$$x = 2 (\text{mod } 5)$$

Therefore,  $\frac{3}{4}(\text{mod } 5) = 2 (\text{mod } 5)$ .

(b)  $90 \div 10 (\text{mod } 4) = \frac{90}{10}(\text{mod } 5) = 9 (\text{mod } 4) = 1 (\text{mod } 4)$ .

(c) Let the unknown be  $x$ . Then,

$$\frac{5}{3} = x(\text{mod } 7)$$

$$5 = 3x (\text{mod } 7)$$

$$3x = 5 (\text{mod } 7)$$

Now, let us find the sum of a multiple of 7 and 5 which is divisible by 3. That is,

$$3x = 5 + 7 (\text{mod } 7)$$

$$3x = 12 (\text{mod } 7) \dots\dots\text{dividing both side by 3 gives}$$

$$x = 4 (\text{mod } 7)$$

Therefore,  $\frac{5}{3}(\text{mod } 7) = 4 (\text{mod } 7)$ .

(d)  $7 \div 2 (\text{mod } 5) = \frac{7}{2}(\text{mod } 5)$

Let the unknown be  $x$ . Then,

$$\frac{7}{2} = x \pmod{5}$$

$$7 = 2x \pmod{5}$$

$$2x = 7 \pmod{5}$$

$2x = 2 \pmod{5}$  .....dividing both side by 2 gives

$$x = 1 \pmod{5}$$

Therefore,  $7 \div 2 \pmod{5} = 1 \pmod{5}$ .

### EXAMPLE 28

Using modular arithmetic to solve for  $x$ .

(a)  $5x + 3 \equiv 2 \pmod{11}$

(b)  $3x \equiv 4 \pmod{7}$

#### Solution

(a)  $5x + 3 \equiv 2 \pmod{11}$

$5x + 3 - 3 \equiv 2 - 3 \pmod{11}$  .....subtracting 3 from both sides give

$$5x \equiv -1 \pmod{11}$$

$5x \equiv -1 + 11 \pmod{11}$  .....adding multiple of 11 to  $-1$

$5x \equiv 10 \pmod{11}$  .....dividing both side by 2 gives

$$x \equiv 2 \pmod{11}.$$

(b)  $3x \equiv 4 \pmod{7}$

$3x \equiv 4 + 7 \pmod{7}$  .....adding multiple of 7 to 4 gives

$3x \equiv 11 \pmod{7}$  .....it is not divisible by 3. So, continue in adding multiple of 7.

$$3x \equiv 11 + 7 \pmod{7}$$

$3x \equiv 18 \pmod{7}$  .....it is divisible by 3. So, divide both sides by 3.

$$x \equiv 6 \pmod{7}.$$

### Exercise

1. Simplify the following.

(a)  $20 \pmod{6}$

(c)  $72 \pmod{12}$

(b)  $29 \pmod{6}$

(d)  $100 \pmod{11}$

2. Divide each of the following in the given modulus.

(a)  $20 \div 5 \pmod{10}$

(c)  $24 \div 8 \pmod{5}$

(b)  $12 \div 3 \pmod{2}$

(d)  $9 \div 5 \pmod{4}$

3. Simplify the following.

(a)  $\frac{2}{3} \pmod{7}$

(c)  $\frac{9}{4} \pmod{5}$

(b)  $\frac{6}{5} \pmod{7}$

(d)  $11 \div 3 \pmod{8}$

4. Find the remainder.

(a)  $x \equiv 22 \pmod{3}$

(c)  $x \equiv -2 \pmod{7}$

(b)  $x \equiv 135 \pmod{9}$

(d)  $x \equiv -313 \pmod{8}$

5. Using modular arithmetic solve for  $x$ .

(a)  $4x \equiv 5 \pmod{9}$

(c)  $3x + 1 \equiv 4 \pmod{7}$

(b)  $2x \equiv 3 \pmod{5}$

(d)  $2x - 1 \equiv 4 \pmod{9}$

## KEY TERMS

- Modular arithmetic
- Polygonal arithmetic
- Polygonal numbers
- Triangular number
- Pentagonal number
- Hexagonal number
- Cyclic variable
- Modulus
- Square number
- Division algorithm

## SUMMARY

- A number which corresponds to the number of points on a regular polygon and discrete geometric pattern of equally spaced points is called a **figurate number**.
- Two integers  $a$  and  $b$  are congruent modulo  $m$  if they differ by an integer multiple of  $m$ , that is,  $b - a = km$  for some  $k \in \mathbb{Z}$ . This equivalence is written  $a \equiv b \pmod{m}$ . We read “ $a \equiv b \pmod{m}$ ” as “ $a$  is **congruent** to  $b$  modulo  $m$ ”, which means  $m \mid a - b$ .
- Let  $n$  be a positive integer. For all  $a, b \in \mathbb{Z}$ 
  - (a)  $a \equiv a \pmod{m}$
  - (b) If  $a \equiv b \pmod{m}$ , then  $b \equiv a \pmod{m}$ .
  - (c) If  $a \equiv b \pmod{m}$  and  $b \equiv c \pmod{m}$ , then  $a \equiv c \pmod{m}$ .

- The sum of two or more numbers in a given modulo is found by first adding the numbers before converting to the given modulo. That is, if  $a \equiv b \pmod{m}$  and  $c \equiv d \pmod{m}$ , then  $a + c \equiv b + d \pmod{m}$ .
- The difference of two numbers in a given modulo is found by first subtracting the numbers before converting to the given modulo. That is, if  $a \equiv b \pmod{m}$  and  $c \equiv d \pmod{m}$ , then  $a - c \equiv b - d \pmod{m}$ .
- The product of two or more numbers in a given modulo is found by first multiplying the numbers before converting to the given modulo. That is, if  $a \equiv b \pmod{m}$  and  $c \equiv d \pmod{m}$ , then

$$a \times c \equiv b \times d \pmod{m}.$$

- If the value of  $A$  is increased by a multiple of  $B$  (say,  $kB$ , where  $k$  is any integer), then

$$A \pmod{B} = (A + kB) \pmod{B}; \text{ where } k \text{ is any integer.}$$

- Addition property of modulo arithmetic is if  $A$ ,  $B$  and  $C$  are positive integers, then

$$(A + B) \pmod{C} = (A \pmod{C} + B \pmod{C}) \pmod{C}$$

- Subtraction property of modulo arithmetic is if  $A$ ,  $B$  and  $C$  are positive integers, then

$$(A - B) \pmod{C} = (A \pmod{C} - B \pmod{C}) \pmod{C}$$

- Multiplication property of modulo arithmetic is if  $A$ ,  $B$  and  $C$  are positive integers, then

$$(A \times B) \pmod{C} = (A \pmod{C} \times B \pmod{C}) \pmod{C}$$

- Let  $a$  be any integer and  $d$  be a positive integer. Then, there exists unique  $q$  and  $r$  ( $0 \leq r < d$ ) such that

$$a = dq + r.$$

$d$  is called the **divisor**,  $a$  is called the **dividend**,  $q$  is called the **quotient** and  $r$  is called the **remainder**.

- Let  $a$  be any integer and  $d$  be a positive integer. If  $a = dq + r$ , then  $a \pmod{d} = r$ .

### EXERCISES

- Use the  $n^{\text{th}}$  term formula for polygonal numbers to find the following.
  - the 7<sup>th</sup> term of triangular numbers.
  - 31<sup>st</sup> term of square numbers.

- (c) 27<sup>th</sup> term of pentagonal numbers.  
 (d) 17<sup>th</sup> term of hexagonal numbers.
2. Simplify the following.
- (a)  $39 \pmod{7}$                       (c)  $166 \pmod{8}$                       (e)  $275 \pmod{8}$   
 (b)  $87 \pmod{9}$                       (d)  $107 \pmod{11}$                       (f)  $486 \pmod{12}$
3. Find the sum of the following in the modul given.
- (a)  $43 + 29 \pmod{7}$                       (c)  $345 - 205 \pmod{9}$   
 (b)  $59 + 92 \pmod{11}$                       (d)  $129 - 106 \pmod{13}$
4. Evaluate each of the following in the modulo given.
- (a)  $18 - 122 + 23 \pmod{13}$                       (c)  $89 - 57 - 35 \pmod{11}$   
 (b)  $45 - 23 + 17 \pmod{4}$                       (d)  $188 + 125 - 314 \pmod{15}$
5. Find the product of the following in the modulo given.
- (a)  $19 \times 9 \pmod{8}$                       (c)  $125 \times 16 \pmod{11}$   
 (b)  $27 \times 21 \pmod{9}$                       (d)  $39 \times 15 \pmod{14}$
6. If  $A = 29$  and  $B = 21$ , then verify that  $A \pmod{B} = (A + kB) \pmod{B}$  for  $k = 5$ .
7. If  $A = 27$ ,  $B = 19$  and  $C = 12$ , then verify each of the following.
- (a)  $(A + B) \pmod{C} = (A \pmod{C} + B \pmod{C}) \pmod{C}$   
 (b)  $(A - B) \pmod{C} = (A \pmod{C} - B \pmod{C}) \pmod{C}$   
 (c)  $(A \times B) \pmod{C} = (A \pmod{C} \times B \pmod{C}) \pmod{C}$
8. Draw an addition  $\oplus$  table for arithmetic modulo 6.  
 Using your table,
- (i) Evaluate,
- (a)  $(2 \oplus 4) \oplus (3 \oplus 4)$                       (c)  $2 \oplus (3 \oplus 4)$   
 (b)  $(2 \oplus 3) \oplus 4$                       (d)  $[(1 \oplus 2) \oplus (3 \oplus 4)] \oplus 4$
- (ii) Find the truth set of:
- (a)  $3 \oplus n = 2$                       (b)  $n \oplus 3 = 4$
9. Draw a multiplication  $\otimes$  table for arithmetic modulo 5.  
 Using your table,
- (i) Evaluate,
- (a)  $(3 \otimes 5) \otimes (4 \otimes 5)$                       (c)  $4 \otimes (5 \otimes 6)$   
 (b)  $(2 \otimes 5) \otimes 6$                       (d)  $[(2 \otimes 3) \otimes (4 \otimes 5)] \otimes 6$

(ii) Find the truth set of:

(a)  $6 \otimes n = 0$

(c)  $n \otimes 5 = 2$

(b)  $n \otimes n = 1$

(d)  $4 \otimes n = 3$

10. If today is Sunday, then

(a) What day of the week will it be 36 days from now?

(b) What day of the week was it 29 days ago?

11. If your birthday was on a Saturday last year, then

(a) on what day will your birthday fall this year?

(b) on what day of the week were you born?

12. Divide each of the following in the given modulus.

(a)  $24 \div 6 \pmod{8}$

(c)  $32 \div 8 \pmod{6}$

(b)  $18 \div 9 \pmod{2}$

(d)  $19 \div 15 \pmod{4}$

13. Simplify the following.

(a)  $\frac{3}{5} \pmod{6}$

(b)  $\frac{7}{4} \pmod{7}$

(c)  $\frac{11}{6} \pmod{5}$

(d)  $\frac{13}{6} \pmod{8}$

14. Find the value of  $x$

(a)  $x \equiv 29 \pmod{4}$

(c)  $x \equiv 4 \pmod{9}$

(b)  $x \equiv 225 \pmod{5}$

(d)  $x \equiv 135 \pmod{8}$

15. Using modular arithmetic to solve for  $x$ .

(a)  $3x \equiv 6 \pmod{9}$

(c)  $6x - 2 \equiv 0 \pmod{4}$

(b)  $4x \equiv 3 \pmod{5}$

(d)  $3x + 5 \equiv 2 \pmod{6}$



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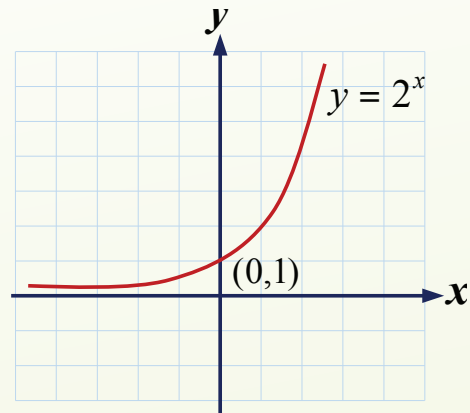
# CHAPTER

# 2

## INDICES AND LOGARITHMS

### Chapter Contents

- 2.1 Exponents (Indices)
- 2.2 Laws of Exponents (Indices)
- 2.3 Properties of Exponents (Indices)
- 2.4 Rational and Negative Exponents (Indices)
- 2.5 Exponential Functions and Equation
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- 2.7 Logarithms
- 2.8 Laws and Properties of Logarithms
- 2.9 Logarithmic Functions and Equations
- 2.10 Logarithm Table
- 2.11 Evaluating Logarithmic Expression
- 2.12 Base 10 Logarithms
- 2.13 Logarithmic of Numbers Greater than 10
- 2.14 Logarithms of Numbers Between 0 and 1
  - Key Terms
  - Summary
  - Exercises



$$\log_b X = \frac{\log X}{\log b}$$

## Chapter Outcomes

By the end of this chapter, Learners will be able to:

- define and discuss indices (notation, laws);
- define and discuss properties of indices;
- define and discuss exponential growth;
- discuss and solve negative powers;
- define and discuss properties of indices;
- define and discuss rational powers;
- define and discuss logarithms apply the logarithms functions to solve problems;
- define and apply base ten logarithms;
- solve logarithmic of numbers greater than 10;
- solve logarithmic of numbers between 0 and 1;
- define and discuss laws of logarithms.

## Introduction

In many applications of mathematics, we can express numbers as power of some given base. We can reverse this question and ask, “what exponent of 3 gives 27? Our attention is then turned to the index itself. This leads to the notion of a logarithm, which is simply index. Logarithms appear in all sorts of calculations in engineering and science, business and economics. In this unit, we will learn about indices and logarithms and rules for manipulating them through a number of worked examples.

In many applications of mathematics, we can express numbers as a product of certain number of factors all of which are the same. For example,

$$4 \times 4 = 16; 7 \times 7 \times 7 = 343; \text{ and } 7 \times 7 \times 7 \times 7 = 2,401.$$

Mathematics use the idea of *indices* to represent a product of a certain number involving the same factor. Indices provide a compact algebraic notation for repeated multiplication. For example,

$$3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^6.$$

Indices are used to show numbers that have been multiplied by themselves. The index of a number says how many times to use the number in a multiplication. The plural of index is *indices*. For example, in  $5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5$ . The number 5 is written 8 times and can be written as  $5^8$ . Here, 5 is called the “*base*”, 8 is called the index or exponent and  $5^8$  is called the *power*.

### EXAMPLE 1

$$a^2 = a \times a$$

$$a^3 = a \times a \times a$$

In general, for any real number  $a$  and  $n$  is a positive integer,

$$a^n = a \times a \times a \times a \times \dots n \text{ times} .$$

### DEFINITION

If  $a$  is any real number and  $n$  is any natural number, then the  $n^{\text{th}}$  power of  $a$ , denoted by  $a^n$ , read as “ $a$  to the power of  $n$ ” is defined as:

$$a^n = \underbrace{a \times a \times a \times \dots \times a}_{n \text{ factors each equal to } a}$$

In the expression  $a^n$ ,  $a$  is called the base,  $n$  is called the index or exponent and  $a^n$  is the power.

**EXAMPLE 2**

Find the index and base of each expression.

(a)  $(-2a)^7$

(b)  $-5y^4$

**Solution**

(a) The index is 7 and the base is  $-2a$ .

(b) Clearly,  $-5y^4 = -5 \times y^4 = -5 \times y \times y \times y \times y$ .

Therefore, the index is 4 and the base is only  $y$ .

**ACTIVITY 1**

1. Identify the base and the index and find the value of each of the following powers.

(a)  $34$

(c)  $\left(\frac{3}{4}\right)^3$

(e)  $(3x)^4$

(b)  $(-2)^6$

(d)  $-(-2)^5$

(f)  $(-2x)^7$

2. Evaluate each of the following powers.

(a)  $(-1)^2$

(d)  $(-1)^5$

(g)  $(-2)^2$

(j)  $(-2)^5$

(b)  $(-1)^3$

(e)  $(-1)^7$

(h)  $(-2)^3$

(k)  $(-2)^7$

(c)  $(-1)^4$

(f)  $(-1)^6$

(i)  $(-2)^4$

(l)  $(-2)^6$

3. What do you conclude from:

(a)  $(-1)^n$ , when  $n$  is even?

(b)  $(-1)^n$ , when  $n$  is odd?

4. For any real number  $a$ , show that:

(a)  $(-a)^5 = -a^5$

(b)  $a^8 = (-a)^8$

**EXAMPLE 3**

Evaluate:

(a)  $(-3)^6$

(b)  $-3^6$

(c)  $(-5)^3$

(d)  $-(-5)^3$

**Solution**

(a)  $(-4)^6 = -4 \times -4 \times -4 \times -4 \times -4 \times -4 = 4,096$ .

(b)  $-4^6 = -1 \times 4^6 = -1 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 = -1 \times 4096 = -4,096$ .

(c)  $(-5)^3 = -5 \times -5 \times -5 = -5 \times 25 = -125$ .

(d)  $-(-5)^3 = -1 \times (-5)^3 = -1 \times -5 \times -5 \times -5 = 5 \times 25 = 125$ .

Note that, in the expression  $(-4)^6$  the base is  $-4$  but in the expression  $-4^6$  the base is only 4.

**EXAMPLE 4**

What is the base in  $(-3p)^5$ ?

**Solution**

The base in  $(-3p)^5$  is  $-3p$ . Thus,

$$\begin{aligned} (-3p)^4 &= -3p \times -3p \times -3p \times -3p \times -3p \\ &= (-3 \times -3 \times -3 \times -3 \times -3) \times (p \times p \times p \times p \times p) \\ &= -243 \times p^5 = -243p^5. \end{aligned}$$

**EXAMPLE 5**

In the expression  $-3p^5$ , what base does the index 5 refer to ?

**Solution**

Clearly,  $-3p^5 = -3 \times p^5 = -3 \times p \times p \times p \times p \times p$ .

Therefore, the index 5 in  $-3p^5$  refers only to the base  $p$ .

**EXERCISES**

- Identify the base and the index and then find the value of each of the following powers.
 

(a) $(-5)^4$	(f) $(-2p)^3$	(j) $-(-0.2)^5$
(b) $-3^7$	(g) $11^3$	(k) $\left(-\frac{1}{2}x\right)^4$
(c) $(0.5)^3$	(h) $-2^9$	(l) $\left(\frac{1}{2}x\right)^4$
(d) $-(-4)^3$	(i) $\left(-\frac{2}{3}\right)^5$	
(e) $-(-2p)^3$		
- Evaluate each of the following powers.
 

(a) $(-1)^{23}$	(c) $(-1)^{45}$	(e) $(-1)^{73}$
(b) $(-1)^{32}$	(d) $(-1)^{54}$	(f) $(-1)^{99}$
- Write each of the following expressions more concisely by using an index.
 

(a) $q \times q \times q \times q \times q \times q$
(b) $(2xy) \times (2xy) \times (2xy) \times (2xy) \times (2xy)$
(c) $\left(\frac{a}{b}\right) \times \left(\frac{a}{b}\right) \times \left(\frac{a}{b}\right) \times \left(\frac{a}{b}\right) \times \left(\frac{a}{b}\right)$
(d) $a^2 \times a^2 \times a^2 \times a^2 \times a^2$

4. What is the base in each of the following powers?

(a)  $(-5x)^3$

(c)  $(-0.2p)^5$

(b)  $-7y^6$

(d)  $0.5q^5$

5. For any real number  $p$ , show that:

(a)  $(-2p)^5 \neq -2p^5$

(b)  $(2p)^8 = (-2p)^8$

The detail of the laws of exponents for exponents different from positive integers will be done in the next sections, in this section we will only see positive integral exponents.

For any real number  $a$ ,

(i)  $a^m \cdot a^n = a^{m+n}$

#### EXAMPLE 6

(a)  $2^5 \times 2^3 = 2^{5+3} = 2^8$

(b)  $3^7 \times 3^2 = 3^{7+2} = 3^9$

(ii)  $\frac{a^m}{a^n} = a^{m-n}$

#### EXAMPLE 7

$$\frac{2^9}{2^5} = 2^{9-5} = 2^4$$

(iii)  $(a^m)^n = a^{m \cdot n}$

#### EXAMPLE 8

$$(2^5)^3 = 2^{5 \times 3} = 2^{15}$$

(iv)  $a^n \cdot b^n = (ab)^n$

#### EXAMPLE 9

$$2^7 \times 3^7 = (2 \times 3)^7 = 6^7$$

$$(v) \quad \frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$$

**EXAMPLE 10**

$$\frac{2^7}{3^7} = \left(\frac{2}{3}\right)^7$$

$$(vi) \quad a^0 = 1, a \neq 1$$

**EXAMPLE 11**

$$(a) \quad 5^0 = 1$$

$$(b) \quad 2^0 = 1$$

**EXERCISES**

Simplify each of the following and write your answer by using a single exponent.

$$1. \quad 53 \times 52$$

$$3. \quad \frac{2^5 \times 2^3}{2^4}$$

$$5. \quad 24 \times 34$$

$$2. \quad \frac{2^7}{2^3}$$

$$4. \quad \left(\frac{2}{5}\right)^7 \times \frac{5^9}{2^3}$$

$$6. \quad 256 \times 2$$

Recall that to simplify an expression means to rewrite it by combining terms or indices. To simplify expressions involving indices, we use the **properties of indices**. The properties of indices may be combined to simplify expressions.

Doing the group work 1 will help you to understand the properties of indices to simplify expressions involving indices.

Consider the product  $p^3 \times p^4$ . Both terms have the same base,  $p$ , but they are raised to different exponents. Expand each expression and then rewrite the resulting expression as follows:

$$p^3 \times p^4 = \underbrace{p \times p \times p}_{3 \text{ factors}} \times \underbrace{p \times p \times p \times p}_{4 \text{ factors}} = \underbrace{p \times p \times p \times p \times p \times p \times p}_{7 \text{ factors}} = p^7$$

$$\text{The result is that } p^3 \times p^4 = p^{3+4} = p^7$$

### Group Work 1

1. Simplify each of the following.

(a)  $3^2 \times 3^4$

(b)  $2^5 \times 2^6$

(c)  $4^{-2} \times 4^3$

(d)  $\left(\frac{2}{x}\right)^4 \times \left(\frac{2}{x}\right)$

(e)  $\frac{3^7}{3^4}$

(f)  $(2^3)^4$

(g)  $(2 \times 3)^4$

(h)  $\left(\frac{2}{3}\right)^5$

(i)  $(2y)^3 \times (2y)^2$

(j)  $q^4 \times q^5 \times q^3$

2. Show that  $(3 \times 4)^3 = 3^3 \times 4^3$  and  $(3 + 4)^3 \neq 3^3 + 4^3$

### EXAMPLE 12

Simplify each of the following.

(a)  $p^4 \times p^5$

(b)  $(-3)^5 \times -3$

(c)  $(2x)^3 \times (2x)^4$

(d)  $a^2 \times a^5 \times a^3$

#### Solution

(a)  $p^4 \times p^5 = p^{4+5} = p^9$

(b)  $(-3)^5 \times -3 = (-3)^5 \times (-3)^1 = (-3)^{5+1} = (-3)^6$

(c)  $(2x)^3 \times (2x)^4 = (2x)^{3+4} = (2x)^7$

(d)  $a^2 \times a^5 \times a^3 = (a^2 \times a^5) \times a^3 = (a^{2+5}) \times a^3 = a^7 \times a^3 = a^{7+3} = a^{10}$

Here, we can also get the same result by adding the three indices in one step. That is,

$$a^2 \times a^5 \times a^3 = a^{2+5+3} = a^{10}$$

### The Product Property of Indices

For any real number  $a$  and natural numbers  $m$  and  $n$ , the product property of indices states that

$$a^m \times a^n = a^{m+n}$$

The *quotient Properties of indices* allows us to simplify an expression that divides two numbers with the same base but different indices. We can simplify an expression

of the form  $\frac{a^m}{a^n}$ , where  $m > n$ .

### EXAMPLE 13

Simplify the expression  $\frac{a^9}{a^4}$ .

**Solution**

Let us perform the division by canceling common factors of the divisor and the dividend as follows:

$$\begin{aligned}\frac{a^9}{a^4} &= \frac{a \times a \times a \times a \times a \times a \times a \times a \times a}{a \times a \times a \times a} = \frac{\cancel{a} \times \cancel{a} \times \cancel{a} \times \cancel{a} \times a \times a \times a \times a \times a}{\cancel{a} \times \cancel{a} \times \cancel{a} \times \cancel{a}} \\ &= \frac{a \times a \times a \times a \times a}{1} = a^5.\end{aligned}$$

The result is  $\frac{a^9}{a^4} = a^{9-4} = a^5$ .

**EXAMPLE 14**

Simplify each of the following.

$$(a) \frac{(-2)^{13}}{(-2)^8} \quad (b) \frac{p^{27}}{p^{15}} \quad (c) \frac{(2b)^7}{(2b)^3} \quad (d) \frac{(-4)^9}{-4}$$

**Solution**

$$\begin{aligned}(a) \frac{(-2)^{13}}{(-2)^8} &= (-2)^{13-8} = (-2)^5 & (c) \frac{(2b)^7}{(2b)^3} &= (2b)^{7-3} = (2b)^4 \\ (b) \frac{p^{27}}{p^{15}} &= p^{27-15} = p^{12} & (d) \frac{(-4)^9}{-4} &= \frac{(-4)^9}{(-4)^1} = (-4)^{9-1} = (-4)^8\end{aligned}$$

**The Quotient Property of Indices**

For any real number  $a$  and natural numbers  $m$  and  $n$  such that  $m > n$ , the quotient property of indices states that

$$\frac{a^m}{a^n} = a^{m-n}$$

Suppose an indicial expression is raised to some power. Can we simplify the result? Yes. To do this, we use the power rule of indices. Consider the expression  $(a^3)^2$ . The expression inside the parentheses is multiplied three times because it has an index of 3. Then, the result is multiplied four times because the given expression has an index of 4.

$$(a^3)^2 = \underbrace{a^3 \times a^3}_{2 \text{ factors}} = \underbrace{a \cdot a \cdot a}_{3 \text{ factors}} \times \underbrace{a \cdot a \cdot a}_{3 \text{ factors}} = a \cdot a \cdot a \cdot a \cdot a \cdot a = a^6$$

The index of the final answer is the product of the indices. That is,

$$(a^3)^2 = a^{3 \times 2} = a^6.$$

### The Power Property of Indices

For any real number  $a$  and natural numbers  $m$  and  $n$ , the power property of indices states that

$$(a^m)^n = a^{m \times n}$$

#### EXAMPLE 15

Simplify each of the following.

(a)  $(3^4)^5$

(b)  $(r^5)^3$

(c)  $((5x)^4)^7$

(d)  $((-2x)^3)^6$

#### Solution

(a)  $(3^4)^5 = 3^{4 \times 5} = 3^{20}$

(b)  $(r^5)^3 = r^{5 \times 3} = r^{15}$

(c)  $((5x)^4)^7 = (5x)^{4 \times 7} = (5x)^{28}$

(d)  $((-2x)^3)^6 = (-2x)^{3 \times 6} = (-2x)^{18}$

#### ACTIVITY 2

1. Simplify each of the following.

(a)  $5^7 \times 5^7$

(b)  $8^{15} \times 8^{12}$

(c)  $((2x)^3)^5$

(d)  $3p^3 \times 2p \times 4p^5$

(e)  $\frac{(-6)^{11}}{6^7}$

(f)  $\frac{(3m^2)^7}{m^9}$

(g)  $\frac{x^{12} \times x^8}{x^{15}}$

(h)  $\frac{(2x)^7 \times 2xy^5}{(2xy)^3}$

(i)  $\left(\frac{a^9}{a^5}\right)^4 \times a^3$

(j)  $(p^2q)^4 \times (q^3p)^5$

2. Simplify each of the following.

(a)  $3p \times 3p \times 3p \times 3p \times 3p$

(b)  $(-2xyz) \times (-2xyz) \times (-2xyz) \times (-2xyz)$

(c)  $(ab^2) \times (ab)^2 \times (ab^2) \times (ab)^2 \times (ab^2)$

(d)  $\frac{2a^2b^3}{3ab^2} \times \frac{(2ab^2)^3}{(a^2b)^2}$

3. Evaluate each of the following indicial expression, give the final answer as an exact simplified result.

(a)  $4^2 + (-4)^3$

(b)  $\frac{12x^5}{5y^2} \times \frac{(5y^2)^2}{3x^2}$

(c)  $\frac{(4ab^2)^3}{(2a)^3} \times \frac{(ab)^4}{a^2b}$

(d)  $(2pq^2)^4 \times \frac{(p^2q)}{2p^5q}$

### The Power of a Product Property of Indices

For any real numbers  $a$  and  $b$  and a natural number  $n$ , the power of a product property of indices states that

$$(ab)^n = a^n b^n$$

#### EXAMPLE 16

Simplify each of the following.

(a)  $(xy^3)^2$

(b)  $(2x)^7$

(c)  $(-3p)^5$

(d)  $(a^2b^3)^3$

**Solution**

(a)  $(xy^3)^2 = (x)^2(y^3)^2 = x^2 \cdot y^{3 \times 2} = x^2y^6.$

(b)  $(2x)^7 = (2)^7 \cdot (x)^7 = 128 \cdot x^7 = 128x^7.$

(c)  $(-3p)^5 = (-3)^5 \cdot (p)^5 = -243 \cdot p^5 = -243p^5.$

(d)  $(a^2b^3)^3 = (a^2)^3 \cdot (b^3)^3 = a^{2 \times 3} \cdot b^{3 \times 3} = a^6b^9.$

#### EXAMPLE 17

Simplify the expression  $\left(\frac{p}{q}\right)^4$ .

**Solution**

let us simplify this expression as follows:

$$\left(\frac{p}{q}\right)^4 = \underbrace{\frac{p}{q} \times \frac{p}{q} \times \frac{p}{q} \times \frac{p}{q}}_{4 \text{ factors}} = \frac{p \times p \times p \times p}{q \times q \times q \times q} = \frac{\underbrace{p \times p \times p \times p}_{4 \text{ factors}}}{\underbrace{q \times q \times q \times q}_{4 \text{ factors}}} = \frac{p^4}{q^4}.$$

### The Power of a Quotient Property of Indices

For any real numbers  $a$  and  $b$  and a natural number  $n$ , the power of a quotient property of indices states that

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

#### EXAMPLE 18

Simplify each of the following.

(a)  $\left(\frac{4}{3}\right)^3$

(b)  $\left(\frac{p^2}{q^3}\right)^3$

(c)  $\left(\frac{(-2p)}{2q}\right)^3$

(d)  $\left(\frac{3x}{2}\right)^4$

**Solution**

$$(a) \left(\frac{4}{3}\right)^3 = \frac{4^3}{3^3} = \frac{64}{27}$$

$$(b) \left(\frac{p^2}{q^3}\right)^3 = \frac{(p^2)^3}{(q^3)^3} = \frac{p^{2 \times 3}}{q^{3 \times 3}} = \frac{p^6}{q^9}$$

$$(c) \left(\frac{-2p}{2q}\right)^3 = \frac{(-2p)^3}{(2q)^3} = \frac{(-2)^3 \cdot (p)^3}{(2)^3 \cdot (q)^3} = \frac{-8 \cdot p^3}{8 \cdot q^3} = \frac{-8p^3}{8q^3} = -\frac{p^3}{q^3}$$

$$(d) \left(\frac{3x}{2}\right)^4 = \frac{(3x)^4}{2^4} = \frac{(3)^4 \cdot (x)^4}{2^4} = \frac{81 \cdot x^4}{16} = \frac{81x^4}{16} = \frac{81}{16}x^4.$$

Recall that in the quotient property of indices, we made the condition  $m > n$  so that the difference  $m - n$  would never be zero or negative. What would happen if  $m = n$ ? In this case, we would use the zero index property to simplify the expression to 1.

**EXAMPLE 19**

Consider the expression  $\frac{p^7}{p^7}$ . Let us simplify it as follows:

$$\frac{p^7}{p^7} = \frac{p \times p \times p \times p \times p \times p \times p}{p \times p \times p \times p \times p \times p \times p} = \frac{\cancel{p} \times \cancel{p} \times \cancel{p} \times \cancel{p} \times \cancel{p} \times \cancel{p} \times \cancel{p}}{\cancel{p} \times \cancel{p} \times \cancel{p} \times \cancel{p} \times \cancel{p} \times \cancel{p} \times \cancel{p}} = 1$$

If we were to simplify the given expression using the quotient property, we would have

$$\frac{p^7}{p^7} = p^{7-7} = p^0$$

If we equate the two results, the result is  $p^0 = 1$ .

**The Zero Index Property of Indices**

For any nonzero real numbers  $a$ , the zero-index property of indices states that

$$a^0 = 1.$$

**EXAMPLE 20**

Simplify each of the following.

$$(a) \frac{p}{p^5}$$

$$(b) \frac{-4z^3}{z^3}$$

**Solution**

(a)  $\frac{p^5}{p^5} = p^{5-5} = p^0 = 1$

(b)  $\frac{-4z^3}{z^3} = -4 \times \frac{z^3}{z^3} = -4 \times z^{3-3}$   
 $= -4 \times z^0 = -4 \times 1 = -4.$

**EXAMPLE 21**

Simplify each of the following.

(a)  $\frac{(a^2b)^6}{(a^2b) \times (a^2b)^5}$

(b)  $\frac{7(x^2y)^3}{(x^2y)^3}$

**Solution**

(a)  $\frac{(a^2b)^6}{(a^2b) \times (a^2b)^5} = \frac{(a^2)^6 \times (b)^6}{(a^2b) \times (a^2)^5 (b)^5} = \frac{a^{12} \times b^6}{a^2 \times b \times a^{10} \times b^5} = \frac{a^{12} \times b^6}{a^{2+10} \times b^{1+5}} = \frac{a^{12} \times b^6}{a^{12} \times b^6}$   
 $= \frac{a^{12}}{a^{12}} \times \frac{b^6}{b^6} = a^{12-12} \times b^{6-6} = a^0 \times b^0 = 1 \times 1 = 1.$

(b)  $\frac{7(x^2y)^3}{(x^2y)^3} = \frac{7(x^2)^3 \times (y)^3}{(x^2)^3 \times (y)^3} = \frac{7x^{2 \times 3} \times y^3}{x^{2 \times 3} \times y^3} = \frac{7x^6 \times y^3}{x^6 \times y^3} = 7 \times \frac{x^6}{x^6} \times \frac{y^3}{y^3} = 7 \times x^{6-6} \times y^{3-3}$   
 $= 7 \times x^0 \times y^0 = 7 \times 1 \times 1 = 7.$

**EXERCISES**

1. Simplify each expression and write the answer in simplest form.

(a)  $(6m^2n)^3 \times mn$

(c)  $(8^2 \times 2^3) \times (3 \times 3^0)^3$

(b)  $(-3a^2)^3 \times -3a$

(d)  $(p^4q^3)^5 \times (pq)^2$

2. Simplify each expression.

(a)  $(3^2)^3$

(c)  $\frac{169^m}{13^m}$

(e)  $\frac{-12(pq^3)^5}{3(q^2p)^5}$

(b)  $\frac{4^{2n+3}}{4^{2n-2}}$

(d)  $\frac{(a^2b)^3}{(a^2b)^3}$

(f)  $(a^2b)^5 \times \left(\frac{a^5}{b}\right)^2$

3. Evaluate each of the following indicial expressions and give the final answer as an exact simplified result.

(a)  $\frac{(3m^2n)^3}{(mn^2) \times (4m^2)^2}$

(b)  $\frac{5p^5q^3 \times 3(pq^3)^2}{15p^3q^2}$

(c) 
$$\frac{12(x^3y^2z)^5}{(4x^2z)^2}$$

(d) 
$$\frac{9(p^{12}q) \times (p^2q)^4}{6(p^4q)^5}$$

4. Simplify each of the following expressions.

(a) 
$$\frac{(2x)^5(3y)^3}{(3xy)^4}$$

(c) 
$$\left(\frac{x^{-2}y^3z^{-3}}{x^4y^{-3}z^{-2}}\right)^{-3}, \text{ where } xyz \neq 0$$

(b) 
$$\frac{(6ab)^6(12ab)^7}{(3ab)^5(4^3a^3b)^2}, \text{ where } ab \neq 0$$

(d) 
$$\left(\frac{p^{-6}q^{-3}}{r^3s^3}\right)^{-\frac{4}{3}} \left(\frac{p^2r^{-4}s^{-2}}{q^{-4}}\right)^{-\frac{3}{2}}$$

## Rational exponents

Raising a number to a power is reversed by finding the root of a number. We indicate roots by using rational exponents or radicals. In this sub-unit we will review definitions and rules concerning rational exponents.

Rational exponents are exponents of numbers that are expressed as a rational number, that is,  $a^{\frac{m}{n}}$ ,  $a$  is the base and  $\frac{m}{n}$  is the rational exponent where  $n \neq 0$ .

### DEFINITION

If  $n$  is a positive even integer and  $a$  is positive, then  $a^{\frac{1}{n}}$  denotes the positive real  $n$ th root of  $a$  and is called the principal  $n$ th root of  $a$ . If  $n$  is a positive odd integer and  $a$  is any real number, then  $a^{\frac{1}{n}}$  denotes the real  $n$ th root of  $a$ .

### ACTIVITY 3

1. Evaluate each of the following expressions.

(a)  $4^{\frac{1}{2}}$

(c)  $(-8)^{\frac{1}{3}}$

(e)  $(16)^{\frac{1}{4}}$

(b)  $8^{\frac{1}{3}}$

(d)  $(-16)^{\frac{1}{4}}$

(f)  $(-32)^{\frac{2}{5}}$

2. Using the rule  $a^m \times a^n = a^{m+n}$ , simplify each of the following.

(a)  $6^{\frac{1}{2}} \times 6^{\frac{1}{2}}$

(c)  $2^{\frac{1}{4}} \times 2^{\frac{1}{4}} \times 2^{\frac{1}{4}} \times 2^{\frac{1}{4}}$

(b)  $8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}}$

(d)  $27^{\frac{1}{3}} \times 27^{\frac{1}{3}}$

**EXAMPLE 22**

Evaluate each of the following expressions.

(a)  $9^{\frac{1}{2}}$

(b)  $27^{\frac{1}{3}}$

(c)  $(-64)^{\frac{1}{3}}$

(d)  $16^{\frac{3}{4}}$

**Solution**

(a)  $9^{\frac{1}{2}} = (9)^{\frac{1}{2}} = (3^2)^{\frac{1}{2}} = 3^{\frac{2 \times 1}{2}} = 3^{\frac{2}{2}} = 3^1 = 3$

(b)  $27^{\frac{1}{3}} = (27)^{\frac{1}{3}} = (3^3)^{\frac{1}{3}} = 3^{\frac{3 \times 1}{3}} = 3^{\frac{3}{3}} = 3^1 = 3$

(c)  $(-64)^{\frac{1}{3}} = (-64)^{\frac{1}{3}} = ((-4)^3)^{\frac{1}{3}} = (-4)^{\frac{3}{3}} = (-4)^1 = -4$

(d)  $16^{\frac{3}{4}} = (16)^{\frac{3}{4}} = (2^4)^{\frac{3}{4}} = 2^{4 \times \frac{3}{4}} = 2^{\frac{12}{4}} = 2^3 = 8.$

We have defined  $a^{\frac{1}{n}}$  as the  $n^{\text{th}}$  root of  $a$ . We extend this definition to  $a^{\frac{m}{n}}$  which is defined as the  $m^{\text{th}}$  power of the  $n^{\text{th}}$  root of  $a$ . A rational power indicates both a root and a power.

**DEFINITION**

If  $m$  and  $n$  are positive integers, then

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m,$$

provided that  $a^{\frac{1}{n}}$  is a real number.

Some examples of rational power are:  $2^{\frac{2}{3}}$ ,  $7^{\frac{5}{9}}$ ,  $12^{\frac{12}{7}}$ ,  $19^{\frac{19}{12}}$ ,  $27^{\frac{13}{17}}$ , etc.

**Note:**  $a^{\frac{1}{n}}$  is not real when  $a$  is negative and  $n$  is even. According to the definition of rational power, expressions such as  $(-16)^{\frac{3}{2}}$ ,  $(-29)^{\frac{3}{4}}$ ,  $(-64)^{\frac{5}{8}}$  and  $(-1)^{\frac{2}{3}}$  are not defined because each of them involves an even root of a negative number.

The exponent indicated in a rational power can be evaluated in either order. That is,  $\left(a^{\frac{1}{n}}\right)^m = \left(a^m\right)^{\frac{1}{n}}$  provided that  $a^{\frac{1}{n}}$  is a real number. For example,  $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = 2^2 = 4$  or  $8^{\frac{2}{3}} = \left(8^2\right)^{\frac{1}{3}} = 64^{\frac{1}{3}} = 4.$

A negative rational exponent indicates reciprocal just as a negative integral exponent does.

**EXAMPLE 23**

$$(a) 8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{\left(8^{\frac{1}{3}}\right)^2} = \frac{1}{2^2} = \frac{1}{4}$$

$$(b) 27^{-\frac{2}{3}} = \frac{1}{27^{\frac{2}{3}}} = \frac{1}{\left(27^{\frac{1}{3}}\right)^2} = \frac{1}{3^2} = \frac{1}{9}$$

**EXAMPLE 24**

Evaluate each of the following expressions.

$$(a) 49^{\frac{3}{2}} \qquad (b) 512^{\frac{3}{4}} \qquad (c) (3125)^{\frac{2}{5}} \qquad (d) 128^{-\frac{4}{7}}$$

**Solution**

$$(a) 49^{\frac{3}{2}} = \left(49^{\frac{1}{2}}\right)^3 = 7^3 = 343$$

$$(b) 512^{\frac{3}{4}} = \left(512^{\frac{1}{4}}\right)^3 = \left((4^4)^{\frac{1}{4}}\right)^3 = 4^3 = 64$$

$$(c) 3125^{\frac{2}{5}} = \left(3125^{\frac{1}{5}}\right)^2 = \left((5^5)^{\frac{1}{5}}\right)^2 = 5^2 = 25$$

$$(d) 128^{-\frac{4}{7}} = \left(128^{\frac{1}{7}}\right)^{-4} = \left((2^7)^{\frac{1}{7}}\right)^{-4} = 2^{-4} = \frac{1}{2^4} = \frac{1}{16}.$$

The rules of integer exponents also hold true for the rational powers as well.

**Rules for Rational Exponents**

The following rules are valid for all real numbers  $a$  and  $b$  and rational numbers  $\frac{m}{n}$  and  $\frac{p}{q}$ , provided that all indicated powers are real and no denominator is zero.

$$(i) a^{\frac{m}{n}} \times a^{\frac{p}{q}} = a^{\frac{m+p}{n}}$$

$$(iv) a^{\frac{m}{n}} \times b^{\frac{m}{n}} = (ab)^{\frac{m}{n}}$$

$$(ii) a^{\frac{m}{n}} \div a^{\frac{p}{q}} = a^{\frac{m-p}{n}}$$

$$(v) \left(\frac{a}{b}\right)^{\frac{m}{n}} = \left(\frac{b}{a}\right)^{\frac{m}{n}}$$

$$(iii) \left(a^{\frac{m}{n}}\right)^{\frac{p}{q}} = a^{\frac{m \times p}{n \times q}}$$

$$(vi) a^{\frac{m}{n}} \div b^{\frac{m}{n}} = (a \div b)^{\frac{m}{n}}$$

(vii)  $a^{-\frac{m}{n}} = \left(\frac{b}{a}\right)^{\frac{m}{n}}$

(viii) If  $a^{\frac{m}{n}} = b$ , then  $a = b^{\frac{n}{m}}$

**Group Work 2**

1. Simplify each of the following.

(a)  $\left(\frac{3}{4}\right)^{-1} \times \left(\frac{4}{3}\right)^{-1}$

(c)  $(-125)^{\frac{4}{3}} \times (25)^{-\frac{3}{2}}$

(b)  $81^{-\frac{5}{4}} \times 9^{\frac{5}{2}}$

(d)  $\left(\frac{32}{81}\right)^4 \times \left(\frac{16}{27}\right)^{-4}$

2. Solve for  $x$ .

(a)  $x^{\frac{3}{2}} = 8$

(c)  $x = (-243)^{\frac{2}{5}}$

(b)  $2x^{\frac{5}{3}} = -64$

(d)  $x^{\frac{4}{3}} = 625$

**EXAMPLE 25**

Evaluate each of the following.

(a)  $8^{\frac{1}{6}} \times 8^{\frac{1}{2}}$

(c)  $9^{\frac{4}{5}} \times (-27)^{\frac{4}{5}}$

(b)  $(-32)^{\frac{1}{10}} \times (-32)^{\frac{1}{5}}$

(d)  $\frac{25^{\frac{2}{3}}}{25^{\frac{1}{6}}}$

**Solution**

(a)  $8^{\frac{1}{6}} \times 8^{\frac{1}{2}} = 8^{\frac{1}{6} + \frac{1}{2}} = 8^{\frac{1+3}{6}} = 8^{\frac{4}{6}} = 8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = 2^2 = 4.$

(b)  $(-32)^{\frac{1}{10}} \times (-32)^{\frac{1}{5}} = (-32)^{\frac{1}{10} + \frac{1}{5}} = (-32)^{\frac{3}{10}} = \left((-2)^5\right)^{\frac{3}{10}} = (-2)^{5 \times \frac{3}{10}} = (-2)^3 = -8.$

(c)  $9^{\frac{4}{5}} \times (-27)^{\frac{4}{5}} = (9 \times -27)^{\frac{4}{5}} = (-243)^{\frac{4}{5}} = \left((-3)^5\right)^{\frac{4}{5}} = (-3)^{5 \times \frac{4}{5}} = (-3)^4 = 81.$

(d)  $\frac{25^{\frac{2}{3}}}{25^{\frac{1}{6}}} = 25^{\frac{2}{3} - \frac{1}{6}} = 25^{\frac{4-1}{6}} = 25^{\frac{3}{6}} = 25^{\frac{1}{2}} = (5^2)^{\frac{1}{2}} = 5^{2 \times \frac{1}{2}} = 5^1 = 5.$

**EXAMPLE 26**Solve for  $x$ .

(a)  $x^{-\frac{3}{2}} = 125$

(b)  $3x^{\frac{4}{3}} = 243$

(c)  $x = (-32)^{\frac{3}{5}}$

(d)  $x^{-\frac{8}{3}} = 256$

**Solution**

(a) If  $x^{\frac{3}{2}} = 125$ , then  $x = 125^{\frac{2}{3}}$ . Thus,  $x = 125^{\frac{2}{3}} = \left((125)^{\frac{1}{3}}\right)^{-2} = 5^{-2} = \frac{1}{5^2} = \frac{1}{25}$ .

Therefore,  $x = \frac{1}{25}$ .

(b) If  $3x^{\frac{4}{3}} = 243$ , then  $x^{\frac{4}{3}} = 81$  and then  $x = 81^{\frac{3}{4}}$ . Thus,  $x = 81^{\frac{3}{4}} = \left(81^{\frac{1}{4}}\right)^3 = 3^3 = 27$ .

(c)  $x = (-32)^{\frac{3}{5}} = \left((-32)^{\frac{1}{5}}\right)^3 = \left(\left((-2)^5\right)^{\frac{1}{5}}\right)^3 = (-2)^3 = -8$ .

(d) If  $x^{\frac{8}{3}} = 256$ , then  $x = 256^{\frac{3}{8}}$ .

Thus,  $x = 256^{\frac{3}{8}} = \left(512^{\frac{1}{8}}\right)^{-3} = \left(2^8\right)^{\frac{1}{8}}^{-3} = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$ .

**EXERCISES**

1. Evaluate each of the following.

(a)  $8^{\frac{2}{3}}$

(d)  $49^{\frac{3}{2}}$

(g)  $27^{\frac{4}{3}}$

(j)  $125^{\frac{4}{5}}$

(b)  $8^{\frac{4}{3}}$

(e)  $16^{\frac{3}{4}}$

(h)  $10000^{\frac{3}{4}}$

(k)  $(-125)^{\frac{4}{3}}$

(c)  $9^{\frac{3}{2}}$

(f)  $32^{\frac{3}{5}}$

(i)  $4^{\frac{5}{2}}$

(l)  $128^{\frac{3}{7}}$

2. Use your calculator to evaluate each of the following, correct to 2 decimal places where necessary.

(a)  $63^{\frac{3}{2}}$

(d)  $729^{\frac{4}{3}}$

(g)  $100^{\frac{2}{5}}$

(b)  $32^{\frac{4}{3}}$

(e)  $81^{\frac{3}{5}}$

(h)  $98^{\frac{4}{5}}$

(c)  $298^{\frac{3}{4}}$

(f)  $1024^{\frac{9}{10}}$

3. Evaluate each of the following.

(a)  $32^{\frac{3}{4}} \times 32^{-\frac{11}{20}}$

(c)  $\left(\frac{64}{125}\right)^{-3} \times \frac{32}{25}$

(b)  $27^{\frac{2}{3}} \times (-125)^{\frac{2}{3}}$

(d)  $10^3 \times 10^{\frac{2}{3}} \times (100)^{-\frac{1}{2}}$



(e) 
$$\frac{9^{\frac{2}{3}} \times 3^{\frac{5}{3}}}{6^2}$$

(g) 
$$\frac{8^{\frac{2}{3}} \times 125^{-\frac{2}{3}}}{625^{\frac{3}{4}} \times 16^{\frac{3}{4}}}$$

(f) 
$$\frac{9^{\frac{3}{4}}}{32^{\frac{2}{3}}} \times \frac{32^{\frac{13}{15}}}{9^{\frac{1}{4}}}$$

(h) 
$$\left(\frac{p^{-6}q^{-3}}{r^3s}\right)^{-\frac{4}{3}} \left(\frac{p^2r^{-4}s^{-2}}{q^{-4}}\right)^{-\frac{3}{2}}$$

4. Simplify each of the following.

(a)  $a^{\frac{2}{3}} \times a^{\frac{4}{3}}$

(d)  $(4^{-2}a^{12})^{-\frac{1}{2}} \times (9^{-1}a^4)^{\frac{1}{2}}$

(g)  $\frac{[(x^2y)^6]^{\frac{1}{3}}}{[(x^2y)^{\frac{1}{3}}]^6}$

(b)  $(x^6y^3)^{\frac{2}{3}}$

(e)  $\left(\frac{a^{\frac{3}{2}} \times a^{\frac{2}{3}}}{a^2}\right)^3$

(h)  $\frac{[(b^{-8})^{\frac{1}{4}}]^{-1} b^{\frac{2}{3}}}{b^{\frac{1}{3}} \times b^2}$

(c)  $(8x^6)^{\frac{2}{3}} \times (64x^{12})^{\frac{1}{6}}$

(f)  $\left(\frac{a^{-4}b^{-8}c^{-2}}{a^{-12}b^{-4}c^{-6}}\right)^{-\frac{1}{4}}$

(h)  $\frac{[(b^{-8})^{\frac{1}{4}}]^{-1} b^{\frac{2}{3}}}{b^{\frac{1}{3}} \times b^2}$

5. Solve for  $x$ .

(a)  $x^{\frac{3}{4}} = 27$

(c)  $x^{\frac{5}{3}} = 1024$

(b)  $6x^{\frac{9}{2}} = 3072$

(d)  $x^{\frac{5}{3}} = 243$

## Negative exponents

We define negative power as the multiplicative inverse of the base raised to the positive opposite of the power.

### Group Work 3

1. Let  $\frac{3^4}{3^7} = \frac{3 \times 3 \times 3 \times 3}{3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3} = \frac{\cancel{3} \times \cancel{3} \times \cancel{3} \times \cancel{3}}{\cancel{3} \times \cancel{3} \times \cancel{3} \times \cancel{3} \times 3 \times 3 \times 3} = \frac{1}{3 \times 3 \times 3} = \frac{1}{3^3}$ . Simplify the following.

(a)  $\frac{2^3}{2^5}$

(b)  $\frac{4^5}{4^5}$

(c)  $\frac{5^4}{5^7}$

(d)  $\frac{2^3}{3^4} \times \frac{3^2}{2^4}$

(e)  $\frac{3^9}{3^5}$

2. Compare the following pairs of numbers.

(a)  $\frac{2^5}{2^7}$  and  $\frac{1}{4}$

(c)  $\frac{5^3}{5^3}$  and 1

(b)  $\frac{3^4}{3^7}$  and  $\frac{3^7}{3^4}$

(d)  $\frac{2^2}{2^3}$  and  $\left(\frac{2}{3}\right)^2$

3. Discuss the relationship between:

(a)  $\frac{1}{2^5}$  and  $2^{-5}$

(b)  $3^{-4}$  and  $\frac{1}{3^4}$

4. Completing the following table. What do you notice when  $2^n$  is divided by 2 each time? Does this pattern continue when  $n$  is zero or negative?

$n$	-4	-3	-2	-1	0	1	2	3	4
$2^n$	$2^{-4}$	$2^{-3}$	$2^{-2}$	$2^{-1}$	$2^0$	$2^1$	$2^2$	$2^3$	$2^4$
Value	$\frac{1}{16}$ or $\frac{1}{2^4}$		$\frac{1}{4}$ or $\frac{1}{2^2}$	$\frac{1}{2}$		2	4		16

A **negative exponent** is defined as the multiplicative inverse of the base raised to the power which is opposite to the given power. For example,  $2^{-3}$ ,  $3^{-5}$  and  $7^{-4}$  stand for  $\frac{1}{2^3}$ ,  $\frac{1}{3^5}$  and  $\frac{1}{7^4}$ , respectively.

#### ACTIVITY 4

1. Write each of the following negative powers into fraction.

(a)  $5^{-3}$

(c)  $8^{-5}$

(e)  $13^{-4}$

(b)  $4^{-7}$

(d)  $10^{-4}$

(f)  $100^{-2}$

2. Write each of the following into negative power.

(a)  $\frac{1}{6^5}$

(b)  $\frac{1}{10^3}$

(c)  $\frac{1}{2^{10}}$

(d)  $\frac{1}{3^5}$

(e)  $\frac{1}{12^2}$

(f)  $\frac{1}{5^{11}}$

3. Compare the following pairs of number.

(a)  $\left(\frac{2}{3}\right)^2$  and  $\left(\frac{3}{2}\right)^{-2}$

(b)  $\left(\frac{5}{3}\right)^3$  and  $\left(\frac{3}{5}\right)^{-3}$

#### DEFINITION

Any non-zero number raised to a negative exponent is the reciprocal of the same power with positive exponent. That is, for  $a \neq 0$  and  $n \in \mathbb{N}$ ,  $a^{-n} = \frac{1}{a^n}$ .

#### EXAMPLE 27

Simplify and write the answer as a non-negative exponent.

(a)  $3^{-4}$

(b)  $2^{-5}$

(c)  $5^{-1}$

(d)  $\frac{2^3}{2^7}$

**Solution**

$$\begin{aligned}
 \text{(a)} \quad 3^{-4} &= \frac{1}{3^4} = \frac{1}{81} & \text{(b)} \quad 2^{-5} &= \frac{1}{2^5} = \frac{1}{32} & \text{(c)} \quad 5^{-1} &= \frac{1}{5^1} = \frac{1}{5} \\
 \text{(d)} \quad \frac{2^3}{2^7} &= \frac{2 \times 2 \times 2}{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2} = \frac{\cancel{2} \times \cancel{2} \times \cancel{2}}{\cancel{2} \times \cancel{2} \times \cancel{2} \times 2 \times 2 \times 2 \times 2} = \frac{1}{2 \times 2 \times 2 \times 2} = \frac{1}{2^4} \\
 &= 2^{-4} \\
 &= 2^{3-7}.
 \end{aligned}$$

We have a set of rules or laws of negative exponents which makes the process of simplification easy.

**Rule 1:** The negative exponent rule states that for every number “ $a$ ” with negative exponent ( $-n$ ), take the reciprocal of the base and multiply it according to the value of the exponent. That is,

$$a^{-n} = \frac{1}{a^n} = \frac{1}{a \times a \times a \times \dots \times a \text{ (} n \text{ times)}} = \frac{1}{a} \times \frac{1}{a} \times \frac{1}{a} \times \dots \times \frac{1}{a} \text{ (} n \text{ times)} = \left(\frac{1}{a}\right)^n.$$

**Rule 2:** The rule for a negative exponent in the denominator suggests that for every number “ $a$ ” in the denominator and its negative exponent ( $-n$ ), the result can be written as:

$$\frac{1}{a^{-n}} = a^n = a \times a \times a \times \dots \times a \text{ (} n \text{ times)}.$$

**EXAMPLE 28**

Simplify each of the following.

$$\text{(a)} \quad \left(\frac{3}{4}\right)^{-1} \qquad \text{(b)} \quad \left(\frac{3}{2}\right)^{-2}$$

**Solution**

$$\begin{aligned}
 \text{(a)} \quad \left(\frac{3}{4}\right)^{-1} &= \frac{1}{\left(\frac{3}{4}\right)^1} = \frac{1}{\frac{3}{4}} = \frac{4}{3}. \\
 \text{(b)} \quad \left(\frac{3}{2}\right)^{-2} &= \frac{1}{\left(\frac{3}{2}\right)^2} = \frac{1}{\frac{3}{2} \times \frac{3}{2}} = \frac{1}{\frac{3 \times 3}{2 \times 2}} = \frac{1}{\frac{9}{4}} = \frac{4}{9} = \frac{2 \times 2}{3 \times 3} = \frac{2}{3} \times \frac{2}{3} = \left(\frac{2}{3}\right)^2.
 \end{aligned}$$

**Note:** If  $a$  and  $b$  are non-zero real numbers, then it is always true that for  $n > 0$ .

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

### EXAMPLE 29

Verify that:

(a)  $\left(\frac{2}{5}\right)^{-4} = \left(\frac{5}{2}\right)^4$

(b)  $\left(\frac{3}{10}\right)^{-3} = \left(\frac{10}{3}\right)^3$

**Solution**

(a)  $\left(\frac{2}{5}\right)^{-4} = \frac{1}{\left(\frac{2}{5}\right)^4} = \frac{1}{\frac{2}{5} \times \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5}} = \frac{1}{\frac{2^4}{5^4}} = \frac{5^4}{2^4} = \left(\frac{5}{2}\right)^4$ .

(b)  $\left(\frac{3}{10}\right)^{-3} = \frac{1}{\left(\frac{3}{10}\right)^3} = \frac{1}{\frac{3}{10} \times \frac{3}{10} \times \frac{3}{10}} = \frac{1}{\frac{3^3}{10^3}} = \frac{10^3}{3^3} = \left(\frac{10}{3}\right)^3$ .

### EXERCISES

1. Write each of the following negative powers as a nonnegative exponent.

(a)  $5^{-4}$

(c)  $7^{-5}$

(e)  $\frac{5^0}{5^1}$

(g)  $(-2)^{-5}$

(b)  $3^{-9}$

(d)  $\frac{3^5}{3^8}$

(f)  $\frac{4^{10}}{4^{19}}$

(h)  $(-3)^{-3}$

2. Write each of the following into negative power.

(a)  $\frac{1}{3^7}$

(c)  $\frac{1}{2^{15}}$

(e)  $\frac{1}{2^8}$

(b)  $\frac{1}{10^4}$

(d)  $\frac{1}{3^{14}}$

(f)  $\frac{1}{7^{11}}$

3. Simplify and give the answer in simplest fractional form.

(a)  $\left(\frac{6}{7}\right)^{-1}$

(c)  $\left(\frac{27}{3}\right)^{-2}$

(e)  $\left(\frac{1}{3^{-2}}\right)^{-1}$

(g)  $(-3)^{-3}$

(b)  $\left(\frac{3}{5}\right)^{-3}$

(d)  $\left(\frac{8}{3}\right)^{-3}$

(f)  $(-2)^{-5}$

4. Evaluate each of the following.

(a)  $2^{-2} + 3^{-2}$

(c)  $\frac{1}{2^{-3}} + \frac{1}{4^{-2}}$

(b)  $2^{-4} - 4^{-3}$

(d)  $\left(\frac{3}{5}\right)^{-3} \times \left(\frac{10}{3}\right)^{-2}$

5. Simplify the expression  $(2^4 + 3^2)^{-2}$  by using the rule of negative exponents.

## Introduction

### DEFINITION

A function  $f$  of the form  $f(x) = a^x$ ,  $a > 0$  and  $a \neq 0$  is called an exponential function with base  $a$ .

### EXAMPLE 30

$f(x) = 2^x$  is an exponential function.

Some of the order pairs of this function are (1, 2), (2, 4), (3, 8), (4, 16), (5, 32),  $\left(-1, \frac{1}{2}\right)$ ,  $\left(\frac{1}{2}, \sqrt{2}\right)$ , ....

### EXAMPLE 31

$g(x) = \left(\frac{1}{3}\right)^x$  is an exponential function.

Here  $g(1) = \frac{1}{3}$ ,  $g(2) = \frac{1}{9}$ ,  $g(-1) = 3$ ,  $g(-2) = 9$ , ... or some of the ordered pairs of  $g$  are  $\left(1, \frac{1}{3}\right)$ ,  $\left(2, \frac{1}{9}\right)$ ,  $(-1, 3)$ ,  $\left(2, \frac{1}{9}\right)$ ,  $(-2, 9)$

### Note

- Domain of  $f(x) = a^x$ ,  $a > 0$  is the set of all real numbers.
- Range of  $f(x) = a^x$ ,  $a > 0$  and  $a \neq 1$  is the set of positive real numbers and if  $a = 1$ , then the range is  $\{1\}$ .

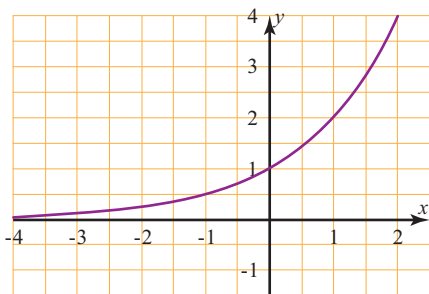
## Graphs of exponential functions

### Graph of $y = 2^x$

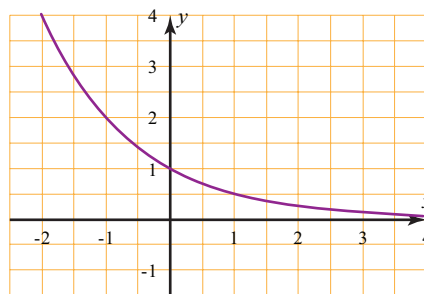
Consider the exponential function  $f(x) = 2^x$  or  $y = 2^x$

$y = 2^x$	$x$	-5	-4	-3	-2	-1	0	1	2	3	4	5
	$y$	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32

The domain of  $y = 2^x$  is the set of all real numbers, the function is continuous. By using the ordered pairs obtained from this table and using the fact that the graph is continuous, we can sketch the graph by connecting the points by a smooth curve.



Graph of  $y = 2^x$



Graph of  $y = 2^{-x}$

### Graph of $y = \left(\frac{1}{2}\right)^x$ or $y = 2^{-x}$

To sketch the graph of  $f(x) = \left(\frac{1}{2}\right)^x$  or  $y = \left(\frac{1}{2}\right)^x$  or  $y = 2^{-x}$

$y = \left(\frac{1}{2}\right)^x$	$x$	-3	-2	-1	0	1	2	3
	$y$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

If the graph of  $y = 2^x$  and  $y = 2^{-x}$  are plotted on the same coordinate axes, we can see that one is the mirror image of the other about the  $y$ -axis.

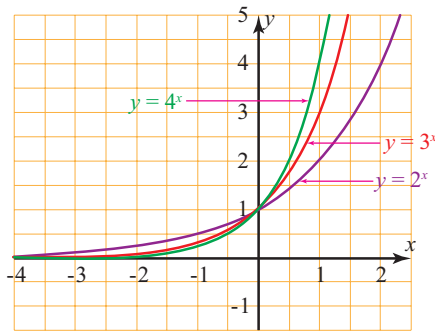
#### Note

From the graph of  $y = 2^x$  and  $y = \left(\frac{1}{2}\right)^x$ .

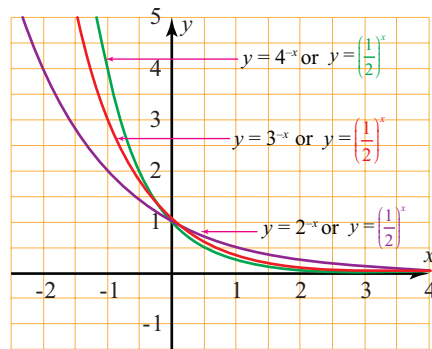
- In the exponential function  $y = 2^x$  .....base 2
  - If  $x > 0$ , then  $y > 1$ .

- (b) If  $x < 0$ , then  $0 < y < 1$ .  
 (c) If  $x = 0$ , then  $y = 1$ .
2. In the exponential function  $y = \left(\frac{1}{2}\right)^x$   
 (a) If  $x > 0$ , then  $0 < y < 1$ .  
 (b) If  $x < 0$ , then  $y > 1$ .  
 (c) If  $x = 0$ , then  $y = 1$ .
3. The function  $y = 2^x$  is increasing while the function  $y = \left(\frac{1}{2}\right)^x$  is decreasing.

Consider the exponential functions  $y = 2^x$ ,  $y = 3^x$  and  $y = 4^x$  drawn using the same coordinate axes (a) and  $y = \left(\frac{1}{2}\right)^x$ ,  $y = \left(\frac{1}{3}\right)^x$  and  $y = \left(\frac{1}{4}\right)^x$  drawn on the same coordinate axes (b).



a.



b.

We can make the following comparisons

- For  $x > 0$ 
  - $4^x > 3^x > 2^x$
  - $\left(\frac{1}{2}\right)^x > \left(\frac{1}{3}\right)^x > \left(\frac{1}{4}\right)^x$
- For  $x < 0$ 
  - $4^x < 3^x < 2^x$
  - $\left(\frac{1}{2}\right)^x < \left(\frac{1}{3}\right)^x < \left(\frac{1}{4}\right)^x$

**EXAMPLE 32**

(a) If  $x = 2$ , then  
 $4^2 > 3^2 > 2^2$

(b) If  $x = -2$ , then  
 $4^{-2} < 3^{-2} < 2^{-2}$

**EXAMPLE 33**

(a) If  $x = -2$ , then  
 $\left(\frac{1}{2}\right)^2 > \left(\frac{1}{3}\right)^2 > \left(\frac{1}{4}\right)^2$

(b) If  $x = 2$ , then  
 $\left(\frac{1}{2}\right)^{-2} < \left(\frac{1}{3}\right)^{-2} < \left(\frac{1}{4}\right)^{-2}$

### The exponential function $y = a^x$

Let us see the behaviors of  $y = a^x$ ,  $a > 0$ , and its graph.

In the exponential function  $y = a^x$ ,  $a > 0$ .

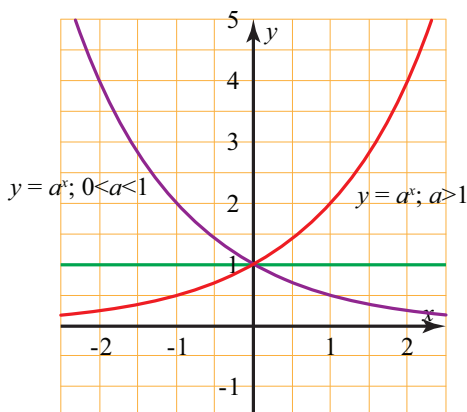
1. The  $y$ -intercept is  $a^0 = 1$  and the graph of  $y = a^x$  cross the  $y$ -axis at  $(0, 1)$ .
2. No  $x$ -intercept (*the graph will not cross the  $x$ -axis*).

The graph of  $y = a^x$  is completely above the  $x$ -axis.

The  $x$ -axis or the line  $y = 0$  is a horizontal asymptote.

$y = a^x$  will never be 0.

- (a) For  $0 < a < 1$ , the graph approaches the positive  $x$ -axis as  $x$  increases to very large number and far up for large negative  $x$ .
  - (b) For  $a > 1$ , the graph approaches the negative  $x$ -axis as  $x$  decreases and the graph will be far up as  $x$  increases.
3. The function is continuous, and graph will take the following pattern.



- 4.
- (a) For  $a > 1$ , the function  $y = a^x$  is increasing.  
If  $b > c$ , then  $a^b > a^c$ .
  - (b) For  $0 < a < 1$ , the function  $y = a^x$  is decreasing.  
If  $b > c$ , then  $a^b < a^c$ .
  - (c) For  $a = 1$ , the function  $y = a^x$  is constant.
5. For  $a > 1$
- (a) If  $x > 0$ , then  $a^x > 1$ .
  - (b) If  $x < 0$ , then  $0 < a^x < 1$ .
6. For  $0 < a < 1$
- (a) If  $x > 0$ , then  $0 < a^x < 1$ .
  - (b) If  $x < 0$ , then  $a^x > 1$ .
7. For  $a > b > 1$
- (a) If  $x < 0$ , then  $a^x < b^x$ .
  - (b) If  $x > 0$ , then  $a^x > b^x$ .
8. For  $0 < a < b < 1$
- (a) If  $x < 0$ , then  $a^x > b^x$ .
  - (b) If  $x > 0$ , then  $a^x < b^x$ .
9.  $y = a^x$  is a one-to-one function

### Exponential equations

In exponential equations there is a variable exponent. In solving exponential equations, we will use the exponential properties and in solving such equations it is important to note that if  $a^x = 1$ , then  $x = 0$ . It is due to  $a^0 = 1$ , where  $a \neq 0$ .

#### EXAMPLE 34

Solve  $2^x = 32$

#### Solution

We can write  $32 = 2^5$

$$2^x = 2^5 \Rightarrow x = 5$$

Therefore, Solution set (SS) =  $\{5\}$ .

#### EXAMPLE 35

Solve  $3^x = 81$

#### Solution

We can write,  $81 = 3^4$

$$3^x = 3^4 \Rightarrow x = 4$$

Therefore, SS =  $\{4\}$ .

**EXAMPLE 36**

Solve  $2^{x+1} = 8$

**Solution**We can write  $8 = 2^3$ 

$$\Rightarrow 2^{x+1} = 2^3$$

$$\Rightarrow x + 1 = 3$$

$$x = 2$$

Therefore,  $SS = \{2\}$ .**EXAMPLE 37**

Solve  $6^{2x+1} \cdot 2^{2x+1} \cdot 3^{2x+1} = \sqrt{6}$

**Solution**

$$(6 \times 2 \times 3)^{2x+1} = 6^{\frac{1}{2}}$$

$$\Rightarrow 36^{2x+1} = (36)^{\frac{1}{2} \times \frac{1}{2}}$$

$$\Rightarrow 2x + 1 = \frac{1}{4}$$

$$\Rightarrow 2x = \frac{-3}{4}$$

$$\Rightarrow x = \frac{-3}{8}$$

Therefore,  $SS = \left\{ \frac{-3}{8} \right\}$ .**EXAMPLE 38**

Solve  $\left(\frac{16}{81}\right)^x \cdot \left(\frac{8}{27}\right)^{2x-1} = \frac{4}{9}$

**Solution**

$$\left(\frac{2}{3}\right)^{4x} \cdot \left(\frac{2}{3}\right)^{3(2x-1)} = \left(\frac{2}{3}\right)^2$$

$$\Rightarrow \left(\frac{2}{3}\right)^{4x+3(2x-1)} = \left(\frac{2}{3}\right)^2$$

$$\Rightarrow 4x + 3(2x - 1) = 2$$

$$\Rightarrow 4x + 6x - 3 = 2$$

$$\Rightarrow 10x = 5$$

$$x = \frac{1}{2}$$

$$\text{Therefore, SS} = \left\{ \frac{1}{2} \right\}.$$

**EXAMPLE 39**

$$\text{Solve the equation } \left(\frac{16}{81}\right)^{2x-1} \cdot \left(\frac{9}{4}\right)^{3x} = \left(\frac{3}{2}\right)^{x-1}$$

**Solution**

$$\left(\frac{2}{3}\right)^{4(2x-1)} \cdot \left(\frac{2}{3}\right)^{-2(3x)} = \left(\frac{2}{3}\right)^{-(x-1)}$$

**Note:**

$$\frac{a}{b} = \left(\frac{b}{a}\right)^{-1}$$

$$\Rightarrow \left(\frac{2}{3}\right)^{8x-4-6x} = \left(\frac{2}{3}\right)^{-(x-1)} \dots\dots\dots \text{(It is a linear Equation)}$$

$$\Rightarrow 8x - 4 - 6x = -(x - 1)$$

$$\Rightarrow 2x - 4 = -x + 1$$

$$\Rightarrow 3x = 5$$

$$\Rightarrow x = \frac{5}{3}$$

$$\text{Therefore, S.S} = \left\{ \frac{5}{3} \right\}.$$

**EXAMPLE 40**

$$\text{Solve the equation } 4^{x-1} = \sqrt{2 \frac{x-1}{x}}, x \neq 0$$

**Solution**

$$(4^{x-1})^2 = 2^{\frac{x-1}{x}} \dots\dots\dots \text{squaring both sides}$$

$$\Rightarrow (2^{2(x-1)})^2 = 2^{\frac{x-1}{x}}$$

$$\Rightarrow 4x - 4 = \frac{x-1}{x}$$

$$\Rightarrow 4x^2 - 4x = x - 1$$

$$\Rightarrow 4x^2 - 5x + 1 = 0 \dots\dots\dots \text{(It is a quadratic Equation)}$$

$$\Rightarrow 4x^2 - 4x - x + 1 = 0$$

$$\Rightarrow 4x(x-1) - 1(x-1) = 0$$

$$\Rightarrow (x-1)(4x-1) = 0$$

$$\Rightarrow x = 1 \text{ or } x = \frac{1}{4}, \text{ Therefore, SS} = \left\{1, \frac{1}{4}\right\}.$$

**EXERCISES**

Solve each of the following equations

1.  $8^{x-1} = 32^{2x-3}$

2.  $8^{-\frac{2}{3}} = 16^{2(x-5)}$

3.  $3^{\frac{x}{5}} \times 3^{\frac{x-10}{10}} = 27$

4.  $\left(\frac{3}{2}\right)^{2x-1} \cdot \left(\frac{27}{8}\right)^{-3x+3} = \left(\frac{1}{2}\right)^{3x} \cdot 3^{3x}$

5.  $\left(\frac{4}{9}\right)^{-x+3} \cdot \left(\frac{8}{27}\right)^{5-3x} = \left(\frac{3}{2}\right)^{\frac{3}{2}}$

 6. Draw the graph of  $y = 2^x$  and  $y = 2^{-x}$  using the same coordinate axes.

 7. What is the domain of  $y = 3^x$ , what is its range?

 8. For what values of  $x$  is  $2^{-x}$  greater than 1?

 9. For what values of  $x$  is  $2^{-x} > 3^{-x}$ ?

 10. If  $a^x > b^x$ , for  $x > 0$ , then how do you compare  $a$  and  $b$ ?

**Exponential growth** is the increase in number or size at a constantly growing rate. In exponential growth, a population's per individual growth rate stays the same regardless of the population size, making it grow faster and faster until it becomes

large. Exponential growth is a pattern of data that shows greater increases with passing time creating the curve of an exponential function.

### Opening Problem

- Suppose there are 15 fishes in an aquarium. After one year they had increased to 135.
- If the fishes are growing exponentially, then:
- At the end of five years, how many fishes will there be in the aquarium?
- At the end of ten years, how many fishes will there be in the aquarium?
- How many years will it take for the number of fishes to be double?
- What will be the exponential growth formula to represent the given word problem?



Figure 1.

What is exponential growth? If something is increasing or growing rapidly as a result of a constant rate of growth applied to it, then it is experiencing exponential growth.

Exponential growth can be modeled with the exponential function

$$y = f(x) = ab^x, \text{ where } a > 1 \text{ and } b > 1.$$

In the equation  $y = f(x) = ab^x$ ,  $x$  is the **exponent**,  $a$  is the **starting amount** when  $x = 0$  and  $b$  is the **base, rate or growth factor** and it is a **constant** and it is greater than 1.

### EXAMPLE 41

The figure 2 represents an exponential growth. In fact, it is the graph of the exponential function  $y = 2^x$ . The general form of an exponential function is  $y = ab^x$ . Thus, if  $y = 2^x$ , then  $a = 1$  and  $b = 2$ .

The table given below shows some points that we could have used to graph this exponential growth. Try to locate some of these points on the graph as shown in Figure.2.

$x$	-4	-3	-2	-1	0	1	2	3	4
$y$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

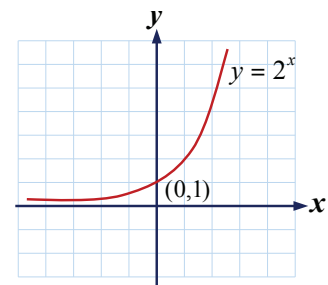


Figure 2.

## ACTIVITY 5

- Determine whether each of the following functions represents exponential growth or not.
 

(a) $y = x^3$	(d) $f(x) = 5\left(\frac{2}{3}\right)^x$
(b) $y = 2^{-x}$	(e) $y = \frac{1}{8}x^2$
(c) $y = 4^x$	(f) $y = 10(0.2)^y$
- Suppose a population of mice rises exponentially by a factor of four every year starting with 4 in the first year, then 16 in second year, 64 in the third year and so on.
  - By what factor is the population growing in each year?
  - What will be the population of the mice in the fifth year?
  - How many years will it take for the number of mice to be 512?
  - Drive the exponential growth formula to represent the given word problem?

## EXAMPLE 42

Suppose there are 8 mice in a cage. They had increased to 72 after two months. Assuming the growth continuous exponentially.

- What will be the exponential growth formula to represent the given word problem?
- How many mice will there be in the cage at end of five months?
- How many mice will there be in the cage at end of ten months?
- How many months will it take for the number of mice to be 648?

**Solution**

- (a) Recall that the general form of an exponential function is  $y = f(t) = ab^t$ .

Here,  $a = 8$  mice,  $t = 2$  months and after two months,  $f(2) = 72$  mice. Thus,

$$72 = 8(b)^2 \text{ implies } 9 = b^2 \text{ and } b = 3.$$

Therefore, the exponential growth formula will be  $y = 8(3)^t$ .

- (b)  $f(5) = 8(3)^5 = 1,944$  mice.

- (c)  $f(10) = 8(3)^{10} = 8 \times 59,049 = 472,392$  mice.

- (d)  $648 = 8(3)^t$  implies  $81 = 3^t$  and then  $3^4 = 3^t$ . Hence,  $t = 4$ .

Therefore, it will take 4 months for the number of mice to be 648.

There are many real-life applications of exponential growth such as *population growth*, *compound interest* and *bacteria growth*.

## 1. Population Growth

Population growth is the increase in the number of people in a population. The **population growth rate** is the rate at which the number of individuals in a population increases in a given time period.

For example, suppose the population of a certain city was 2.5 million in 2000. Then, every year after that, the population has grown by 2%. This is an example of exponential growth. Notice that the rate of growth is 2% or 0.02 and it is constant. This is important since the rate of growth cannot change.

Let us find the exponential function.

*Year 2001 or 1 year after:*

$$\begin{aligned} 2,500,000 + 2,500,000 \times 0.02 &= 2,500,000(1 + 0.02) \\ &= 2,500,000(1.02) \\ &= 2,500,000(1.02)^1. \end{aligned}$$

*Year 2002 or 2 years after:*

$$\begin{aligned} 2,500,000(1.02) + 2,500,000(1.02) \times 0.02 &= 2,500,000(1.02) [1 + 0.02] \\ &= 2,500,000(1.02) (1.02) \\ &= 2,500,000(1.02)^2. \end{aligned}$$

*Year 2002 or 3 years after:*

$$\begin{aligned} 2,500,000(1.02)^2 + 2,500,000(1.02)^2(1+0.02) &= 2,500,000(1.02)^2(1 + 0.02) \\ &= 2,500,000(1.02)^2(1.02) \\ &= 2,500,000(1.02)^3 \end{aligned}$$

Following this pattern, suppose that  $x$  is the number of years since 2000, 2,500,000 is the starting amount and 1.02 is the rate or growth factor. Then,  $y = 2,500,000(1.02)^x$ .

Comparing this exponential function with  $y = ab^x$ , we see that  $a = 2,500,000$ ,  $b = 1.02$ .

The growth of population,  $p$ , at the end of  $t$  years with growth rate  $r$  can be given by the formula:

$$p = p_0(1 + r)^t,$$

where  $p_0$  is the original number of populations.

**EXAMPLE 43**

A certain town has a population of 400,000 that is increasing at the rate of 5% each year. Approximate the population of the town after 6 years.

**Solution**

**Given:**  $P_0 = 400,000$ ,  $r = 5\% = 0.05$  and  $t = 6$  years. Required:  $P = ?$

$$P = P_0(1 + r)^t \Rightarrow P = 400,000(1 + 0.05)^6 = 400,000(1.05)^6 \\ = 536,038.$$

Therefore, the population of the town at the end of 6 years is approximately 536,038.

**2. Compound Interest**

When *interest* is added to the *principal* at the end of each period, it counted into principal. The total amount due at the end of the period is called the **compound amount**. For example, David lends L\$5,000 to his friend at annual interest of 6% compounded annually. Notice that the rate of growth is 6% or 0.06 and it is constant. Let us find the exponential function.

**1 year after:**

$$5,000 + 5,000 \times 0.06 = 5,000(1 + 0.06) = 5,000(1.06) = 5,000(1.06)^1.$$

**2 years after:**

$$5,000(1.06) + 5,000(1.06) \times 0.06 = 5,000(1.06) [1 + 0.06] = 5,000(1.06) (1.06) \\ = 5,000(1.06)^2.$$

**3 years after:**

$$5,000(1.06)^2 + 5,000(1.06)^2 (0.06) = 5,000(1.06)^2 + 5,000(1.06)^2 (0.06) \\ = 5,000(1.06)^2(1 + 0.06) \\ = 5,000(1.06)^2(1.06) \\ = 5,000(1.06)^3.$$

Following this pattern, suppose that  $x$  is the number of years, 5,000 is the starting amount and 1.06 is the rate or growth factor. Then, the exponential function will be  $y = 5,000(1.06)^x$ . Comparing this exponential function with  $y = ab^x$ , we can see that  $a = 5,000$ ,  $b = 1.06$ .

If a principal  $P$  is invested at an annual rate of  $r$ , compounded annually, then the amount after  $t$  years is given by

$$A = P(1 + r)^t$$

**EXAMPLE 44**

John deposits L\$10,000 in a bank that pays 8% compound interest annually. How much money will he have after 11 years without withdrawal?

**Solution**

**Given:**  $P = 10,000$ ,  $r = 8\% = 0.08$  and  $t = 11$ . Required:  $A = ?$

$$A = P(1 + r)^t \Rightarrow A = 10,000(1 + 0.08)^{11} = 10,000(1.08)^{11} \\ = 23,316.40.$$

Therefore, after 11 years, John will have L\$23,316.40.

**3. Bacterial Growth**

One of the most prevalent applications of exponential functions involves growth and decay models.

An exponential growth or decay function is a function that grows or shrinks at a constant growth rate. The equation can be written in the form

$$y = ab^x,$$

where,  $a$  is the initial or starting value,  $b$  is the growth factor and  $x$  is the number of time intervals that have passed.

**EXAMPLE 45**

There were 72 fishes in a pond. They had increased to 216 after six months. If the fishes are growing exponentially, then how many fishes will there be in the pond at the end of two years?

**Solution**

Here,  $a = 72$ ,  $x = 0.5$  year and  $y = 216$ . Thus, the exponential growth function is,

$$y = 72(b)^x$$

Since there were 216 fishes after six months, we have

$$216 = 72b^{\frac{1}{2}} \text{ implies that } 3 = b^{\frac{1}{2}} \text{ and then squaring both sides give } b = 9.$$

Here, we can observe that  $b = 9$  and  $9 > 1$ , as it is exponential growth. Now, we have to find the number of fishes at the end of 2 years. So, we substitute  $x = 2$  and  $b = 9$  in the equation  $y = 72(b)^x$ .

$$y = 72(9)^x = 72(9)^2 = 72 \times 81 = 5,832.$$

Therefore, at the end of two years, there will be 5,832 fishes in the pond.

## EXERCISES

1. James invest L\$25,000 in a bank which pays 8% compound interest. How much money will he have after 10 years without withdrawal?
2. In a certain culture there were 30 bacteria initially. They had increased to 60 after one hours. If the bacteria is growing exponentially, then how many bacterias will there be at the end of 8 hours?
3. A sum of money deposited at compound interest double itself in 5 years. If the interest is being compounded annually, in how many years will it amount to four times itself?
4. In 2010, there were 100 inhabitants in a certain town. If population has increased by 4% every year, then how many residents will there be in 5 years?
5. The bear population increases at a rate of 5% per year. There are 1500 bear this year. Write an exponential growth function that models the bear population. How many bears will there be in 10 years?

The ***logarithm of a number*** is the index or exponent by which another value must be raised to produce an equivalent value of the given number. For example, in the indicial expression,  $4^3 = 64$ , the base is 4 and the index or exponent is 3. We write this expression in logarithm as  $\log_4 64 = 3$ .

We read this as “*logarithm of 64 to the base 4 is 3.*”

## HISTORICAL NOTE

Logarithms were developed in the 17<sup>th</sup> century by the Scottish mathematician, John Napier (1550–1617). He was the first mathematician to devise the idea of the logarithm, naming it logarithm from the Greek root’s *logos*, meaning *proportion* and *arithmus* meaning *number* because he used it to relative numbers to another value.



*John Napier*

The logarithm has changed very much since its introduction into the world of mathematics, it began as a way to make the multiplication and division of large numbers into an easier form looking up value in a table and then adding them for multiplication and subtracting them for division and from there it has evolved into the logarithm known today as the *inverse* of the ***exponent***.

In general, the logarithm of a positive real number  $x$  with respect to base  $a$  and  $a \neq 1$  is the index or exponent by which  $a$  must be raised to yield  $x$ .

#### EXAMPLE 46

- (a) Since  $3^4 = 81$ , we can say that  $\log_3 81 = 4$ .  
 (b) Since  $10^5 = 100,000$ , we can say that  $\log_{10} 100,000 = 5$ .

The following activity will help you learn how to convert indicial or exponential statements to logarithmic statements and vice versa.

#### ACTIVITY 6

1. Complete the following table.

Indicial (exponential) statement	Logarithmic statement
$5^3 = 125$	$\log_5 125 = 3$
$2^7 = 128$	
	$\log_{10} 1000 = 3$
$7^3 = 343$	
	$\log_6 1,296 = 4$
$10^4 = 10,000$	
	$\log_2 512 = 9$

2. Convert each of the following indicial (exponential) statements to logarithmic statements.

(a)  $2^{10} = 1,024$       (b)  $6^4 = 216$       (c)  $8^{\frac{2}{3}} = 4$       (d)  $81^{\frac{1}{4}} = 3$

3. Convert each of the following logarithmic statements to indicial (exponential) statements.

(a)  $\log_8 16 = \frac{4}{3}$       (b)  $\log_2 256 = 8$       (c)  $\log_5 125 = 3$       d.  $\log_{27} 9 = \frac{2}{3}$

In general, for a fixed positive number  $b \neq 1$  and for every real number  $a > 0$

$$c = \log_b a \text{ if and only if } b^c = a.$$

From the above statement, we can note that every logarithmic statement can be converted into an indicial (exponential statement) and vice versa.

**Note:** The value of  $\log_b a$  is the answer to the question “to what power must  $b$  be raised to produce  $a$ ”.

**EXAMPLE 48**

Write each of the following in logarithmic statement.

(a)  $16^{\frac{1}{4}} = 2$

(b)  $9^4 = 6,561$

(c)  $1000^{\frac{2}{3}} = 100$

(d)  $8^4 = 4,096$

**Solution**

(a) From  $16^{\frac{1}{4}} = 2$ , we deduce that  $\log_{16} 2 = \frac{1}{4}$ .

(b) From  $9^4 = 6,561$ , we deduce that  $\log_9 6561 = 4$ .

(c) From  $1000^{\frac{2}{3}} = 100$ , we deduce that  $\log_{100} 1000 = \frac{2}{3}$ .

(d) From  $8^4 = 4,096$ , we deduce that  $\log_8 4096 = 4$ .

**EXAMPLE 49**

Write each of the following as an exponential statement.

(a)  $\log_2 128 = 7$

(b)  $\log_5 625 = 4$

(c)  $\log_{64} \frac{1}{4} = -\frac{1}{3}$

(d)  $\log_{243} 3 = \frac{1}{5}$

**Solution**

Recall that  $\log_b a = y$  if and only if  $b^y = a$ . Thus,

(a)  $\log_2 128 = 7$  means  $2^7 = 128$ .

(b)  $\log_5 625 = 4$  means  $5^4 = 625$ .

(c)  $\log_{64} \frac{1}{4} = -\frac{1}{3}$  means  $(64)^{-\frac{1}{3}} = \frac{1}{4}$ .

(d)  $\log_{243} 3 = \frac{1}{5}$  means  $(243)^{\frac{1}{5}} = 3$ .

**EXAMPLE 47**

Evaluate each of the following.

(a)  $\log_4 1024$

(b)  $\log_7 343$

(c)  $\log_{100} 10$

(d)  $\log_{81} 27$

**Solution**

(a) To find  $\log_4 1024$ , let us answer the question that “to what power must 4 be raised to get 1024?”

Since  $4^5 = 1024$ , we get  $\log_4 1024 = 5$ .

- (b) To find  $\log_7 343$ , let us answer the question that “to what power must 7 be raised to get 343?”

Since  $7^3 = 343$ , the logarithm of 343 to base 7 is 3. Thus,  $\log_7 343 = 3$ .

- (c) To find  $\log_{100} 10$ , let us answer the question that “to what power must 100 be raised to get 10?”

Since  $100^{\frac{1}{2}} = 10$ , the logarithm of 10 to base 100 is  $\frac{1}{2}$ . Thus,  $\log_{100} 10 = \frac{1}{2}$ .

- (d) To find  $\log_{81} 27$ , let us answer the question that “to what power must 81 be raised to get 27?”

Since  $81^{\frac{3}{4}} = 27$ , the logarithm of 27 to base 81 is  $\frac{3}{4}$ . Thus,  $\log_{81} 27 = \frac{3}{4}$ .

## EXERCISES

1. Convert each of the following indicial (exponential) statements to logarithmic statements

(a)  $10^6 = 1000000$

(f)  $7^4 = 2401$

(b)  $2^{-5} = \frac{1}{32}$

(g)  $9^{-2} = \frac{1}{81}$

(c)  $3125^{\frac{3}{5}} = 125$

(h)  $27^{\frac{2}{3}} = 9$

(d)  $64^{\frac{2}{3}} = 16$

(i)  $10^1 = 10$

(e)  $243^{-\frac{3}{5}} = \frac{1}{27}$

(j)  $32^{-\frac{3}{5}} = \frac{1}{8}$

2. Convert each of the following logarithmic statements to indicial (exponential) statements

(a)  $\log_{0.3} 0.027 = 3$

(f)  $\log_3 \frac{1}{3} = -1$

(b)  $\log_3 81 = 4$

(g)  $\log_{81} 9 = \frac{1}{2}$

(c)  $\log_8 32 = \frac{5}{3}$

(h)  $\log_5 5 = 1$

(d)  $\log_{10} 0.1 = -1$

(i)  $\log_7 1 = 0$

(e)  $\log_2 16 = 4$

(j)  $\log_{27} 3 = \frac{1}{3}$

3. Evaluate each of the following.

(a)  $\log_3 243$

(b)  $\log_5 625$

(c)  $\log_4 16$

(d)  $\log_4 4^{-3}$

(e)  $\log_{10} 1$

(f)  $\log_{27} 243$

(g)  $\log_{10} \frac{1}{100}$

(h)  $\log_{10} 100$

(i)  $\log_8 16$

(j)  $\log_8 128$

There are a number of rules known as the laws of logarithms. These allow expressions involving logarithms to be rewritten in a variety of different ways. The laws apply to logarithms of any base but the same base must be used throughout a calculation.

### Group work 2.4

The following Group Work will help you to observe different laws while performing different types of logarithmic operations.

1. Compute

(a)  $\log_2 32 + \log_2 16$ ; compare the result with  $\log_2 (32 \times 16)$ .

(b)  $\log_{10} 1000 + \log_{10} 0.001$ ; compare the result with  $\log_{10} (1000 \times 0.001)$ .

(c)  $\log_3 27 + \log_3 81$ ; compare the result with  $\log_3 (27 \times 81)$ .

(d)  $\log_5 625 \times \frac{1}{25}$ ; compare the result with  $\log 625 - \log 25$ .

From your answers, can you suggest a possible for simplification for  $\log_a x + \log_a y$ ?

2. Compute

(a)  $\log_2 32 - \log_2 64$ ; compare the result with  $\log_2 (32 \div 64)$ .

(b)  $\log_{10} 10000 - \log_{10} 100$ ; compare the result with  $\log_{10} (10000 \div 100)$ .

(c)  $\log_3 243 - \log_3 27$ ; compare the result with  $\log_3 (243 \div 27)$ .

(d)  $\log_5 125 - \log_5 \frac{1}{25}$ ; compare the result with  $\left(125 \div \frac{1}{25}\right)$ .

From your answers, can you suggest a possible answer for the simplification of  $\log_a x - \log_a y$ ?

3. Compute

(a)  $3\log_4 4$ ; compare the result with  $\log_4(4)^3$ .

(b)  $2\log_2 16$ ; compare the result with  $\log_2(16)2$ .

(c)  $\frac{1}{2}\log_3 81$ ; compare the result with  $\log_3(81)^{\frac{1}{2}}$ .

From your answers, can you suggest a possible answer for the simplification of  $m \log_a x$ ?

4. Compute

(a)  $\log_4 4$

(c)  $\log_{25} 25$

(b)  $\log_{10} 10$

(d)  $\log_{100} 100$

From your answers, can you suggest a possible answer for the simplification of  $\log_a a$ , where  $a > 0$  and  $a \neq 1$ ?

5. Compute

(a)  $\log_2 1$

(c)  $\log_{10} 1$

(b)  $\log_5 1$

(d)  $\log_{100} 1$

From your answers, can you suggest a possible answer for the simplification of  $\log_a 1$ , where  $a > 0$  and  $a \neq 1$ ?

### Suppose

$$x = a^m \quad \text{and} \quad y = a^n,$$

then the equivalent logarithmic forms are

$$\log_a x = m \quad \text{and} \quad \log_a y = n.$$

Using the product property of indices, we get

$$xy = a^m \times a^n = a^{m+n}.$$

Now the logarithmic form of the statement  $xy = a^{m+n}$  is  $\log_a xy = m + n$ . But,  $m = \log_a x$  and  $n = \log_a y$  so, putting these results together, we have

$$\log_a xy = m + n = \log_a x + \log_a y.$$

### I. Product Law of Logarithm

If  $a$ ,  $x$  and  $y$  are positive real numbers, where  $a \neq 1$ , then

$$\log_a (xy) = \log_a x + \log_a y.$$

**EXAMPLE 50**

Use the product law of logarithm to simplify each of the following.

(a)  $\log_2 8 + \log_2 5$

(b)  $\log_{10} A + \log_{10} B$

(c)  $\log_4 x + \log_4 x$

**Solution**

(a)  $\log_2 8 + \log_2 5 = \log_2 (8 \times 5) = \log_2 40.$

(b)  $\log_{10} A + \log_{10} B = \log_{10} (A \times B) = 1 \log_{10} AB = \log_{10} AB.$

(c)  $\log_4 x + \log_4 x = \log_4 (x \times x) = \log_4 x^2.$

**EXAMPLE 51**

Show that  $\log_3 27 + \log_3 9 = \log_3 (27 \times 9).$

**Solution**

$\log_3 27 = 3 \text{ and } \log_3 9 = 2.$

$\log_3 27 + \log_3 9 = 3 + 2 = 5; \text{ and}$

$\log_3 (27 \times 9) = \log_3 243 = 5.$

Hence,  $\log_3 27 + \log_3 9 = \log_3 (27 \times 9)$

$$3 + 2 = 5.$$

Therefore,  $\log_3 27 + \log_3 9 = \log_3 (27 \times 9)$  is verified.

**Suppose**

$$x = a^m \text{ and } y = a^n,$$

then their equivalent logarithmic forms are

$$\log_a x = m \text{ and } \log_a y = n$$

Consider  $x \div y$ .

Using the quotient property of indices, we get

$$\frac{x}{y} = \frac{a^m}{a^n} = a^{m-n}.$$

In logarithmic form  $\log_a \left( \frac{x}{y} \right) = m - n$ . But,  $m = \log_a x$  and  $n = \log_a y$  so, putting these results together, we have

$$\log_a \left( \frac{x}{y} \right) = m - n = \log_a x - \log_a y.$$

This leads to the following law.

## II. Quotient Law of Logarithm

If  $a$ ,  $x$  and  $y$  are positive real numbers, where  $a \neq 1$ , then

$$\log_a \left( \frac{x}{y} \right) = \log_a x - \log_a y.$$

### EXAMPLE 52

Use the quotient law of logarithm to simplify the following.

(a)  $\log_3 162 - \log_3 3$

(b)  $\log_2 560 - \log_2 7 - \log_2 5$

**Solution**

(a)  $\log_3 162 - \log_3 3 = \log_3 \left( \frac{162}{3} \right) = \log_3 81 = 4.$

(b)  $\log_2 560 - \log_2 7 - \log_2 5 = \log_2 560 - (\log_2 7 + \log_2 5) = \log_2 560 - \log_2 (7 \times 5)$   
 $= \log_2 560 - \log_2 35 = \log_2 \left( \frac{560}{35} \right) = \log_2 16 = 4.$

Suppose  $x = a^n$ , or equivalently  $\log_a x = n$ . Suppose we raise both sides of  $x = a^n$  to the power  $m$ :

$$x^m = (a^n)^m.$$

Using the *power property* of indices, we can write this as

$$x^m = a^{nm}.$$

Thinking of the quantity  $x^m$  as a single term, the logarithmic form is:

$$\log_a x^m = mn = m \log_a x.$$

This leads to the *power law of logarithm*.

## III. Power Law of Logarithm

If  $a$  and  $x$  are positive real numbers, where  $a \neq 1$ , and  $k$  is any real number, then

$$\log_a x^k = k \log_a x.$$

### EXAMPLE 53

Use the power law of logarithm to simplify the following.

(a)  $\log_3 (4x)^2$

(b)  $3\log_2 8$

**Solution**

(a)  $\log_3 (4x)^2 = 2\log_3 4x$

(b)  $3\log_2 8 = \log_2 8^3$  or  $\log_2 512$

In addition to the above laws of logarithms, there are other important laws of logarithms.

### Six Additional Laws of Logarithm

If  $a$  and  $x$  are positive real numbers, where  $a \neq 1$ , and  $k$  is any real number, then

- |                                       |                       |
|---------------------------------------|-----------------------|
| (i) $\log_a x^k = k \log_a x$         | (iv) $\log_a a^x = x$ |
| (ii) $\log_a \frac{1}{x} = -\log_a x$ | (v) $\log_a a = 1$    |
| (iii) $a^{\log_a x} = x$              | (vi) $\log_a 1 = 0$   |

#### EXAMPLE 54

Evaluate each of the following logarithms.

(a)  $\log_3 27 + \log_3 81$

(b)  $\log_3 2 - \log_3 162$

#### Solution

(a)  $\log_3 27 + \log_3 81 = \log_3 3^4 = 3\log_3 3 + 4\log_3 3 = (3 + 4)\log_3 3 = 7\log_3 3 = 7(1) = 7.$

(b)  $\log_3 2 - \log_3 162 = \log_3 \left( \frac{1}{81} \right) = \log_3 3^{-4} = -4 \log_3 3 = -4(1) = -4.$

### ACTIVITY 7

1. Using one of the laws of logarithm, simplify each of the following.

(a)  $\log_5 x^3 - \log_5 x^2$

(b)  $5\log 2 + 4\log 3 - 3\log 4$

2. Determine the value of the following logarithms.

(a)  $\log_3 27$

(c)  $\log_{125} 25$

(e)  $\log_{64} 512$

(b)  $\log_2 128$

(d)  $\log_{81} 3$

(f)  $\log_{49} 343$

3. Evaluate each of the following

(a)  $\log_5 1$

(c)  $\log_4 4^{12}$

(e)  $16^{\log_4 8}$

(b)  $\log_7 7$

(d)  $b^{\log_b 9}$

(f)  $\log_{25} 5^3$

4. Write the following expressions in terms of  $\log x$ ,  $\log y$  and  $\log z$ .

(a)  $\log x^3 y$

(c)  $\log \frac{x^3 y^2}{z}$

(b)  $\log xyz^2$

(d)  $\log \frac{x}{yz}$

5. Given that  $\log 2 = x$ ,  $\log 3 = y$  and  $\log 5 = z$ , express the following expressions in terms of  $x$ ,  $y$  and  $z$ .

(a)  $\log 12$

(c)  $\log \frac{15}{4}$

(b)  $\log 200$

(d)  $\log 120$

The next law of logarithms, tells us how logarithms to different bases are related.

### Change of Base Law of Logarithm

If  $x$ ,  $a$  and  $b$  are positive real numbers, where  $a \neq 1$  and  $b \neq 1$ , then

$$(i) \quad \log_b x = \frac{\log_a x}{\log_a b}$$

$$(ii) \quad \log_b x = \frac{1}{\log_x b}.$$

#### EXAMPLE 55

Simplify each of the following logarithmic expressions using change of base.

(a)  $\log_{32} 16$

(b)  $\log_{27} 243$

**Solution**

$$(a) \quad \log_{32} 16 = \frac{\log_2 16}{\log_2 32} = \frac{\log_2 2^4}{\log_2 2^5} = \frac{4 \log_2 2}{5 \log_2 2} = \frac{4(1)}{5(1)} = \frac{4}{5}.$$

$$\text{Therefore, } \log_{32} 16 = \frac{4}{5}.$$

$$(b) \quad \log_{27} 243 = \frac{\log_3 243}{\log_3 27} = \frac{\log_3 3^5}{\log_3 3^3} = \frac{5 \log_3 3}{3 \log_3 3} = \frac{5(1)}{3(1)} = \frac{5}{3}.$$

$$\text{Therefore, } \log_{27} 243 = \frac{5}{3}.$$

This law enables us to calculate the logarithm of a number to any base from a calculator which calculates the logarithm of a number to only one base.

#### EXAMPLE 56

If your calculator has only logarithms to base 10 on it, how can you find  $\log_5 9$ ?

**Solution**

By change of base of law of logarithm, we have

$$\log_5 9 = \frac{\log_{10} 9}{\log_{10} 5} = \frac{\log 9}{\log 5}.$$

This last expression can be evaluated by any calculator which can evaluate logarithms to base 10. So, using scientific calculator, we can obtain that  $\log_9 = 0.9542$  and  $\log_5 = 0.6990$ . Thus,

$$\log_5 9 = \frac{\log 9}{\log 5} = \frac{0.9542}{0.6990} \approx 1.3651.$$

Therefore,  $\log_5 9 \approx 1.3651$ .

### EXERCISES

1. Evaluate each of the following expressions.

(a)  $\log_{128} 8$

(d)  $\log_6 \frac{1}{36}$

(b)  $\log_{\frac{1}{27}} 9$

(e)  $\log_{27} \frac{1}{81}$

(c)  $\log_4 \frac{1}{64}$

(f)  $\log_{\frac{1}{2}} \frac{1}{8}$

2. Simplify each of the following expressions.

(a)  $\log_2 (2^x 2^{x+y})$

(b)  $\log_7 49^{uv}$

(c)  $\log_2 8 \times \log_8 256$

(d)  $7 + \log_9 3^{p+q}$

3. Evaluate each of the following logarithms.

(a)  $\log_{12} 27(\log_{10} 4 + \log 10^3) \log_3 10$

(b)  $16^{\log_4 3} + 2^{1-\log_2 4} - 5^{\log_5 49} + \log_{\frac{1}{2}} 8$

(c)  $\log_a a^2 - 4 \log_a \frac{1}{a}$

(d)  $\log_5 75 - \log_5 3 + \log_{27} 81$

4. Simplify each of the following.

(a)  $\log_5 25 + \log_5 125$

(c)  $\log_8 16 + \log_8 52 - \frac{1}{\log_{13} 8}$

(b)  $\log_8 256 - \log_8 32$

(d)  $\log 8 + \log 5 - \log 4$

5. If  $\log 2 = 0.3010$  and  $\log 3 = 0.4771$ , then find each of the following without using a calculator or a logarithm table.
- (a)  $\log 5$  (c)  $\log 30$   
 (b)  $\log 24$  (d)  $\log 36$
6. Given that  $\log 2 = x$ ,  $\log 3 = y$  and  $\log 5 = z$ , express the following expressions in terms of  $x$ ,  $y$ , and  $z$ .
- (a)  $\log 1.5$  (c)  $\log \frac{15}{4}$   
 (b)  $\log 22.5$  (d)  $\log 112.5$
7. Rewrite the logarithmic expression  $\log_a b \times \log_b c$  as single logarithm.
8. Let  $x = 2^p$  and  $y = 4^q$ . Then, express  $\log_2 (x^3 y)$  in terms of  $p$  and  $q$ .
9. Let  $\log 2 = x$  and  $\log 5 = y$ , then find  $\log_{125} 200$  in terms of  $x$  and  $y$ .
10. Using a scientific calculator, find the following logarithms.
- (a)  $\log_3 18$  (c)  $\log_{20} 19$   
 (b)  $\log_8 5$  (d)  $\log_9 25$

Recall that a function of the form  $y = f(x) = a^x$ ;  $a > 0$ ,  $a \neq 1$  is called an exponential function. Its domain is the set of all real numbers. In this sub-unit, we introduce logarithmic functions.

More precisely logarithmic function can be defined as follows:

#### DEFINITION

If  $x$  and  $a$  are positive real numbers such that  $a \neq 1$ , then  $y = f(x) = \log_a x$  is called the logarithmic function with base  $a$ , and  $y = \log_a x$  if and only if  $a^y = x$ .

**Domain** of logarithmic function is the set of all positive real numbers. That is, domain of logarithmic function is  $\{x \in \mathbb{R} : x > 0\}$  and its **range** is the set of all real numbers,  $\mathbb{R}$ .

#### EXAMPLE 57

- (a)  $y = \log_2 x$  is a logarithmic function  
 (b)  $y = \log_3 (x - 2)$  is a logarithmic function

The expression  $y = \log_a x$  is equivalent to  $a^y = x$  and indicates that *the logarithm  $y$  is the exponent to which  $a$  must be raised to obtain  $x$* . The expression  $y = \log_a x$  is called the **logarithmic form** of the equation, and the expression  $a^y = x$  is called the **exponential form** of the equation.

**EXAMPLE 58**

(a)  $\log_2 16$  is an exponent to which 2 must be raised to obtain 16. We can write this as  $2^x = 16$ . Clearly this exponent is 4. Thus  $\log_2 16 = 4$ .

(b)  $\log_{\frac{1}{3}} 27$  is an exponent to which  $\frac{1}{3}$  must be raised to obtain 27. We can write this as  $\left(\frac{1}{3}\right)^x = 27$ . Solving this equation for  $x$ , we get  $3^{-x} = 3^3$  and  $-x = 3$  or  $x = -3$ .

Thus,  $\log_{\frac{1}{3}} 27 = -3$ .

**Note:**  $\log_a 0$  is not defined and if  $x < 0$ , then  $y = \log_a x$  is not defined.

**EXAMPLE 58**

Express each of the following equations in an exponential form.

(a)  $\log_4 64 = 3$

(b)  $\log_{27} 81 = \frac{4}{3}$

**Solution**

From the definition of the logarithmic function, we know  $\log_a x = y$  if and only if  $a^y = x$ . Thus,

(a)  $\log_4 64 = 3$  if and only if  $4^3 = 64$ .

(b)  $\log_{27} 81 = \frac{4}{3}$  if and only if  $27^{\frac{4}{3}} = 81$  because  $27^{\frac{4}{3}} = (3^3)^{\frac{4}{3}} = 3^{3 \times \frac{4}{3}} = 3^4 = 81$ .

**EXAMPLE 58**

Express each of the following equations in a logarithmic form.

(a)  $5^4 = 625$

(b)  $32^{-\frac{3}{5}} = \frac{1}{8}$

**Solution**

From the definition of the logarithmic function, we know  $a^y = x$  if and only if  $\log_a x = y$ . Thus,

(a)  $5^4 = 625$  if and only if  $\log_5 625 = 4$ .

(b)  $32^{-\frac{3}{5}} = \frac{1}{8}$  if and only if  $\log_{32} \frac{1}{8} = -\frac{3}{5}$ .

To study the properties of the graph of  $y = \log_a x$ ;  $a > 1$  and  $y = \log_a x$ ;  $0 < a < 1$ , let us sketch the graphs of  $y = \log_2 x$  and  $y = \log_{\frac{1}{2}} x$ .

To sketch the graph of  $y = \log_2 x$  by converting it to exponential form:

**Group Work 2.5**

- Draw the graphs of  $f(x) = \log_3 x$  and  $f(x) = \log_{\frac{1}{3}} x$  using:
  - Different coordinate systems
  - The same coordinate system.
- Evaluate each of the following.
 

(a) $\log_2 1$	(c) $\log_a 1$ , where $a > 0$ , $a \neq 1$
(b) $\log_{10} 10$	(d) $\log_a a$ , where $a > 0$ , $a \neq 1$
- Find the domain and range of each of the following logarithmic functions.
 

(a) $y = \log_{10} x$	(c) $y = \log_5(x - 1)$
(b) $y = \log_3 2x$	(d) $y = \log_{10}(2 - x)$

Exponential form of  $y = \log_2 x$  is  $x = 2^y$ . Let us consider some ordered pairs which belonging to  $y = \log_2 x$  as listed in the table below.

$x$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16
$y = \log_2 x$	-4	-3	-2	-1	0	1	2	3	4

Using the table of values obtained above, we can plot the points and then connect them with a smooth curve to sketch the graph of  $y = \log_2 x$  as shown in Figure 3.

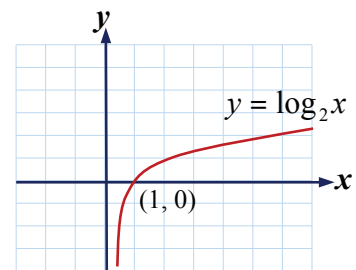


Figure 3.

In a similar manner, let us consider some ordered pairs  $y = \log_{\frac{1}{2}} x$  as listed in the table below.

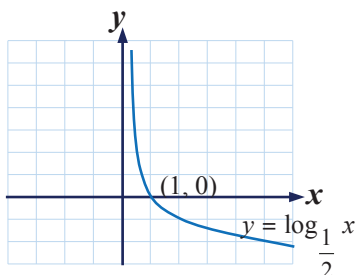


Figure 4.

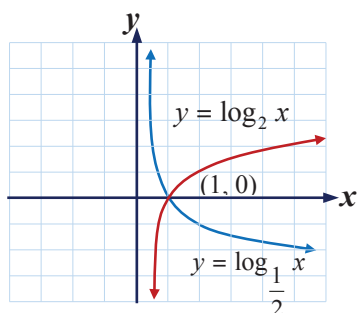


Figure 5.

$x$	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
$y = \log_{\frac{1}{2}} x$	-4	-3	-2	-1	0	1	2	3	4

Using the table of values, we can plot the points and connect them with a smooth curve to sketch the graph of  $y = \log_{\frac{1}{2}} x$  as shown in Figure 4.

If we sketch the graphs of  $y = \log_2 x$  and  $y = \log_{\frac{1}{2}} x$  in the same coordinate system, we can obtain the graph as shown in Figure 5.

The graph of  $y = \log_{\frac{1}{2}} x$  can be obtained from the graph of  $y = \log_2 x$  by reflecting the graph of  $y = \log_2 x$  about the  $x$ -axis. In other words, the graph

of  $y = \log_{\frac{1}{2}} x$  is the mirror image of the graph of  $y = \log_2 x$  and they are symmetric about the  $x$ -axis as shown in Figure 5.

The Activity 8 will help you to summarize the basic properties of the graphs of  $y = \log_2 x$  and  $y = \log_{\frac{1}{2}} x$ .

### ACTIVITY 8

Study the graphs of the logarithmic functions  $y = \log_2 x$  and  $y = \log_{\frac{1}{2}} x$  to answer each of the following questions.

- What is the domain of each function?
- What is the range of each function?
- Find the point where the two graphs intersect.
- For which value of  $x$  is  $y = \log_2 x$  negative? Positive?

- (e) For which value of  $x$  is  $y = \log_{\frac{1}{2}} x$  negative? Positive?
- (f) Does  $y = \log_2 x$  increase as  $x$  increases?
- (g) Do the graphs cross or touch the  $y$ -axis?
- (h) Is the graph of each function smooth curve and continuous?

From the above discussions, we can deduce that a logarithmic function defined by  $y = \log_a x$  with  $a > 1$ , has similar properties with the graph of  $y = \log_2 x$ . Observe Figure 6.

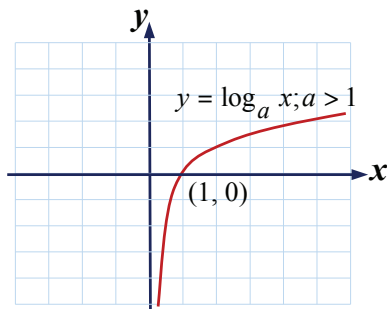


Figure 6.

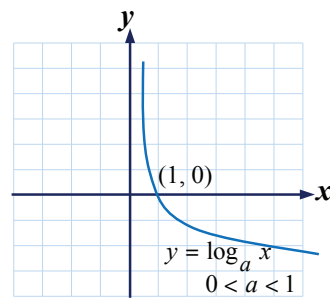


Figure 7.

Similarly, if  $0 < a < 1$  the graph of  $y = \log_a x$  has similar property with the graph of  $y = \log_{\frac{1}{2}} x$ .

Therefore, graph of  $y = \log_a x$ , for  $0 < a < 1$  behaves as in the case of  $y = \log_{\frac{1}{2}} x$  as shown in Figure 7.

Thus, the graph of the logarithmic function

$g(x) = \log_{\frac{1}{2}} x$  can be obtained from the graph of the logarithmic function  $f(x) = \log_a x$  by reflecting the

graph of the logarithmic function  $f(x) = \log_a x$  about the  $x$ -axis. In other words, the graph of  $g(x) = \log_{\frac{1}{2}} x$  is the mirror image of the graph of  $f(x) = \log_a x$  and they are symmetric about the  $x$ -axis as shown in Figure 8.

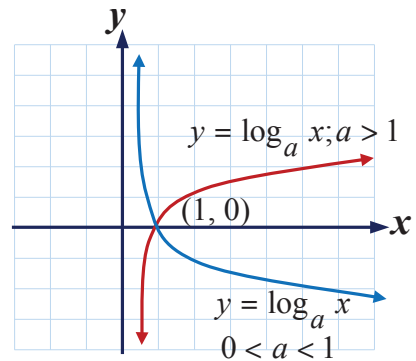


Figure 8.

**Basic Properties of the Logarithmic Functions**  $f(x) = \log_b x$ ;  $b > 0$  and  $b \neq 1$ 

1. The domain is the set of all positive real numbers,  $\{x \in \mathbb{R} : x > 0\}$ .
2. The range is the set of all real numbers,  $\mathbb{R}$ .
3. The graph intersects the  $x$ -axis at the point  $(1, 0)$ . Thus, the  $x$ -intercept is  $(1, 0)$ .
4. The graph never cross or touch the  $y$ -axis. Thus, there are no  $y$ -intercepts.
5. The graph is smooth curves and continuous with no corners and gaps.
6. If  $b > 1$ , then the logarithmic function is an increasing function.
7. If  $0 < b < 1$ , then the logarithmic function  $f$  is a decreasing function.
8. (i) If  $0 < x < 1$ , then  $y = \log_2 x < 0$  and  $y = -\log_2 x > 0$ .  
(ii) If  $x > 1$ , then  $y = \log_2 x > 0$  and  $y = -\log_2 x < 0$ .

**Logarithmic equations**

An equation is called logarithmic equation, if the variable is part of a number whose logarithm is to be taken.

In this section, we will see how to give solutions of equations which involve logarithms by applying the properties studied.

In solving logarithmic equation, note that if  $\log_a x = 0$ , then  $x = 1$ . In solving any log equation you note that  $x > 0$ .

**EXAMPLE 58**

Find the truth set of  $\log(x - 7) = 0$

**Solution**

$$\log(x - 7) = 0$$

Here  $x - 7 > 0$  or  $x > 7$ . The universe is  $\{x : x > 7\}$ .

$$\log(x - 7) = 0 \text{ implies } x - 7 = 1 \text{ or } x = 8.$$

Therefore, S.S =  $\{8\}$ .

**EXAMPLE 58**

$$\text{Solve } \log(2x - 7) = 0, \text{ domain} = \left\{x : x > \frac{7}{2}\right\}.$$

**Solution**

$$2x - 7 = 1$$

$$2x = 8$$

$$x = 4$$

Therefore, S.S = {4}.

**EXAMPLE 58**

Find the truth set of  $\log_2(29x - 5) = \log_2(x - 7) + 5$

**Solution**

$$\log_2(29x - 5) - \log_2(x - 7) - \log_2 32 = 0 \quad \dots \text{ use } \log_2 32 = 5$$

$$\log_2 \frac{(29x - 5)}{(x - 7)(32)} = 0$$

$$\frac{29x - 5}{(x - 7)(32)} = 1$$

$$32x - 224 = 29x - 5$$

$$32x - 29x = -5 + 224$$

$$3x = 219$$

$$x = \frac{219}{3} = 73$$

Therefore, S.S = {73}.

**EXERCISES**

1. Express each of the following equations in an exponential form.

(a)  $\log_7 49 = 2$

(c)  $\log_9 243 = \frac{5}{2}$

(b)  $\log_2 128 = 7$

(d)  $\log_{64} 512 = \frac{3}{2}$

2. Express each of the following equations in an logarithmic form.

(a)  $7^3 = 343$

(c)  $3^6 = 729$

(b)  $64^{\frac{2}{3}} = 16$

(d)  $81^{-\frac{3}{4}} = \frac{1}{27}$

3. Evaluate each of the following.
- |                           |                        |
|---------------------------|------------------------|
| (a) $\log_{100} 1$        | (d) $\log_{81} 27$     |
| (b) $\lg_{10} 1000$       | (e) $\log_{32} 256$    |
| (c) $\log_9 \frac{1}{27}$ | (f) $\log_{100} 10000$ |
4. Find the domain and range of each of the following logarithmic functions.
- |                                |                             |
|--------------------------------|-----------------------------|
| (a) $y = \log_4 x$             | (c) $y = \log_3 (x - 3)$    |
| (b) $y = \log_{\frac{3}{2}} x$ | (d) $y = \log_{10} (1 - x)$ |
5. Draw the graphs of the logarithmic functions  $y = \log_5 x$  and  $y = \log_{\frac{1}{5}} x$  using the same coordinate system.
- What is the domain and range of each function?
  - Find the point where each function crosses the  $x$ -axis.
  - Which function is increasing and which is decreasing?
  - For which value(s) of  $x$  is  $y = \log_5 x$  negative? Positive?
  - For which value(s) of  $x$  is  $y = \log_{\frac{1}{5}} x$  negative? Positive?
6. Solve for  $x$ .
- |                       |                         |
|-----------------------|-------------------------|
| (a) $\log_{10} x = 1$ | (c) $\log_8 x = 3$      |
| (b) $\log_2 x = 4$    | (d) $\log_{32} 128 = x$ |

In general, when we write a number  $N$  in the scientific notation of the form  $a \times 10^n$  the “ $a$ ” part is a number greater or equal to 1 but less than 10 and hence its logarithm can be read from a common logarithm table. This logarithm is a positive decimal and is called the ***mantissa*** of the logarithm of the number. The factor,  $10^n$  is an integral power of 10 and hence its logarithm is its exponent,  $n$ . This integral value of logarithm of the number is called the ***characteristic*** of the logarithm. Therefore, the common logarithm of a number is equal to the ***characteristic*** plus the ***mantissa***.

The common logarithm table is a table of mantissas. The mantissa can be given to any number of decimal place, but in most of the logarithm tables it is often given only to *four decimal places*.

In a logarithm table the first column headed by  $N$  lists the numbers from 1.0 to 9.9. The remaining columns are headed respectively by the digits from 0 to 9 as shown in the table below.

N	0	1	2	3	4	5	6	7	8	9
1.0	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1.9	2788	2810	2833	2856	2878	<b>2900</b>	2923	2945	2967	2989
2.0	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2.9	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
3.0	4771	4786	4814	4800	4829	4843	4857	4871	4886	4900
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
3.9	5911	5922	5933	5944	<b>5955</b>	5966	5977	5988	5999	6010
4.0	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
4.9	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
5.0	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
5.9	7709	7716	7723	7731	7738	7745	7752	7760	<b>7767</b>	7774
6.0	7782	7789	7796	7803	7810	7818	7825	7832	7839	<b>7846</b>
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
9.9	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

The base ten logarithm of a number is the power to which 10 is raised to give that number.

### EXAMPLE 58

Find the common logarithm of each of the following numbers using common logarithm table.

- (a) 19,500                      (b) 598,000                      (c) 60.90                      (d) 3,940,000

#### Solution

- (a) First, express 19,500 in scientific notation. That is,  $19500 = 1.95 \times 10^4$ .

Now, let us find the row which starts with 1.9 and find the column headed by 5. Thus, we read 2900 at the intersection of the row which starts with 1.9 and the column headed by 5 as shown in the table given above. Thus,  $\log 1.95 = 0.2900$ .

$19500 = 1.95 \times 10^4 \dots$  (Expressing the given number in scientific notation).

$$\begin{aligned}
 &= 10^{0.29} \times 10^4 \dots\dots \text{(From common logarithm table, we obtain} \\
 &\quad \log 1.95 = 0.2900)\text{.} \\
 &= 10^{0.29+4} \dots\dots\dots \text{(By product property of exponents).} \\
 &= 10^{4.29}\text{.}
 \end{aligned}$$

Therefore,  $\log 19500 = 4.29$

Here, 4 is the characteristics and 0.29 is the mantisa of the logarithm.

- (b) First, express 598,000 in scientific notation. That is,  $598,000 = 5.98 \times 10^5$ .

Now, let us find the row which starts with 5.9 and find the column headed by 8. Thus, we read 7767 at the intersection of the row which starts with 5.9 and the column headed by 8 as shown in the table given above. Thus,  $\log 5.98 = 0.7767$ .

$$\begin{aligned}
 598,000 &= 5.98 \times 10^5 \dots \text{(Expressing the given number in scientific notation).} \\
 &= 10^{0.7767} \times 10^5 \dots\dots\dots \text{(From common logarithm table, we obtain} \\
 &\quad \log 5.98 = 0.7767)\text{.} \\
 &= 10^{0.7767+5} \dots\dots\dots \text{(By product property of exponents).} \\
 &= 10^{5.7767}\text{.}
 \end{aligned}$$

Therefore,  $\log 598.000 = 5.7767$ .

Here, 5 is the characteristics and 0.7767 is the mantissa of the logarithm.

- (c) First, express 60.90 in scientific notation. That is,  $60.90 = 6.09 \times 10^1$ .

Now, let us find the row which starts with 6.0 and find the column headed by 9. Thus, we read 7846 at the intersection of the row starts with 6.0 and the column headed by 9 as shown in the table given above. Thus,  $\log 6.09 = 0.7846$ .

$$\begin{aligned}
 60.90 &= 6.09 \times 10^1 \dots\dots \text{(Expressing the given number in scientific notation).} \\
 &= 10^{0.7846} \times 10^1 \dots\dots\dots \text{(From common logarithm table, we obtain} \\
 &\quad \log 6.09 = 0.7846)\text{.} \\
 &= 10^{0.7846+1} \dots\dots\dots \text{(By product property of exponents).} \\
 &= 10^{1.7846}\text{.}
 \end{aligned}$$

Therefore,  $\log 60.90 = 1.7846$ .

Here, 1 is the characteristics and 0.7746 is the mantissa of the logarithm.

- (d) First, express 3,940,000 in scientific notation. That is,  $3,940,000 = 3.94 \times 10^6$ .

Now, let us find the row which starts with 3.9 and find the column headed by 4. Thus, we read 5955 at the intersection of the row which starts with 3.9 and the column headed by 4 as shown in the table given above. Thus,  $\log 3.94 = 0.5955$ .

$$3,940,000 = 3.94 \times 10^6 \dots \text{(Expressing the given number in scientific notation).}$$

$$\begin{aligned}
 &= 10^{0.5955} \times 10^6 \dots\dots \text{(From common logarithm table, we obtain} \\
 &\quad \log 3.94 = 0.5955). \\
 &= 10^{0.5955+6} \dots\dots\dots \text{(By product property of exponents).} \\
 &= 10^{6.5955}.
 \end{aligned}$$

Therefore,  $\log 3,940,000 = 6.5955$ .

Here, 6 is the characteristics and 0.5955 is the mantissa of the logarithm.

### EXERCISES

- Find the common logarithm of each of the following numbers using common logarithm table.
 

(a) 19.50	(d) 89,400
(b) 1090	(e) 959,000
(c) 9,080	(f) 6,790,000
- Write the mantissa and characteristic after finding the logarithm of each of the following numbers.
 

(a) 75	(d) 405,900
(b) 890	(e) 955,400
(c) 95,000	(f) 2,785,000

To evaluate and simplify logarithmic expressions we can always use the properties discussed in section 2.8

### EXAMPLE 58

The following statements describe the relationship between logarithmic form an exponential form.

(a) $25^{\frac{1}{2}} = 5 \Leftrightarrow \log_{25} 5 = \frac{1}{2}$	(c) $\sqrt[4]{625} = 5 \Leftrightarrow \log_{625} 5 = \frac{1}{4}$
(b) $\left(\frac{4}{9}\right)^{\frac{1}{2}} = \frac{2}{3} \Leftrightarrow \log_{\frac{4}{9}} \frac{2}{3} = \frac{1}{2}$	(d) $27^{-\frac{1}{3}} = \frac{1}{3} \Leftrightarrow \log_{27} \frac{1}{3} = \frac{-1}{3}$

**EXAMPLE 58**

Evaluate each of the following logarithms

(a)  $\log_{\frac{4}{9}} \frac{27}{8}$

(b)  $\log_7 \sqrt[3]{49}$

(c)  $\log_{\frac{1}{8}} \sqrt[3]{\frac{1}{32}}$

**Solution**

$$\begin{aligned} \text{(a) Let } \log_{\frac{4}{9}} \frac{27}{8} = x &\Leftrightarrow \left(\frac{4}{9}\right)^x = \frac{27}{8} \Leftrightarrow \left(\frac{9}{4}\right)^{-x} = \left(\frac{27}{8}\right) \\ &\Leftrightarrow \left(\frac{3}{2}\right)^{-2x} = \left(\frac{3}{2}\right)^3 \Leftrightarrow -2x = 3 \Leftrightarrow x = -\frac{3}{2} \end{aligned}$$

$$\text{Therefore, } \log_{\frac{4}{9}} \frac{27}{8} = -\frac{3}{2}.$$

$$\text{(b) Let } \log_7 \sqrt[3]{49} = y \Leftrightarrow 7^y = \sqrt[3]{49} \Leftrightarrow 7^y = 49^{\frac{1}{3}} \Leftrightarrow 7^y = 7^{\frac{2}{3}} \Leftrightarrow y = \frac{2}{3}$$

$$\text{Therefore, } \log_7 \sqrt[3]{49} = \frac{2}{3}.$$

$$\begin{aligned} \text{(c) Let } \log_{\frac{1}{8}} \sqrt[3]{\frac{1}{32}} = y &\Leftrightarrow \left(\frac{1}{8}\right)^y = \left(\frac{1}{32}\right)^{\frac{1}{3}} \Leftrightarrow (2^{-3})^y = (2^{-5})^{\frac{1}{3}} \\ &\Leftrightarrow 2^{-3y} = 2^{-\frac{5}{3}} \Leftrightarrow -3y = -\frac{5}{3} \Leftrightarrow y = \frac{5}{9} \end{aligned}$$

$$\text{Therefore, } \log_{\frac{1}{8}} \sqrt[3]{\frac{1}{32}} = \frac{5}{9}.$$

As we have seen from the definition of logarithmic function, it is the inverse of exponential function. Thus, the basic laws of exponents have their counter parts for logarithms. A summary of these laws was given in section 2.8:

**EXERCISES**

1. Evaluate each of the following by using the above properties.

(a)  $\log_2 (16 \times 32)$

(b)  $2 \log_2 8$

(c)  $10\log_{10} 100$

(d)  $\log_2 \left( \frac{1}{32} \right)$

(e)  $\log_2 \left( \frac{128}{8} \right)$

(f)  $\log_4 32$

(g)  $\log_{\frac{1}{2}} 8$

(h)  $\log_2 4 \times \log_4 2$

(i)  $\log_{2^3} 8$

The **common logarithm** of a number  $x$  is the power or exponent to which “10” has to be raised to be equal to  $x$ .

Logarithms to base 10 are called **common logarithms**.

In the expression  $10^x$  the exponent  $x$  can be negative. For example,  $\frac{1}{10}$  and  $\frac{1}{100}$  are the same as  $10^{-1}$  and  $10^{-2}$ , respectively. Their logarithms are the exponents  $-1$  and  $-2$ .

### ACTIVITY 9

- If the base of the logarithm is 10, then:
  - What is the logarithm of 100,000?
  - What is the logarithm of  $10^7$ ?
- When the base changes from 10 to  $a$ , then:
  - What is the logarithm of 1?
  - What is the logarithm of  $a^n$ ?
- Compute the following common logarithms.
  - $\log 10$
  - $\log 1$
  - $\log \frac{1}{100}$
  - $\log 0.00001$

### EXAMPLE 58

- $\log_{10} 1000 = 3$  is equivalent to  $1000 = 10^3$ , where 10 is the base, 3, is the exponent and 1000 is the number.
- $\log_{10} \frac{1}{1000} = -3$  is equivalent to  $\frac{1}{1000} = 10^{-3}$ , where 10 is the base,  $-3$  is the exponent and  $\frac{1}{1000}$  is the number.

A common logarithm is usually written without indicating its base, 10. That is,  $\log_{10} x$  is simply denoted by  $\log x$ . For example,  $\log_{10} 19$  can be written as  $\log 19$  and  $\log_{10} 100$  can be written as  $\log 100$ .

**DEFINITION**

For any  $x > 0$ , the common logarithm of  $x$  is defined as logarithm of  $x$  to the base 10. That is,  $\log_{10} x$  or  $\log x$ .

**EXAMPLE 58**

Find the following common logarithms.

(a)  $\log 10,000$

(c)  $\log 0.00001$

(b)  $\log 1,000,000$

(d)  $\log 0.000001$

**Solution**

(a)  $\log_{10} 10,000 = \log 10,000 = 4$ , because  $10^4 = 10,000$ .

(b)  $\log_{10} 1,000,000 = \log 1,000,000 = 6$ , because  $10^6 = 1,000,000$ .

(c)  $\log_{10} 0.00001 = \log 0.00001 = -5$ , because  $0.00001 = \frac{1}{100000} = \frac{1}{10^5} = 10^{-5}$ .

(d)  $\log_{10} 0.000001 = \log 0.000001 = -6$ , because  $0.000001 = \frac{1}{1000000} = \frac{1}{10^6} = 10^{-6}$ .

**EXERCISES**

1. Write each of the following numbers in an exponential form with base 10.

(a) 10,000

(c)  $\frac{1}{1,000,000}$

(b) 10,000,000

(d) 0.00000001

2. Find the logarithm of each of the following numbers.

(a) 10,000,000

(c) 0.0000001

(b)  $10^8$

(d)  $10^{-7}$

3. Compute the following common logarithms.

(a)  $\log 0.1$

(b)  $\log 10000000$

(c)  $\log \frac{1}{10000} \log 0.0000001$

4. Compute the following common logarithms.

- |                              |  |
|------------------------------|--|
| (a) $\log(100 \times 1000)$  | (e) $\log\left(\frac{1}{10} \times \frac{1}{100}\right)$ |
| (b) $\log 100 + \log 1000$   | (f) $\log \frac{1}{10} + \log \frac{1}{100}$             |
| (c) $\log(10^5 \times 10^7)$ | (g) $\log(0.001 \times 0.00001)$                         |
| (d) $\log 10^5 + \log 10^7$  | (h) $\log 0.001 + \log 0.00001$                          |

5. Use the definition of the logarithmic function to find the value of  $x$ .

- |                        |                         |
|------------------------|-------------------------|
| (a) $\log_{10} x = 3$  | (c) $\log_x 1000 = 3$   |
| (b) $\log_{10} x = -5$ | (d) $\log_x 10000 = -4$ |

## Using scientific calculator

The “Log” function on a scientific calculator is a key that allows us to work with logarithms. Logarithms are ways to figure out what exponents we need to multiply into a specific number.

For example, using the “Log” function on the number 100 would reveal that we have to multiply the base number of 10 by itself two times to be equal to the number 100.

The following steps are important to find the logarithm of a number using scientific calculator.

**Step 1 :** Type the number into the scientific calculator.

**Step 2:** Press the “Log” button on the calculator. The number immediately seen on the screen is the exponent for the original number we entered.

Assuming that the base number is 10, to find the given original number we have to multiply 10 by itself the number of times we see on the screen.



### EXAMPLE 58

Find the logarithm of 10,000 using the scientific calculator.

#### Solution

**Step 1:** Type the number 10,000 into scientific calculator.

**Step 2:** Press the “Log” button on the calculator.

So, pressing “Log” will result in the number “4” appearing on the screen.

To check it, perform the product:  $10^4 = 10 \times 10 \times 10 \times 10 = 10000$ .

**EXAMPLE 58**

Find the common logarithm of each of the following numbers using scientific calculator.

- (a) 190                      (b) 97.50                      (c) 6,470                      (d) 980,000

**Solution**

$$\begin{aligned} \text{(a) } 190 &= 1.9 \times 10^2 \dots\dots \text{ (Expressing the given number in scientific notation).} \\ &= 10^{0.2788} \times 10^2 \dots\dots \text{ (Type 1.9 into the calculator and press "Log",} \\ &\quad \text{you obtain 0.2788).} \\ &= 10^{0.2788+2} \dots\dots \text{ (By product property of exponents).} \\ &= 10^{2.2788} \end{aligned}$$

Therefore,  $\log 190 = 2.2788$ .

$$\begin{aligned} \text{(b) } 97.50 &= 9.75 \times 10^1 \dots\dots \text{ (Expressing the given number in scientific notation).} \\ &= 10^{0.9890} \times 10^1 \dots\dots \text{ (Type 9.75 into the calculator and press} \\ &\quad \text{"Log", you obtain 0.9890).} \\ &= 10^{0.9890+1} \dots\dots \text{ (By product property of exponents).} \\ &= 10^{1.9890} \end{aligned}$$

Therefore,  $\log 97.50 = 1.9890$ .

$$\begin{aligned} \text{(c) } 6470 &= 6.47 \times 10^3 \dots\dots \text{ (Expressing the given number in scientific notation).} \\ &= 10^{0.8109} \times 10^3 \dots\dots \text{ (Type 6.47 into the calculator and press} \\ &\quad \text{"Log", you obtain 0.8109).} \\ &= 10^{0.8109+3} \dots\dots \text{ (By product property of exponents).} \\ &= 10^{3.8109} \end{aligned}$$

Therefore,  $\log 6470 = 3.8109$ .

$$\begin{aligned} \text{(d) } 980,000 &= 9.8 \times 10^5 \dots\dots \text{ (Expressing the given number in scientific notation).} \\ &= 10^{0.9912} \times 10^5 \dots\dots \text{ (Type 9.8 into the calculator and press} \\ &\quad \text{"Log", you obtain 0.9912).} \\ &= 10^{0.9912+5} \dots\dots \text{ (By product property of exponents).} \\ &= 10^{5.9912} \end{aligned}$$

Therefore,  $\log 980,000 = 5.9912$ .

**EXERCISES**

1. Express each of the following numbers in scientific notation.

- (a) 259                      (c) 405.90                      (e) 829,000  
 (b) 5.06                      (d) 87,950                      (f) 2,095,000

2. Use common table of logarithm to compute the following.
- (a)  $\log 2.49$                       (c)  $\log 50$                       (e)  $\log 65$   
 (b)  $\log 1.27$                       (d)  $\log 29.5$                       (f)  $\log 729$
3. Find the common logarithm of each of the following numbers using scientific calculator.
- (a) 2.85                                      (d) 75,800  
 (b) 79.50                                      (e) 247,000  
 (c) 3,950                                      (f) 1,002,000

### ACTIVITY 10

1. How does scientific notation differ from standard notation?
2. Write each of the following numbers in scientific notation.
- (a) 607.8                      (b) 40500                      (c) 768000                      (d) 9,032,000
3. Write each of the following numbers in scientific notation.
- (a) 8329                                      (c)  $2.5 \times 10^3 + 4.8 \times 10^2$   
 (b)  $3.5 \times 10^4 \times 4.8 \times 10^5$                       (d)  $9.45 \times 10^5 - 3.2 \times 10^3$
4. Find common logarithm of each of the following numbers.
- (a) 25                                      (c) 2,950  
 (b) 109                                      (d) 590,000

Scientific notation is a way of expressing any number, especially a very large or a very small number, in a concise and convenient fashion using powers of 10.

A way of expressing numbers  $N$  in the form  $N = a \times 10^n$ , where  $1 < a < 10$  and  $n$  is an integer, is said to be scientific notation.

### EXAMPLE 58

Express each of the following numbers in scientific notation.

- (a) 8,560                                      (c) 56,109  
 (b) 489.5                                      (d) 1,725,000

#### Solution

- (a)  $8,560 = 8.56 \times 10^3$                       (c)  $56,109 = 5.6109 \times 10^4$   
 (b)  $489.5 = 4.895 \times 10^2$                       (d)  $1,725,00 = 1.725 \times 10^6$

Any number  $N$  between 0 and 1 can be expressed in the form  $N = a \times 10^n$ , where  $1 < a < 10$  and  $n$  is a negative integer. In other words, any number  $N$  between 0 and 1 can be expressed in the form

$$N = a \times 10^{-n}, \text{ where } 1 < a < 10.$$

### ACTIVITY 11

- Identify the numbers that are greater than 1 and the numbers that are between 0 and 1.
  - 0.99
  - 1.0001
  - 12.09
  - 0.00009
  - 0.90099
  - 27.0009
- Write each of the following numbers in scientific notation.
  - 0.025
  - 0.00890
  - 0.00005
  - 0.9032
  - 0.0000095
  - 0.10200000
- Find the common logarithm of each of the following numbers.
  - 0.1
  - 0.105
  - 0.00295
  - 0.000099
- Consider the following table and study the relationship between the number of zeros to the right side of the decimal point and the characteristic of the logarithm.

Number	Scientific notation	Characteristic
0.7285	$7.285 \times 10^{-1}$	-1
0.07285	$7.285 \times 10^{-2}$	-2
0.007285	$7.285 \times 10^{-3}$	-3
0.0007285	$7.285 \times 10^{-4}$	-4
0.00007285	$7.285 \times 10^{-5}$	5

- Write the characteristic of the logarithm of each of the following numbers.
  - 0.54
  - 0.1045
  - 0.00030
  - 0.0000205

The characteristic of the logarithm of a number between 0 and 1 is negative.

### EXAMPLE 58

Find the common logarithm of 0.000457.

#### Solution

Since the number of zeros on the right hand-side of the decimal point is 3, the characteristic is -4. Its mantissa is obtained in the table of common logarithm as 0.6599. Therefore,

the characteristic and the mantissa of the logarithm of 0.000457 are  $-4$  and  $0.6599$  respectively.

So, we have a logarithm whose integral part is negative and the decimal part is positive. Therefore,  $\log 0.000457 = -3 + 0.6599 = -2.3401$ .

### EXAMPLE 58

Find the logarithm of each of the following numbers using table of common logarithm.

(a) 0.60

(b) 0.0197

#### Solution

(a) First, express 0.60 in scientific notation. That is,  $0.60 = 6 \times 10^{-1}$ .

Now, let us find the row which starts with 6.0 and find the column headed by 0. Thus, we read 7782 at the intersection of the row which starts with 6.0 and the column headed by 0 as shown in the table given above. Thus,  $\log 6.0 = 0.7782$ .

$$\begin{aligned} 0.60 &= 6.0 \times 10^{-1} \dots\dots\dots \text{(Expressing the given number in scientific notation).} \\ &= 10^{0.7782} \times 10^{-1} \dots\dots\dots \text{(From common logarithm table, we obtain} \\ &\quad \log 6.0 = 0.7782\text{).} \\ &= 10^{0.7782+(-1)} \dots\dots\dots \text{(By product property of exponents).} \\ &= 10^{-0.2218} \end{aligned}$$

Therefore,  $\log 0.60 = -0.2218$ .

$$\begin{aligned} \text{Since } \log 0.60 &= (-1) + 1 - 0.2218 \\ &= (-1) + 0.7782, \end{aligned}$$

$-1$  is the characteristics and  $0.7782$  is the mantissa of the logarithm.

(b) First, express 0.0197 in scientific notation. That is,  $0.0197 = 1.97 \times 10^{-2}$ .

Now, let us find the row which starts with 1.9 and find the column headed by 7. Thus, we read 2945 at the intersection of the row which starts with 1.9 and the column headed by 7 as shown in the table given above. Thus,  $\log 1.97 = 0.2945$ .

$$\begin{aligned} 0.0197 &= 1.97 \times 10^{-2} \dots \text{(Expressing the given number in scientific notation).} \\ &= 10^{0.2945} \times 10^{-2} \dots\dots\dots \text{(From common logarithm table, we obtain} \\ &\quad \log 1.97 = 0.2945\text{).} \\ &= 10^{0.2945+(-2)} \dots\dots\dots \text{(By product property of exponents).} \\ &= 10^{-1.7055} \end{aligned}$$

Therefore,  $\log 0.0197 = -1.7055$ .

$\log 0.0197 = (-2) + 2 - 1.0.7055 = (-2) + 0.2945$ . So,  $-2$  is the characteristics and  $0.2945$  is the mantissa of the logarithm.

**EXAMPLE 58**

Find the common logarithm of each of the following numbers using scientific calculator.

(a) 0.00985

(b) 0.000029

**Solution**

$$\begin{aligned} \text{(a) } 0.00985 &= 9.85 \times 10^{-3} \dots \text{ (Expressing the given number in scientific notation).} \\ &= 10^{0.9934} \times 10^{-3} \quad \text{(Type 9.85 into the calculator and press “Log”,} \\ &\quad \text{you obtain 0.9934).} \\ &= 10^{0.9934+(-3)} \dots \text{ (By product property of exponents).} \\ &= 10^{-2.0066} \end{aligned}$$

Therefore,  $\log 0.00985 = -2.0066$ .

$$\begin{aligned} \text{(b) } 0.000029 &= 2.9 \times 10^{-5} \dots \text{ (Expressing the given number in scientific notation).} \\ &= 10^{0.4624} \times 10^{-5} \text{ (Type 2.7 into the calculator and press “Log”,} \\ &\quad \text{you obtain 0.4624).} \\ &= 10^{0.4624+(-5)} \dots \text{ (By product property of exponents).} \\ &= 10^{-4.5376} \end{aligned}$$

Therefore,  $\log 0.000029 = -4.5376$ .

**EXERCISES**

1. Express each of the following numbers in scientific notation.

(a) 0.25

(f) 0.0000012

(b) 4.95

(g)  $2.8 \times 10^3 \times 3.4 \times 10^{-8}$

(c) 209.50

(h)  $1.528 \times 10^2 + 4.5 \times 10^{-1}$

(d) 0.007950

(i)  $7.25 \times 10^{-3} - 4.59 \times 10^{-3}$

(e) 0.0008295

2. Use common table of logarithm to compute each of the following.

(a)  $\log 0.45$

(d)  $\log 0.00275$

(b)  $\log 0.029$

(e)  $\log 0.000093$

(c)  $\log 0.80$

(f)  $\log 0.729$

3. Find the common logarithm of each of the following numbers using common logarithm table.

(a) 205

(c) 0.25

(e) 0.000405

(b) 7.5

(d) 19

(f) 0.125

- (g) 0.075                      (i) 0.002090                      (k) 0.0959  
 (h) 0.1090                      (j) 0.000450                      (l) 0.000068

4. Find the common logarithm of each of the following numbers using scientific calculator.

- (a) 0.395                      (c) 0.00895                      (e) 0.000054  
 (b) 0.0795                      (d) 0.0005                      (f) 0.00000729

5. Find the mantissa and characteristic of the logarithm of the following numbers.

- (a) 0.45                      (c) 0.0099                      (e) 0.000955  
 (b) 0.09                      (d) 0.000405                      (f) 0.0000785

## KEY TERMS

- Indices
- Exponential growth
- Negative powers
- Rational power
- Logarithmic function
- Logarithm table
- Base
- Power
- Mantissa
- Index
- Characteristics
- Common Logarithm
- Properties of indices
- Population growth
- Bacteria growth
- Laws of logarithms
- Exponent
- Scientific notation
- Compound interest

## SUMMARY

- If  $a$  is any real number and  $n$  is any natural number, then the  $n^{\text{th}}$  power of  $a$ , denoted by  $a^n$ , read as “ $a$  to the power of  $n$ ” is defined as:

$$a^n = \underbrace{a \times a \times a \times \dots \times a}_{n \text{ factors each equal to } a}$$

- In the expression  $a^n$ ,  $a$  is called the base,  $n$  is called the index or exponent and  $a^n$  is the power.
- The growth of population,  $P$ , at the end of  $t$  years with growth rate  $r$  can be given by the formula:

$$P = P_0(1 + r)^t,$$

where  $P_0$  is the original number of populations.

- If a principal  $P$  is invested at an annual rate of  $r$ , compounded annually, then the amount after  $t$  years is given by

$$A = P(1 + r)^t$$

- An exponential growth or decay function is a function that grows or shrinks at a constant percent growth rate. The equation can be written in the form

$$y = ab^x,$$

where,  $a$  is the initial or starting value,  $b$  is the growth factor and  $x$  is the number of time intervals that have passed.

- Any nonzero number raised to a negative exponent is the reciprocal of the same power with positive exponent. That is, for  $a \neq 0$  and  $n > 0$ ,  $a^{-n} = \frac{1}{a^n}$ .

(i) For  $a \neq 0$ ,  $a^{-1} = \frac{1}{a}$ .

- (ii) If  $a$  and  $b$  are nonzero real numbers, then it is always true that for  $n > 0$ .

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

- The Product Property of indices

- (i) For any real number  $a$  and natural numbers  $m$  and  $n$ , the product property of indices states that

$$a^m \times a^n = a^{m+n}$$

- (ii) For any real number  $a$  and natural numbers  $m$  and  $n$  such that  $m > n$ , the quotient property of indices states that

$$\frac{a^m}{a^n} = a^{m-n}$$

- (iii) For any real number  $a$  and natural numbers  $m$  and  $n$ , the power property of indices states that

$$(a^m)^n = a^{m \cdot n}$$

- (iv) For any real numbers  $a$  and  $b$  and a natural number  $n$ , the power of a product property of indices states that

$$(ab)^n = a^n b^n$$

- (v) For any real numbers  $a$  and  $b$  and a natural number  $n$ , the power of a quotient property of indices states that

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

- (vi) For any nonzero real numbers  $a$ , the zero-index property of indices states that

$$a^0 = 1.$$

- If  $n$  is a positive even integer and  $a$  is positive, then  $a^{\frac{1}{n}}$  denotes the positive real  $n^{\text{th}}$  root of  $a$  and is called the principal  $n^{\text{th}}$  root of  $a$ . If  $n$  is a positive odd integer and  $a$  is any real number, then  $a^{\frac{1}{n}}$  denotes the real  $n^{\text{th}}$  root of  $a$ .
- If  $m$  and  $n$  are positive integers, then

$$a^{\frac{m}{n}} = \left( a^{\frac{1}{n}} \right)^m$$

provided that  $a^{\frac{1}{n}}$  is a real number.

- $a^{\frac{m}{n}}$  is not real when  $a$  is negative and  $n$  is even. According to the definition of rational power, expressions such as  $(-16)^{\frac{3}{2}}$ ,  $(-29)^{\frac{3}{4}}$ ,  $(-64)^{\frac{5}{8}}$  and  $(-1)^{\frac{2}{2}}$  are not defined because each of them involves an even root of a negative number.
  - For a fixed positive number  $b \neq 1$  and for every real number  $a > 0$ ,  $c = \log_b a$  if and only if  $b^c = a$
- If  $x$  and  $a$  are positive real numbers such that  $a \neq 1$ , then  $y = f(x) = \log_a x$  is called the logarithmic function with base  $a$ , and  $y = \log_a x$  if and only if  $a^y = x$ .
  - Basic Properties of the Logarithmic Functions  $f(x) = \log_b x$ ;  $b > 0$  and  $b \neq 1$ 
    - The domain is the set of all positive real numbers.,  $\{x \in \mathbb{R} : x > 0\}$ .
    - The range is the set of all real numbers,  $\mathbb{R}$ .
    - The graph intersects the  $x$ -axis at a point  $(1, 0)$ . Thus, the  $x$ -intercept is  $(1, 0)$ .
    - The graph never cross or touch the  $y$ -axis. Thus, there are no  $y$ -intercepts.
    - The graph is smooth curves and continuous with no corners and gaps.
    - If  $a > 1$ , then the logarithmic function is an increasing function.
    - If  $0 < a < 1$ , then the logarithmic function  $f$  is a decreasing function.

(viii) (a) If  $0 < x < 1$ , then  $y = \log_2 x < 0$  and  $y = -\log_2 x > 0$ .

(b) If  $x > 1$ , then  $y = \log_2 x > 0$  and  $y = -\log_2 x < 0$ .

- For any  $x > 0$ , the common logarithm of  $x$  is defined as logarithm of  $x$  to the base 10. That is,  $\log_{10} x$  or  $\log x$ .
- The common logarithm of a number  $x$  is the power or exponent to which “10” has to be raised to be equal to  $x$ . Logarithms to base 10 are called common logarithms.
- A way of expressing numbers  $N$  in the form  $N = a \times 10^n$ , where  $0 < a < 10$  and  $n$  is an integer, is said to be scientific notation. If a number is greater or equal to 1, then  $n$  which is the power of 10, is positive.

### EXERCISES

1. Write the simplified form of each of the following expressions.

- |            |               |                                     |                                     |
|------------|---------------|-------------------------------------|-------------------------------------|
| (a) $2^7$  | (c) $2^{-7}$  | (e) $\left(\frac{2}{3}\right)^3$    | (g) $\frac{3^{-3}}{2^{-3}}$         |
| (b) $-2^7$ | (d) $-2^{-7}$ | (f) $\left(\frac{2}{3}\right)^{-3}$ | (h) $\left(\frac{3}{2}\right)^{-3}$ |

2. Use the properties of indices to simplify each of the following expressions.

- |  |   |                             |  |
|--|---|-----------------------------|--|
| (a) $3^2 \times 3^5$                             | (d) $a^{-5} b^{-5}$                           | (g) $\frac{a^{-6}}{a^{-2}}$ | (j) $\frac{4^x}{4^{2-x}}$  |
| (b) $\left(8^{\frac{1}{3}}\right)^4$             | (e) $(3m^3)^2$                                | (h) $(x^2 y^3)^5$           | (k) $\frac{7^{x-3}}{7^{x-4}}$  |
| (c) $\frac{256^{\frac{3}{4}}}{16^{\frac{3}{4}}}$ | (f) $\left(\frac{xy^{-1}}{2x^{-2}y}\right)^3$ | (i) $2^{2x-3} 2^{4-x}$      | (l) $\left(\frac{p^{-3} q^{-1} r^2}{p^{-5} q^{-3} r^{-2}}\right)^{-2}$ |

3. Simplify:

- |   |  |
|---|--|
| (a) $\left(\frac{81x^{-3}y^{-2}}{27x^{-3}y^{-5}}\right)^{-3} \left(\frac{xy}{x^3y^2}\right)^{-1}$ | (b) $\left(\frac{a^{-2} - b^{-2}}{a^{-1} - b^{-1}}\right) \left(\frac{ab^2}{a^{-1} + b^{-1}}\right)$ |
|---|--|

4. Change each logarithmic form to an equivalent indicial form.

(a)  $\log_3 27 = 3$

(e)  $\log_{16} 32 = \frac{5}{4}$

(b)  $\log_{64} 4 = \frac{1}{3}$

(f)  $\log_{64} 32 = \frac{5}{6}$

(c)  $\log_2 \frac{1}{32} = -5$

(g)  $\log_{10} 1 = 0$

(d)  $\log_{\frac{1}{27}} \frac{1}{81} = \frac{4}{3}$

(h)  $\log_{10} 0.0001 = -4$

5. Solve for  $x$ .

(a)  $\log_2 x = 7$

(e)  $\log_1 128 = x$

(i)  $\log_{27} 81 = x$

(b)  $\log_3 81 = x$

(f)  $\log_{25} \frac{1}{5} = x$

(j)  $\log_{100} x = \frac{3}{2}$

(c)  $\log_9 9 = x$

(g)  $\log_{10} 0.0001 = x$

(d)  $\log_{16} x = \frac{1}{4}$

(h)  $\log_x 8 = 3$

6. Use laws of logarithms to write each of the following expression as a single logarithm.

(a)  $\log_{10} 3 + \log_{10} 25$

(d)  $\log_a x^3 + \log_a bx^2$

(b)  $\log_5 24 - \log_5 8$

(e)  $3\log_a x + 2\log_a y$

(c)  $3\log_3 5 - 2\log_3 15$

(f)  $\log_a x^2y - \log_a xy^2$

7. Given that  $x = 2^p$  and  $y = 4^q$

(a) Express  $32xy$  as a power of 2.

(b) Find  $\log_2 (x^2 y^3)$  in terms of  $p$  and  $q$ .

8. Simplify each of the following logarithms.

(a)  $\log_3 2 \times \log_4 3 \times \log_5 4 \times \log_6 5 \times \log_7 6 \times \log_8 7$

(b)  $\log_8 1024 - \log_8 16$

(c)  $\log_7 21 + \log_7 77 + \log_7 88 - \log_7 121 - \log_7 24$

(d)  $3\log_4^2 + 2\log_9 3 - 6\log_{64} 4$

9. Evaluate each of the following.

(a)  $\frac{\log \frac{9}{8} - 2 \log \frac{3}{8}}{\log 12 - \log 6}$

(c)  $\log_6 25 (\log_{10} 3 + \log_{10} 2) \times \log_5 10$

(b)  $\log 8 + \log 5 - \log 4$

(d)  $3\log_{10} \frac{25}{24} - 2\log_{10} \frac{5}{96} + \log_{10} \frac{6}{25}$

10. Express each of the following numbers in scientific notation.
- |               |   |
|---------------|---|
| (a) 0.28      | (f) 0.0000038                                   |
| (b) 8.95      | (g) $3.5 \times 10^4 \times 2.8 \times 10^{-7}$ |
| (c) 909.50    | (h) $4.58 \times 10^1 + 4.58 \times 10^{-2}$    |
| (d) 0.005790  | (i) $9.35 \times 10^{-4} - 5.28 \times 10^{-4}$ |
| (e) 0.0008925 |   |
11. Find the logarithm of each of the following numbers using table of common logarithm.
- |                 |                     |                     |
|-----------------|---------------------|---------------------|
| (a) $\log 4.39$ | (e) $\log 0.000405$ | (i) $\log 0.002090$ |
| (b) $\log 9.5$  | (f) $\log 125$      | (j) $\log 0.00888$  |
| (c) $\log 0.35$ | (g) $\log 500$      |                     |
| (d) $\log 27$   | (h) $\log 0.99$     |                     |
12. Given that  $\log 2 = x$ ,  $\log 3 = y$  and  $\log 5 = z$ , express the following expressions in terms of  $x$ ,  $y$  and  $z$ .
- |               |                |                 |                 |
|---------------|----------------|-----------------|-----------------|
| (a) $\log 25$ | (b) $\log 4.8$ | (c) $\log 6.25$ | (d) $\log 14.4$ |
|---------------|----------------|-----------------|-----------------|
13. Using a scientific calculator, find the following logarithms.
- |                 |                   |                    |                 |
|-----------------|-------------------|--------------------|-----------------|
| (a) $\log_6 28$ | (b) $\log_{12} 9$ | (c) $\log_{25} 29$ | (d) $\log_9 45$ |
|-----------------|-------------------|--------------------|-----------------|
14. The predicted population of a certain city is given by  $P = 200,000^{0.03t}$ , where  $t$  is the number of years after 2020. Predict the approximate population for the year 2025?
15. If L\$80,000 is invested at a rate of 4% compounded yearly, then what is approximate amount of L\$ at the end of 3 years?

# CHAPTER



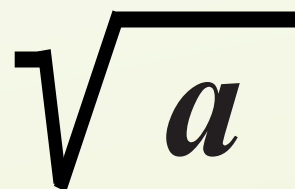
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# 3

## SURDS AND PERCENTAGES

### Chapter Contents

- 3.1 Surds
- 3.2 Reduction of Surds to Basic Form
- 3.3 Addition and Subtraction of Surds
- 3.4 Product and Quotient of Surds
- 3.5 Compound Interest in Relation to Simple Interest
- 3.6 Interest Formula
- 3.7 Depreciation
  - Key Terms
  - Summary
  - Exercises



## **Chapter Outcomes**

By the end of this chapter, you will be able to:

- define and discuss surds;
- simplify surds;
- simplify products and quotients of surds;
- define compound interest in relation to simple interest;
- define and discuss interest formulae;
- define and discuss depreciation and hire purchase.

## Introduction

In this topic, we will learn about square roots of whole numbers which produce irrational numbers, that is, numbers which cannot be written as fractions or as a ratio of two positive integers. These are called surds. The concept of surds with basic operations has been introduced in this topic. In addition, we will learn about simple interest and compound interest.

In chapter 2, you studied about the concept of base and *index* and *laws of indices*. In this sub topic, you will learn about surds which are numbers whose square roots cannot be determined in terms of rational numbers.

### HISTORICAL NOTE

It is known that AL-KHWARIZMI (*the Persian guy who we get the name “Algorithm” from*) identified surds as something special in mathematics and this mute quality of theirs.

In or around 825 AD he referred to the rational numbers as ‘audible’ and irrational as ‘inaudible’... this was later translated to the Latin surdus (“deaf” or “mute”). So, “surd” is derived from the Latin surdus.



Square roots of numbers that cannot be simplified into whole numbers are called **surds**. *Irrational numbers* which are in the form of roots are called **surds**. For example,  $\sqrt{2}$ ,  $\sqrt{3}$ ,  $\sqrt{5}$ ,  $\sqrt{7}$ ,  $\pi$  and  $\sqrt[3]{6}$  are irrational numbers while  $\sqrt{16}$ ,  $\sqrt[3]{8}$ ,  $\sqrt[3]{27}$  and  $\sqrt[3]{32}$  are rational numbers.

### Group Work 3.1

1. Discuss the conclusion that “if it is a root and irrational, it is a surd. But not all roots are surds.”
2. Are indices, surds and logarithms closely related? Present and explain your answer to your group.
3. Distinguish between rational and irrational numbers.

### ACTIVITY 1

1. What is rational number?  
recurring decimal?
2. What is irrational number?
3. What are fractional indices?
4. What is the exact result of  $\sqrt{4}$  and  $\sqrt{5}$ ?
5. Determine whether each of the following numbers is a surd.
 

(a) $\sqrt{2}$	(e) $\sqrt{16}$
(b) $\sqrt{4}$	(f) $\sqrt{19}$
(c) $\sqrt{6}$	(g) $\sqrt{25}$
(d) $\sqrt{13}$	(h) $\sqrt{32}$

Surds are nothing but square root representation of irrational numbers that cannot be expressed in fractional or recurring decimals

### DEFINITION

A general surd is an irrational number of the form  $a\sqrt[n]{p}$ , where  $a$  is a rational number and  $\sqrt[n]{p}$  is an irrational number, where  $\sqrt[n]{p}$  is called a **radical**.

### EXAMPLE 1

Determine whether each of the following numbers is a surd or not.

(a)  $\sqrt{7}$

(c)  $\sqrt{12}$

(b)  $\sqrt{9}$

(d)  $\sqrt{15}$

#### Solution

When we express surds, we have to fix the decimals, and we get nonterminating and nonrecurring decimals.

(a)  $\sqrt{7} = 2.6457513 \dots$ . Since this decimal is not recurring,  $\sqrt{7}$  is a surd.

(b)  $\sqrt{9} = \sqrt{3 \times 3} = 3 \dots$ . Since 3 is a whole number,  $\sqrt{9}$  is not a surd.

(c)  $\sqrt{12} = 3.4641016 \dots$ . Since this decimal is not recurring,  $\sqrt{12}$  is a surd.

(d)  $\sqrt{15} = 3.8729833 \dots$ . Since this decimal is not recurring,  $\sqrt{15}$  is a surd.

As we can see from the above example, the surds have a decimal which goes on forever without repeating, and that makes them irrational numbers.

### Key Point

- If a positive whole number is not a perfect square, then its square root is called a **surd**. A surd cannot be written as a fraction.
- A surd is an example of an irrational number.

### EXERCISES

1. Convert the following rational numbers to decimals.

(a)  $\frac{3}{5}$

(c)  $4\frac{2}{3}$

(e)  $7\frac{3}{4}$

(b)  $5\frac{3}{8}$

(d)  $\frac{7}{12}$

(f)  $10\frac{1}{4}$

2. Convert the following rational numbers to fractions.

- |          |                  |                         |
|----------|------------------|-------------------------|
| (a) 0.12 | (c) $0.\dot{8}$  | (e) $0.4\dot{2}\dot{7}$ |
| (b) 0.96 | (d) $0.5\dot{3}$ | (f) 0.125               |

3. Determine whether the following real numbers are rational or irrational.

- |                 |                          |                            |
|-----------------|--------------------------|----------------------------|
| (a) $\sqrt{8}$  | (c) $\sqrt{36}$          | (e) $\sqrt{\frac{16}{25}}$ |
| (b) $\sqrt{14}$ | (d) $\frac{\sqrt{2}}{4}$ | (f) $\sqrt{54}$            |

4. Determine whether the following real numbers are surds or not.

- |                 |                 |                  |
|-----------------|-----------------|------------------|
| (a) $\sqrt{21}$ | (c) $\sqrt{64}$ | (e) $\sqrt{121}$ |
| (b) $\sqrt{27}$ | (d) $\sqrt{99}$ | (f) $\sqrt{108}$ |

5. Decide if the following numbers are surds.

- |                            |                                  |                                  |
|----------------------------|----------------------------------|----------------------------------|
| (a) $\frac{3\sqrt{9}}{2}$  | (c) $\sqrt{\frac{12}{27}}$       | (e) $\frac{\sqrt{75}}{\sqrt{6}}$ |
| (b) $\frac{-5\sqrt{2}}{3}$ | (d) $\frac{\sqrt{80}}{\sqrt{4}}$ | (f) $\sqrt{\frac{162}{27}}$      |

If  $\sqrt{a}$  cannot be reduced any further, it is said to be in its basic form. For example,  $\sqrt{2}$ ,  $\sqrt{3}$ ,  $\sqrt{5}$ ,  $\sqrt{7}$ ,  $\sqrt{11}$ , etc. are surds in its basic forms.

Before  $\sqrt{a}$  can be reduced further, “a” must contain a factor which is a perfect square.

Let us take some other surds which are a bit complex such as  $\sqrt{32}$ .

$\sqrt{32} = \sqrt{16 \times 2} = \sqrt{4 \times 4 \times 2} = 4\sqrt{2}$ . Here,  $\sqrt{2}$  is a surd. Therefore,  $4\sqrt{2}$  is also a surd.

So, we can say that if any number is in the form of  $a\sqrt{p}$ , where  $\sqrt{p}$  is a surd then  $a\sqrt{p}$  is also a surd.

### ACTIVITY 2

- List all perfect square numbers up to 100.
- Write the highest square factor of 24, 45, 50 or 80.
- Write the following surds in the simplest form.

- |                  |                  |
|------------------|------------------|
| (a) $\sqrt{80}$  | (d) $3\sqrt{44}$ |
| (b) $5\sqrt{24}$ | (e) $5\sqrt{45}$ |
| (c) $\sqrt{98}$  | (f) $\sqrt{500}$ |

Knowing the common square numbers like 4, 9, 16, 25, 36, ..., 100 is very helpful when simplifying surd expressions.

Surds can be simplified if the number within the surd has a *square number* as one of its factors.

### EXAMPLE 2

Write the highest square factor of the following numbers.

(a) 48

(c) 98

(b) 75

(d) 125

#### Solution

(a) The highest square factor of 48 is 16 because  $16 \times 3 = 48$ .

(b) The highest square factor of 75 is 25 because  $25 \times 3 = 75$ .

(c) The highest square factor of 98 is 49 because  $49 \times 2 = 98$ .

(d) The highest square factor of 125 is 25 because  $25 \times 5 = 125$ .

### EXAMPLE 3

Write the following surds in the simplest form.

(a)  $\sqrt{20}$

(b)  $\sqrt{32}$

(c)  $\sqrt{75}$

(d)  $\sqrt{300}$

#### Solution

(a)  $\sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2 \times \sqrt{5} = 2\sqrt{5}$ .

(b)  $\sqrt{32} = \sqrt{16 \times 2} = \sqrt{16} \times \sqrt{2} = 4 \times \sqrt{2} = 4\sqrt{2}$ .

(c)  $\sqrt{75} = \sqrt{25 \times 3} = \sqrt{25} \times \sqrt{3} = 5 \times \sqrt{3} = 5\sqrt{3}$ .

(d)  $\sqrt{300} = \sqrt{100 \times 3} = \sqrt{100} \times \sqrt{3} = 10 \times \sqrt{3} = 10\sqrt{3}$ .

### Key Point

- When simplifying surds, look for square factors of the number inside the square root and then use

(i)  $\sqrt{a \times a} = a$

(ii)  $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$

(iii)  $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

**EXAMPLE 4**

Write the following surds in the simplest form.

(a)  $2\sqrt{18}$                       (b)  $3\sqrt{200}$                       (c)  $\frac{5\sqrt{80}}{6}$                       (d)  $\sqrt{\frac{75}{9}}$

**Solution**

$$(a) \quad 2\sqrt{18} = 2\sqrt{9 \times 2} = 2 \times \sqrt{9} \times \sqrt{2} = 2 \times 3 \times \sqrt{2} = 6\sqrt{2}.$$

$$(b) \quad 3\sqrt{200} = 3\sqrt{100 \times 2} = 3 \times \sqrt{100} \times \sqrt{2} = 3 \times 10 \times \sqrt{2} = 30\sqrt{2}.$$

$$(c) \quad \frac{5\sqrt{80}}{6} = \frac{5\sqrt{16 \times 5}}{6} = \frac{5 \times \sqrt{16} \times \sqrt{5}}{6} = \frac{5 \times 4 \times \sqrt{5}}{6} = \frac{20 \times \sqrt{5}}{6}$$

$$= \frac{\cancel{2} \times 10 \times \sqrt{5}}{\cancel{2} \times 3} = \frac{10\sqrt{5}}{3}.$$

$$(d) \quad \sqrt{\frac{75}{9}} = \frac{\sqrt{75}}{\sqrt{9}} = \frac{\sqrt{25 \times 3}}{\sqrt{3 \times 3}} = \frac{\sqrt{25} \times \sqrt{3}}{3} = \frac{5 \times \sqrt{3}}{3} = \frac{5\sqrt{3}}{3}.$$

**EXAMPLE 5**Express each of the following surds as a square root of a positive integers (or as  $\sqrt{n}$ )

(a)  $3\sqrt{2}$                       (b)  $2\sqrt{3}$                       (c)  $4\sqrt{5}$                       (d)  $5\sqrt{7}$

**Solution**

$$(a) \quad 3\sqrt{2} = \sqrt{3 \times 3} \times \sqrt{2} = \sqrt{9} \times \sqrt{2} = \sqrt{18}.$$

$$(b) \quad 2\sqrt{3} = \sqrt{2 \times 2} \times \sqrt{3} = \sqrt{4} \times \sqrt{3} = \sqrt{12}.$$

$$(c) \quad 4\sqrt{5} = \sqrt{4 \times 4} \times \sqrt{5} = \sqrt{16} \times \sqrt{5} = \sqrt{80}.$$

$$(d) \quad 5\sqrt{7} = \sqrt{5 \times 5} \times \sqrt{7} = \sqrt{25} \times \sqrt{7} = \sqrt{175}.$$

**EXERCISES**

1. Find the highest square factor of each of the following numbers.

(a) 96                      (b) 175                      (c) 162                      (d) 720

2. Simplify the following surds.

(a)  $\sqrt{80}$                       (d)  $\sqrt{128}$                       (g)  $\sqrt{147}$                       (j)  $\sqrt{405}$

(b)  $\sqrt{500}$                       (e)  $\sqrt{175}$                       (h)  $\sqrt{675}$                       (k)  $\sqrt{363}$

(c)  $\sqrt{98}$                       (f)  $\sqrt{320}$                       (i)  $\sqrt{512}$                       (l)  $\sqrt{338}$

3. Simplify the following surds.

(a) $5\sqrt{24}$	(c) $5\sqrt{45}$	(e) $3\sqrt{98}$	(g) $2\sqrt{63}$
(b) $3\sqrt{44}$	(d) $6\sqrt{200}$	(f) $4\sqrt{125}$	(h) $7\sqrt{80}$

4. Simplify the following surds.

(a) $\frac{\sqrt{24}}{3}$	(d) $\frac{\sqrt{28}}{2}$	(g) $\frac{2\sqrt{98}}{7}$	(j) $\frac{4\sqrt{150}}{5}$
(b) $\frac{\sqrt{54}}{6}$	(e) $\frac{5\sqrt{44}}{2}$	(h) $\frac{4\sqrt[7]{98}}{21}$	(k) $\frac{5\sqrt{147}}{14}$
(c) $\frac{\sqrt{80}}{20}$	(f) $\frac{3\sqrt{200}}{25}$	(i) $\frac{6\sqrt{75}}{20}$	

5. Simplify the following.

(a) $\sqrt{\frac{12}{49}}$	(c) $\sqrt{\frac{80}{9}}$	(e) $\sqrt{\frac{56}{76}}$	(g) $\sqrt{\frac{27}{16}}$
(b) $\sqrt{\frac{18}{25}}$	(d) $\sqrt{\frac{45}{72}}$	(f) $\sqrt{\frac{15}{27}}$	(h) $\sqrt{\frac{12}{108}}$

6. Express the following surds as a square root of a positive integer (or as  $\sqrt{n}$ )

(a) $5\sqrt{2}$	(c) $8\sqrt{2}$	(e) $10\sqrt{7}$	(g) $12\sqrt{3}$
(b) $7\sqrt{3}$	(d) $9\sqrt{10}$	(f) $11\sqrt{3}$	(h) $16\sqrt{5}$

Surds can be added or subtracted only if they are like surds, that is, if they have the same value under the radical sign.

### EXAMPLE 6

Simplify each of the following

(a) $3\sqrt{2} + 7\sqrt{2}$	(c) $\sqrt{72} + \sqrt{32}$
(b) $12\sqrt{3} - 7\sqrt{3}$	(d) $\sqrt{675} - \sqrt{192}$

**Solution**

(a)  $3\sqrt{2} + 7\sqrt{2} = (3 + 7)\sqrt{2} = 10\sqrt{2}$ .

(b)  $12\sqrt{3} - 7\sqrt{3} = (12 - 7)\sqrt{3} = 5\sqrt{3}$ .

$$\begin{aligned} \text{(c)} \quad \sqrt{72} + \sqrt{32} &= \sqrt{36 \times 2} + \sqrt{16 \times 2} = \sqrt{36} \times \sqrt{2} + \sqrt{16} \times \sqrt{2} = 6\sqrt{2} + 4\sqrt{2} \\ &= (6 + 4)\sqrt{2} = 10\sqrt{2}. \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad \sqrt{675} - \sqrt{192} &= \sqrt{225 \times 3} - \sqrt{64 \times 3} = \sqrt{225} \times \sqrt{3} - \sqrt{64} \times \sqrt{3} \\ &= 15\sqrt{3} - 8\sqrt{3} = (15 - 8)\sqrt{3} = 7\sqrt{3}. \end{aligned}$$

### Key Point

In general, we have addition or subtraction law of surds with the same radicals.

1. Addition law of surds with the same radicals

$$a\sqrt{b} + c\sqrt{b} = (a + c)\sqrt{b}.$$

2. Subtraction law of surds with the same radicals.

$$a\sqrt{b} - c\sqrt{b} = (a - c)\sqrt{b}.$$

### EXAMPLE 7

Simplify each of the following.

$$\text{(a)} \quad 6\sqrt{3} + 4\sqrt{3} + 5\sqrt{3}$$

$$\text{(c)} \quad \sqrt{243} - \sqrt{12} + 2\sqrt{75}$$

$$\text{(b)} \quad 8\sqrt{5} - 3\sqrt{5} + 2\sqrt{5}$$

$$\text{(d)} \quad \sqrt{200} + \sqrt{8} - \sqrt{32}$$

### Solution

$$\text{(a)} \quad 6\sqrt{3} + 4\sqrt{3} + 5\sqrt{3} = (6 + 4 + 5)\sqrt{3} = 15\sqrt{3}.$$

$$\text{(b)} \quad 8\sqrt{5} - 3\sqrt{5} + 2\sqrt{5} = (8 - 3 + 2)\sqrt{5} = 7\sqrt{5}.$$

$$\begin{aligned} \text{(c)} \quad \sqrt{243} - \sqrt{12} + 2\sqrt{75} &= \sqrt{81 \times 3} - \sqrt{4 \times 3} + 2\sqrt{25 \times 3} \\ &= \sqrt{81} \times \sqrt{3} - \sqrt{4} \times \sqrt{3} + 2\sqrt{25} \times \sqrt{3} \\ &= 9 \times \sqrt{3} - 2 \times \sqrt{3} + 2 \times 5 \times \sqrt{3} = 9\sqrt{3} - 2\sqrt{3} + 10\sqrt{3} \\ &= (9 - 2 + 10)\sqrt{3} = 17\sqrt{3}. \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad \sqrt{200} + \sqrt{8} - \sqrt{32} &= \sqrt{100 \times 2} + \sqrt{4 \times 2} - \sqrt{16 \times 2} \\ &= \sqrt{100} \times \sqrt{2} + \sqrt{4} \times \sqrt{2} - \sqrt{16} \times \sqrt{2} \\ &= 10 \times \sqrt{2} + 2 \times \sqrt{2} - 4 \times \sqrt{2} = 10\sqrt{2} + 2\sqrt{2} - 4\sqrt{2} \\ &= (10 + 2 - 4)\sqrt{2} = 8\sqrt{2}. \end{aligned}$$

**Note**

(I)  $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$

(II)  $\sqrt{a} - \sqrt{b} \neq \sqrt{a-b}$

**EXAMPLE 8**

Simplify each of the following.

(a)  $5\sqrt{2} + 4\sqrt{3} + 6\sqrt{2} - 2\sqrt{3}$

(b)  $8\sqrt{3} + 2\sqrt{7} - 5\sqrt{3} - 4\sqrt{7}$

**Solution**

$$\begin{aligned} \text{(a)} \quad 5\sqrt{2} + 4\sqrt{3} + 6\sqrt{2} - 2\sqrt{3} &= 5\sqrt{2} + 6\sqrt{2} + 4\sqrt{3} - 2\sqrt{3} \\ &= (5+6)\sqrt{2} + (4-2)\sqrt{3} = 11\sqrt{2} + 2\sqrt{3}. \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad 8\sqrt{3} + 2\sqrt{7} - 5\sqrt{3} - 4\sqrt{7} &= 8\sqrt{3} - 5\sqrt{3} + 2\sqrt{7} - 4\sqrt{7} \\ &= (8-5)\sqrt{3} + (2-4)\sqrt{7} = 3\sqrt{3} - 2\sqrt{7}. \end{aligned}$$

**EXERCISES**

1. Simplify each of the following.

(a)  $15\sqrt{6} - 3\sqrt{6} - 4\sqrt{6}$

(d)  $8\sqrt{3} + 5\sqrt{3} - 7\sqrt{3}$

(b)  $7\sqrt{11} + 6\sqrt{11} - 3\sqrt{11}$

(e)  $3\sqrt{5} - 8\sqrt{5} + 2\sqrt{5}$

(c)  $3\sqrt{5} - 7\sqrt{5} + 9\sqrt{5}$

(f)  $5\sqrt{10} - 7\sqrt{10} + 3\sqrt{10}$

2. Simplify each of the following.

(a)  $5\sqrt{6} + \sqrt{24} - 3\sqrt{5}$

(d)  $7\sqrt{3} + 4\sqrt{5} + 3\sqrt{3} - \sqrt{5}$

(b)  $\sqrt{50} + 6\sqrt{3} - \sqrt{8}$

(e)  $5\sqrt{2} + 6\sqrt{3} - 3\sqrt{2} + \sqrt{3}$

(c)  $8\sqrt{10} + 4\sqrt{5} - \sqrt{90}$

(f)  $6\sqrt{5} + 2\sqrt{11} - 3\sqrt{5} - \sqrt{11}$

3. Simplify each of the following.

(a)  $10\sqrt{5} - 4\sqrt{3} - 5\sqrt{3} + 2\sqrt{5}$

(d)  $\sqrt{28} + \sqrt{27} + \sqrt{63} + \sqrt{12}$

(b)  $6\sqrt{11} - 3\sqrt{10} - 4\sqrt{11} - 2\sqrt{10}$

(e)  $8\sqrt{5} - \sqrt{20} + \sqrt{45} - 5\sqrt{6}$

(c)  $3\sqrt{12} + 2\sqrt{8} - \sqrt{50} - \sqrt{75}$

(f)  $\sqrt{72} - \sqrt{8} + 8\sqrt{3} - \sqrt{108}$

4. Expand and simplify each of the following expressions.

(a)  $\sqrt{2}(3 + \sqrt{6})$

(d)  $(2 + \sqrt{3})(1 + \sqrt{3})$

(b)  $\sqrt{6}(\sqrt{2} + \sqrt{8})$

(e)  $(3 - \sqrt{5})(3 + 2\sqrt{5})$

(c)  $\sqrt{3}(\sqrt{27} + \sqrt{12})$

(f)  $(8 + \sqrt{2})(8 - \sqrt{2})$

The product of two unlike surds cannot be rational. Let  $\sqrt{a}$  and  $\sqrt{b}$  be unlike surds, then  $\sqrt{a} \times \sqrt{b}$  cannot be rational number. In like manner, the quotient of two unlike surds cannot be rational. Let  $\sqrt{a}$  and  $\sqrt{b}$  be unlike surds, then  $\frac{\sqrt{a}}{\sqrt{b}}$  cannot be rational number. In this sub-topic, we will learn the product and quotient of surds.

## Product of surds

In multiplying surds, surds are combined using multiplication rule to write as a single surd, simplified where possible.

### ACTIVITY 3

- Use calculator to find a decimal approximation for  $\sqrt{5} \times \sqrt{7}$  and for  $\sqrt{35}$ .
- What do you notice about the result from question 1? Try other pairs of surds to see if your conclusion is correct.
- Simplify the following surds.
 

(a) $\sqrt{3} \times \sqrt{5}$	(b) $\sqrt{6} \times \sqrt{18}$	(c) $\sqrt{5} \times \sqrt{20}$	(d) $\sqrt{80} \times \sqrt{50}$
--------------------------------	---------------------------------	---------------------------------	----------------------------------
- Simplify each of the following.
 

(a) $2\sqrt{3} \times 3\sqrt{5}$	(b) $5\sqrt{7} \times 2\sqrt{7}$	(c) $3\sqrt{2} \times 2\sqrt{8}$	(d) $2\sqrt{3} \times 5\sqrt{27}$
----------------------------------	----------------------------------	----------------------------------	-----------------------------------
- Simplify each of the following.
 

(a) $\sqrt{8} \times \sqrt{6} \times \sqrt{2}$	(c) $\sqrt{3}(\sqrt{12} + \sqrt{3})$
(b) $2\sqrt{2} \times 3\sqrt{2} \times \sqrt{3}$	(d) $2\sqrt{2}(\sqrt{72} - \sqrt{18})$

When simplifying surds such as  $\sqrt{48}$ , we write  $\sqrt{48} = \sqrt{16 \times 3} = \sqrt{16} \times \sqrt{3} = 4 \times \sqrt{3} = 4\sqrt{3}$ , where we use the fact that  $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$ . This can be used in reverse to simplify the product of two surds.

We know that:  $\sqrt{2} \times \sqrt{2} = 2$ ;  $\sqrt{3} \times \sqrt{3} = 3$ ; and  $\sqrt{5} \times \sqrt{5} = 5$ .

So, multiplying surds that have the same number inside the square root gives a *rational number*.

Suppose we were asked to simplify the expression  $\sqrt{300} \times \sqrt{50}$ :

$$\begin{aligned} \sqrt{300} \times \sqrt{50} &= \sqrt{100 \times 3} \times \sqrt{25 \times 2} = \sqrt{100} \times \sqrt{3} \times \sqrt{25} \times \sqrt{2} \\ &= 10 \times \sqrt{3} \times 5 \times \sqrt{2} = 10 \times 5 \times \sqrt{3 \times 2} = 50\sqrt{6}, \end{aligned}$$

which cannot be simplified any further.

When multiplying surds, we simply multiply the numbers inside the square roots.

**Key Point**

- The following formulas may be used to simplify expressions involving surds:

(i)  $\sqrt{a} \times \sqrt{a} = a$

(iii)  $a\sqrt{d} \times b\sqrt{d} = ab\sqrt{d^2}$

(ii)  $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

**EXAMPLE 9**

Simplify each of the following.

(a)  $\sqrt{6} \times \sqrt{24}$       (b)  $3\sqrt{2} \times 2\sqrt{54}$       (c)  $\sqrt{5} \times \sqrt{10} \times \sqrt{2}$       (d)  $\sqrt{2} \times 3\sqrt{2} \times \sqrt{3}$

**Solution**

(a)  $\sqrt{6} \times \sqrt{24} = \sqrt{6 \times 24} = \sqrt{144} = \sqrt{12 \times 12} = \sqrt{12} \times \sqrt{12} = 12.$

(b)  $3\sqrt{2} \times 2\sqrt{54} = 3 \times 2 \times \sqrt{2} \times \sqrt{54}$

$$= 6 \times \sqrt{2 \times 54} = 6 \times \sqrt{36 \times 3} = 6 \times \sqrt{36} \times \sqrt{3} = 6 \times 6 \times \sqrt{3} = 36\sqrt{3}.$$

(c)  $\sqrt{5} \times \sqrt{10} \times \sqrt{2} = \sqrt{5 \times 10 \times 2} = \sqrt{100} = \sqrt{10 \times 10} = \sqrt{10} \times \sqrt{10} = 10.$

(d)  $\sqrt{2} \times 3\sqrt{2} \times \sqrt{3} = 3 \times \sqrt{2 \times 2 \times 3} = 3 \times \sqrt{2} \times \sqrt{2} \times \sqrt{3} = 3 \times 2 \times \sqrt{3} = 6\sqrt{3}.$

**EXAMPLE 10**

Simplify each of the following.

(a)  $\sqrt{3}(\sqrt{18} + \sqrt{8})$

(b)  $2\sqrt{2}(\sqrt{32} - \sqrt{8})$

**Solution**

(a)  $\sqrt{3}(\sqrt{18} + \sqrt{8}) = \sqrt{3} \times \sqrt{18} + \sqrt{3} \times \sqrt{8} = \sqrt{3 \times 18} + \sqrt{3 \times 8} = \sqrt{54} + \sqrt{24}$

$$= \sqrt{9 \times 6} + \sqrt{4 \times 6} = \sqrt{9} \times \sqrt{6} + \sqrt{4} \times \sqrt{6} = 3 \times \sqrt{6} + 2 \times \sqrt{6}$$

$$= 3\sqrt{6} + 2\sqrt{6} = 5\sqrt{6}.$$

(b)  $2\sqrt{2}(\sqrt{32} - \sqrt{8}) = 2\sqrt{2} \times \sqrt{32} - 2\sqrt{2} \times \sqrt{8} = 2\sqrt{2 \times 32} - 2\sqrt{2 \times 8} = 2\sqrt{64} - 2\sqrt{16}$

$$= 2 \times \sqrt{8 \times 8} - 2 \times \sqrt{4 \times 4} = 2 \times \sqrt{8} \times \sqrt{8} - 2 \times \sqrt{4} \times \sqrt{4}$$

$$= 2 \times 8 - 2 \times 4 = 16 - 8 = 8.$$

**Quotient of surds**

In dividing surds, surds are combined using division rule to write them as a single surd, simplified where possible.

**ACTIVITY 4**

- Divide the following surds.
  - $\sqrt{8} \div \sqrt{2}$
  - $\sqrt{18} \div \sqrt{6}$
  - $\sqrt{80} \div \sqrt{45}$
  - $\sqrt{108} \div \sqrt{32}$
- Simplify each of the following.
  - $12\sqrt{3} \div 3\sqrt{2}$
  - $9\sqrt{5} \times 3\sqrt{5}$
  - $3\sqrt{2} \div 2\sqrt{8}$
  - $3\sqrt{6} \div 5\sqrt{27}$
- Determine whether the simplified forms of each of the following expressions are rational or irrational.
  - $\sqrt{80} \div \sqrt{20}$
  - $\sqrt{50} \div \sqrt{12}$
  - $\sqrt{75} \div \sqrt{48}$
  - $3\sqrt{10} \div 2\sqrt{45}$
  - $3\sqrt{32} \div 5\sqrt{8}$
  - $3\sqrt{50} \div 2\sqrt{18}$

Suppose we were asked to simplify the expression  $\sqrt{2000} \div \sqrt{40}$ :

$$\begin{aligned} \sqrt{2000} \div \sqrt{40} &= \frac{\sqrt{2000}}{\sqrt{40}} \\ &= \sqrt{\frac{2000}{40}} = \sqrt{50} = \sqrt{25 \times 2} \\ &= \sqrt{25} \times \sqrt{2} = 5 \times \sqrt{2} = 5\sqrt{2}, \text{ which} \\ &\text{cannot be simplified any further.} \end{aligned}$$

When dividing surds, we simply divide the numbers inside the square roots.

**Key Point**

- The following formulas may be used to simplify expressions involving surds:
  - $\sqrt{a} \div \sqrt{a} = 1.$
  - $\sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} = \sqrt{a \div b}.$
  - $a\sqrt{b} \div c\sqrt{d} = \frac{a\sqrt{b}}{c\sqrt{d}} = \frac{a}{c} \sqrt{\frac{b}{d}}.$

**EXAMPLE 11**

Simplify the following expressions.

$$(a) \sqrt{162} \div \sqrt{27} \quad (b) \sqrt{20} \div \sqrt{45} \quad (c) \frac{12\sqrt{45}}{3\sqrt{5}} \quad (d) \frac{3\sqrt{26}}{5\sqrt{13}}$$

**Solution**

$$\begin{aligned} (a) \quad \sqrt{3}(\sqrt{18} + \sqrt{8}) &= \sqrt{3} \times \sqrt{18} + \sqrt{3} \times \sqrt{8} = \sqrt{3 \times 18} + \sqrt{3 \times 8} = \sqrt{54} + \sqrt{24} \\ &= \sqrt{9 \times 6} + \sqrt{4 \times 6} = \sqrt{9} \times \sqrt{6} + \sqrt{4} \times \sqrt{6} = 3 \times \sqrt{6} + 2 \times \sqrt{6} \\ &= 3\sqrt{6} + 2\sqrt{6} = 5\sqrt{6}. \\ (b) \quad 2\sqrt{2}(\sqrt{32} - \sqrt{8}) &= 2\sqrt{2} \times \sqrt{32} - 2\sqrt{2} \times \sqrt{8} = 2\sqrt{2 \times 32} - 2\sqrt{2 \times 8} = 2\sqrt{64} - 2\sqrt{16} \\ &= 2 \times \sqrt{8 \times 8} - 2 \times \sqrt{4 \times 4} = 2 \times \sqrt{8} \times \sqrt{8} - 2 \times \sqrt{4} \times \sqrt{4} \\ &= 2 \times 8 - 2 \times 4 = 16 - 8 = 8. \end{aligned}$$

**Solution**

$$(a) \sqrt{162} \div \sqrt{72} = \frac{\sqrt{162}}{\sqrt{72}} = \frac{\sqrt{81 \times 2}}{\sqrt{36 \times 2}} = \frac{\sqrt{81} \times \sqrt{2}}{\sqrt{36} \times \sqrt{2}} = \frac{9 \times \sqrt{2}}{6 \times \sqrt{2}} = \frac{9}{6} = \frac{3}{2}.$$

$$(b) \sqrt{20} \div \sqrt{45} = \frac{\sqrt{20}}{\sqrt{45}} = \sqrt{\frac{20}{45}} = \sqrt{\frac{20}{45}} = \sqrt{\frac{4}{9}} = \sqrt{\frac{2}{3} \times \frac{2}{3}} = \frac{2}{3}.$$

$$(c) \frac{12\sqrt{45}}{3\sqrt{5}} = \frac{12 \times \sqrt{9 \times 5}}{3 \times \sqrt{5}} = \frac{12 \times \sqrt{9} \times \sqrt{5}}{3 \times \sqrt{5}} = \frac{12 \times 3 \times \sqrt{5}}{3 \times \sqrt{5}} = \frac{36}{3} \times \frac{\sqrt{5}}{\sqrt{5}} = 12.$$

$$(d) \frac{3\sqrt{26}}{5\sqrt{13}} = \frac{3\sqrt{26}}{5\sqrt{13}} = \frac{3}{5} \sqrt{\frac{26}{13}} = \frac{3}{5} \sqrt{2} = \frac{3\sqrt{2}}{5}.$$

**Rationalizing the Denominator**

Another technique to simplify expressions involving surds is to rationalize the denominator. This means removing a surd from the denominator.

In a fraction, if the denominator is a surd, then we need to convert the denominator into a rational number and this conversion is known as **rationalizing the denominator**. This changes the surd denominator, which is irrational, into a whole number.

The fraction  $\frac{1}{\sqrt{3}}$  is a surd. To rationalize the denominator, we multiply by the fraction  $\frac{\sqrt{3}}{\sqrt{3}}$  which is equal to 1. When we multiply by this number, we do not change the value of our original expression. We obtain

$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

**ACTIVITY 5**

1. Identify the denominator of each of the following expressions.

$$(a) \frac{2}{3} \quad (c) \frac{2\sqrt{5}}{9} \quad (e) \frac{\sqrt{8}}{\sqrt{3}}$$

$$(b) \frac{3\sqrt{2}}{5} \quad (d) \sqrt{\frac{2}{9}} \quad (f) \frac{\sqrt{12}}{3\sqrt{2}}$$

2. Rationalize the denominator of the following surds.

$$(a) \frac{1}{\sqrt{6}} \quad (c) \frac{\sqrt{5}}{\sqrt{45}} \quad (e) \frac{\sqrt{8}}{\sqrt{24}}$$

$$(b) \frac{2}{\sqrt{2}} \quad (d) \frac{4}{\sqrt{6}} \quad (f) \frac{5}{2\sqrt{2}}$$

3. Rationalize the denominator of the following surds.

$$(a) \frac{1}{2 + \sqrt{3}} \quad (c) \frac{3}{\sqrt{3} + \sqrt{2}}$$

$$(b) \frac{4}{3 - \sqrt{2}} \quad (d) \frac{2}{\sqrt{2} - \sqrt{3}}$$

**Key Point**

- A fraction in the form of  $\frac{a}{\sqrt{b}}$ , when the denominator  $\sqrt{b}$  is a surd.
- To convert a surd  $\frac{a}{\sqrt{b}}$  into a rational number, multiply both the numerator and the denominator by  $\sqrt{b}$ . That is,

$$\frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{a \times \sqrt{b}}{\sqrt{b} \times \sqrt{b}} = \frac{a\sqrt{b}}{b}.$$

**EXAMPLE 12**

Rationalize the denominator of the following surds.

(a)  $\frac{8}{\sqrt{6}}$                       (b)  $\frac{6}{\sqrt{3}}$                       (c)  $\frac{10}{\sqrt{15}}$                       (d)  $\frac{\sqrt{8}}{\sqrt{12}}$

**Solution**

$$(a) \frac{8}{\sqrt{6}} = \frac{8}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}} = \frac{8 \times \sqrt{6}}{\sqrt{6} \times \sqrt{6}} = \frac{8\sqrt{6}}{6} = \frac{4\sqrt{6}}{3}.$$

$$(b) \frac{6}{\sqrt{3}} = \frac{6}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{6 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}.$$

$$(c) \frac{10}{\sqrt{15}} = \frac{10}{\sqrt{15}} \times \frac{\sqrt{15}}{\sqrt{15}} = \frac{10 \times \sqrt{15}}{\sqrt{15} \times \sqrt{15}} = \frac{10\sqrt{15}}{15} = \frac{2\sqrt{15}}{3}.$$

$$(d) \frac{\sqrt{8}}{\sqrt{12}} = \frac{\sqrt{8}}{\sqrt{12}} \times \frac{\sqrt{12}}{\sqrt{12}} = \frac{\sqrt{8} \times \sqrt{12}}{\sqrt{12} \times \sqrt{12}} = \frac{\sqrt{96}}{12} = \frac{\sqrt{16 \times 6}}{12} = \frac{\sqrt{16} \times \sqrt{6}}{12} = \frac{4 \times \sqrt{6}}{12} = \frac{\sqrt{6}}{3}.$$

**Note**

If the denominator of a fraction contains an expression of the form:

- $\sqrt{n} + m$ , then rationalize the denominator by multiplying both numerator and denominator by  $\sqrt{n} - m$ .
- $\sqrt{n} - m$ , then rationalize the denominator by multiplying both numerator and denominator by  $\sqrt{n} + m$ .
- $\sqrt{n} + \sqrt{m}$ , then rationalize the denominator by multiplying both numerator and denominator by  $\sqrt{n} - \sqrt{m}$ .
- $\sqrt{n} - \sqrt{m}$ , then rationalize the denominator by multiplying both numerator and denominator by  $\sqrt{n} + \sqrt{m}$ .

**EXAMPLE 13**

Rationalize the denominator of the following surds.

(a)  $\frac{4}{2-\sqrt{3}}$

(b)  $\frac{6}{\sqrt{3}+\sqrt{2}}$

**Solution**

(a) 
$$\frac{4}{2-\sqrt{3}} = \frac{4}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}} = \frac{4(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})} = \frac{4 \times 2 + 4 \times \sqrt{3}}{2^2 - (\sqrt{3})^2} = \frac{8+4\sqrt{3}}{4-3} = 8+4\sqrt{3}.$$

(b) 
$$\begin{aligned} \frac{6}{\sqrt{3}+\sqrt{2}} &= \frac{6}{\sqrt{3}+\sqrt{2}} \times \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}} = \frac{6(\sqrt{3}-\sqrt{2})}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})} = \frac{6\sqrt{3}-6\sqrt{2}}{(\sqrt{3})^2 - (\sqrt{2})^2} \\ &= \frac{6\sqrt{3}-6\sqrt{2}}{3-2} = 6\sqrt{3}-6\sqrt{2}. \end{aligned}$$

**EXERCISES**

1. Simplify each of the following.

(a)  $\sqrt{12} \times \sqrt{3}$

(d)  $\sqrt{20} \times \sqrt{50}$

(g)  $5\sqrt{8} \times 4\sqrt{32}$

(b)  $\sqrt{24} \times \sqrt{75}$

(e)  $6\sqrt{2} \times 3\sqrt{8}$

(h)  $5\sqrt{8} \times 8\sqrt{6}$

(c)  $\sqrt{2} \times \sqrt{90}$

(f)  $6\sqrt{3} \times 8\sqrt{6}$

2. Simplify each of the following.

(a)  $2\sqrt{3}(\sqrt{3}+\sqrt{2})$

(c)  $3\sqrt{5}(\sqrt{45}-\sqrt{20})$

(b)  $3\sqrt{2}(\sqrt{32}+\sqrt{18})$

(d)  $2\sqrt{6}(2\sqrt{3}-\sqrt{12})$

3. Simplify the following expressions.

(a)  $\sqrt{125} \div \sqrt{27}$

(d)  $\sqrt{216} \div \sqrt{18}$

(g)  $40\sqrt{80} \div 8\sqrt{5}$

(b)  $\sqrt{72} \div \sqrt{8}$

(e)  $4\sqrt{6} \div 2\sqrt{2}$

(h)  $9\sqrt{15} \div 3\sqrt{3}$

(c)  $\sqrt{96} \div \sqrt{12}$

(f)  $20\sqrt{8} \div 5\sqrt{2}$

4. Rationalize the denominator of the following surds.

(a)  $\frac{15}{\sqrt{5}}$

(b)  $\frac{25}{\sqrt{10}}$

(c)  $\frac{3\sqrt{2}}{\sqrt{3}}$

(d)  $\frac{\sqrt{6}}{\sqrt{8}}$

5. Rationalize the denominator of the following surds.

(a)  $\frac{4}{3+\sqrt{2}}$

(b)  $\frac{3}{\sqrt{3}-2}$

(c)  $\frac{1}{\sqrt{5}-\sqrt{2}}$

(d)  $\frac{5}{2\sqrt{2}-\sqrt{3}}$

When money is deposited in a bank, it commonly attracts interest; in a similar way, a borrower must normally pay interest on money borrowed. In this sub-topic, we will learn *compound interest* in relation to *simple interest*.

### ACTIVITY 6

1. What does the word percent mean?
2. What is rate of interest?
3. Suppose you deposit L\$250 in a bank account paying 6%:
  - (a) simple interest and leave it for two years. What will be the balance on the account at the end of that time?
  - (b) compound interest and leave it for two years. What will be the balance on the account at the end of that time?
4. In equation 3 above, is there any difference between part(a) and (b) of the the balance of the account at the end of two years? So, what can you say about simple and compound interest?

The money borrowed is called the *principal*, usually denoted by P.

The extra money paid is called *interest*, usually denoted by I.

The amount of interest depends on the principal; that is the interest is certain percentage of the principal. This percent is called the *rate* of interest, or *rate*.

When money is borrowed, the borrower agrees to pay back the principal and the interest within a specified period of time.

There are two basic ways of calculating the amount of interest paid on money deposited: *simple interest* and *compound interest*.

## Simple Interest

If simple interest is paid, interest is calculated only on the principal P, the amount deposited (the original capital sum).

### EXAMPLE 14

John invests L\$2,500 in a bank account. At the end of the year his account is credited with 8% interest. How much interest had his L\$2,500 earned in the year? two years? three years?

#### Solution

In one year, interest = 8% of L\$2,500

$$= \frac{8}{100} \times \text{L\$}2,500 = \text{L\$}200$$

Since the interest is calculated on the original deposit,  
the interest in two years = twice of 8% of L\$2,500

$$= 2 \times \frac{8}{100} \times \text{L\$}2,500 = \text{L\$}400$$

the interest in three years = three times of 8% of L\$2,500

$$= 3 \times \frac{8}{100} \times \text{L\$}2,500 = \text{L\$}600 .$$

### EXAMPLE 15

James invests L\$14,000 in an account that pays  $r\%$  interest. After the first year he receives L\$840 interest. What is the value of  $r$ , the rate of interest?

#### Solution

After one year, the amount of interest =  $r\%$  of L\$14,000

$$\text{L\$}840 = \frac{r}{100} \times \text{L\$}14,000$$

$$\text{L\$}840 \times 100 = r \times \text{L\$}14,000$$

$$r = \frac{\text{L\$}840 \times 100}{\text{L\$}14,000} = 6$$

Therefore, the interest rate is 6%.

## Compound Interest

When the interest due at the end of a certain period is added to the principal and that sum earns interest for the next period, the interest paid is called **compound interest**. This is when interest is added (or compounded) to the principal sum so that interest is paid on the whole amount. Under this method, if the interest for the first year is left in the account, the interest for the second year is calculated on the whole amount so far accumulated.

### EXAMPLE 16

A certain firm deposits L\$25,000 in a bank account paying 8% compound interest and leaves it for three years. At the end three years what will be the balance on the account?

#### Solution

Balance after 0 years = L\$25,000 ..... (Initial deposit )

Interest: 8% of L\$25,000 = L\$2,000

Balance after 1 year = L\$25,000 + L\$2,000 = L\$27,000.

Interest: 8% of L\$27,000 = L\$2,160

Balance after 2 years = L\$27,000 + L\$2,160 = L\$29,160.

Interest: 8% of L\$29,160 = L\$2,332.80

Balance after 3 years = L\$29,160 + L\$2,332.80 = L\$31,492.80.

### EXAMPLE 17

Mary's invests L\$12,000 in a bank account which pays interest at the rate of 6% per annum. Calculate the value of her investment after 4 years.

#### Solution

At an interest rate of 6% per annum, the value of her investment after one year is:

$$\begin{aligned} \text{L\$12,000} + \frac{6}{100} \times \text{L\$12,000} &= \text{L\$12,000} + 0.06 \times \text{L\$12,000} = 1.06 \times \text{L\$12,000} \\ &= \text{L\$12,720.} \end{aligned}$$

After two years, the investment is worth

$$1.06 \times \text{L\$12,720} = \text{L\$13,483.20}$$

and after three years,

$$1.06 \times \text{L\$13,483.20} = \text{L\$14,292.20}$$

At the end of four years, the value of Mary's investment will be:

$$1.06 \times \text{L\$14,292.20} = \text{L\$15,149.73.}$$

### EXERCISES

- Calculate the interest payable and the total amount owing on the following deposits at simple interest.
  - L\$1,500 borrowed for 5 years at 8% per annum.
  - L\$2,000 invested for 4 years at 9.5% per annum.
  - L\$500 borrowed for 2 years at 10% per annum.
  - L\$2,500 invested for 6 months at 8% per annum.
- What is the actual rate of simple interest if L\$4,000 deposited for 3 years attracts interest of L\$1,440?
- For how long would L\$1,000 have to be left in an account paying 5% compound interest per annum to give a balance of L\$2,000?
- A boy borrows L\$100 from his sister and promises to pay back L\$1.00 a month later. What is this as an annual rate of simple interest?

5. Calculate the amount accumulated after three years at compound interest in the following cases.
- L\$500 deposited at 15% per annum.
  - L\$1000 borrowed at 10% per annum.
  - L\$1200 borrowed at 8% per annum.
  - L\$25 000 borrowed at 12% per annum.

The interest formula includes two types of interests. These are simple interest and compound interest. In this sub-topic, we will learn both types of formulas.

### ACTIVITY 7

- What do you call the amount of money a lender receives for lending out money?
- Robert borrowed L\$2,000 from a bank for a period of  $t$  years. If the bank charged simple interest at a rate of 6% per annum, how much interest is to be paid by him to the bank?
- Patricia deposits L\$2,000 in a bank account paying 6% compound interest and leave it for  $t$  years. What will be the balance on the account at the end of  $t$  years compounded yearly?

## Simple Interest Formula

When the money paid as interest is the same every year, the interest is called *simple interest*.

### Key Point

- The formula for calculating simple interest is:  
$$I = P \times r \times t$$
- where,  $I$  is interest earned,  $P$  is principal (amount invested),  $r$  is rate of interest (as a decimal) and  $t$  = time
- The total amount owing can then be calculated by adding  $I$  to  $P$ . That is,  
$$A = P + I$$

### EXAMPLE 18

Michael borrowed L\$35,800 from a bank for a period of 5 years. If the bank charged simple interest at rate of 9% per annum, how much interest was paid by the borrower? Find the total amount paid.

#### Solution

Here, the principal,  $P$  is L\$35,800, rate of interest  $R$  is 9% or 0.09.

Using the simple interest formula, we have

$$I = P \times r \times t = \text{L}\$35,800 \times 0.09 \times 5 = \text{L}\$16,110.$$

Therefore, the man paid L\$16,110 as the interest to the bank.

$$A = P + I = \text{L}\$35,800 + \text{L}\$ 16,110 = \text{L}\$51,910.$$

Therefore, the total amount at the end of the second year is L\$51,910.

The following are the main formulas used in calculating *simple interest*, *principal*, *time*, *rate* and *amount*.

1. To find *simple interest*, we use the formula:

$$I = P \times r \times t$$

2. To find the *principal*, we use the formula:

$$P = \frac{I}{r \times t}$$

3. To find the *time* (number of years) we use the formula:

$$t = \frac{I}{P \times r}$$

4. To find the *rate of interest*, we use the formula:

$$r = \frac{I}{P \times t}$$

5. To find the *amount*, we use the formula:

$$A = P + I$$

### EXAMPLE 19

A sum of money was deposited for 5 years at the rate of 6% simple interest. Calculate the sum deposited if the simple interest received by the depositor is L\$3,685.

#### Solution

Let  $P$  be the sum deposited. Given that  $I = \text{L}\$ 3,685.50$ ,  
 $T = 5$  years and  $R = 6\% = 0.06$ . Thus, using formula (2) we have,

$$P = \frac{I}{r \times t} = \frac{\text{L}\$3,685.50}{0.06 \times 5} = \frac{\text{L}\$3,685.50}{0.3} = \text{L}\$ 12,285.$$

Therefore, the sum deposited was L\$12,285.

### EXAMPLE 20

At what rate of simple interest will the sum of L\$ 2,950 becomes L\$5,428 in 7 years?

**Solution**

Here,  $P = \text{L\$}2,950$ ;  $T = 7$  years and  $A = \text{L\$}5,428$ . Thus,

$$I = A - P = \text{L\$}5,428 - \text{L\$}2,950 = \text{L\$}2,478.$$

Hence, using formula (4), we have

$$r = \frac{I}{P \times t} = \frac{\text{L\$}2,478}{\text{L\$}2,950 \times 7} = \frac{\text{L\$}2,478}{\text{L\$}20,650} = 0.12.$$

Therefore, the rate of interest is 0.12 or 12%.

**EXAMPLE 21**

For how long would  $\text{L\$}5,000$  have to be deposited in an account paying 4% simple interest per annum to give a balance of  $\text{L\$}6,000$ ?

**Solution**

$$I = A - P = \text{L\$}6000 - \text{L\$}5000 = \text{L\$}1000.$$

$$t = \frac{I}{P \times r} = \frac{\text{L\$}1,000}{\text{L\$}5,000 \times 0.04} = \frac{1,000}{200} = 5$$

Therefore, 5 years is required to get the given balance.

**Compound Interest Formula**

The compound interest is calculated after calculating the total amount over a period of time based on the rate of interest and the initial principal.

**EXAMPLE 22**

William invests  $\text{L\$}20,000$  in a bank account which pays compound interest at the rate of 7% per annum. Calculate the value of his investment after 3 years.

**Solution**

At an interest rate of 7% per annum, the value of his investment after one year is

$$\begin{aligned} A &= P + P \times \frac{r}{100} \times t \\ &= \text{L\$}20,000 + \text{L\$}20,000 \times 0.07 \\ &= \text{L\$}20,000(1 + 0.07) \\ &= \text{L\$}20,000(1.07)^1 = \text{L\$}21,400. \end{aligned}$$

After two years, the investment is worth

$$\begin{aligned} A &= \text{L\$}20,000(1.07) + \text{L\$}20,000(1.07) \times 0.07 \\ &= \text{L\$}20,000(1.07)[1 + 0.07] \\ &= \text{L\$}20,000(1.07)(1.07) \\ &= \text{L\$}20,000(1.07)^2 = \text{L\$}22,898. \end{aligned}$$

At the end of three years, the value of William's investment will be:

$$\begin{aligned} A &= \text{L\$}20,000(1.07)^2 + \text{L\$}20,000(1.07)^2 \times 0.07 \\ &= \text{L\$}20,000(1.07)^2[1 + 0.07] \\ &= \text{L\$}20,000(1.07)^2(1.07) \\ &= \text{L\$}20,000(1.07)^3 = \text{L\$}24,500.86. \end{aligned}$$

Similarly, at the end of  $t$  years, the value of William's investment will be:

$$A = \text{L\$}20,000(1.07)^t$$

### Key Point

- If a principal  $P$  is invested at an annual rate of  $r$ , compounded annually, then the amount after  $t$  years is given by

$$A = P(1+r)^t$$

### EXAMPLE 23

Linda deposits L\$10,000 in a bank that pays 8% compound interest. How much money will she have after 11 years?

#### Solution

**Given:**  $P = 10,000$ ,  $r = 8\% = 0.08$  and  $t = 11$ . **Required:**  $A = ?$

$$A = P(1+r)^t \Rightarrow A = 10,000(1+0.08)^{11} = 10,000(1.08)^{11} = 23,316.40.$$

Therefore, after 11 years, Linda will have L\$23,316.40

Suppose a sum of money called *principal*  $P$ , is deposited in a bank that pays an annual interest rate  $r$  compounded at a regular interval of time  $n$  times per year. Then, the compounding period is  $\frac{12}{n}$  months. The interest rate per period is given by  $\frac{1}{n}$ . In  $t$  years there are  $nt$  compounding periods.

### Key Point

- If a sum of money called *principal*  $P$ , is deposited in a bank, that pays an annual interest rate  $r$  compounded  $n$  times per year, the total accumulated amount  $A$  in  $t$  years is given by:

$$A = P \left( 1 + \frac{r}{n} \right)^{nt}$$

### EXAMPLE 24

If Susan invested L\$12,000 at compound interest 8%, then what will be the amount of money at the end of 3 years compounded,

- (a) yearly                                      (b) semiannually?                                      (c) quarterly?

#### Solution

- (a) **Given:**  $P = \text{L}\$12,000$ ,  $r = 8\% = 0.08$ ,  $n = 1$  and  $t = 3$ . **Required:**  $A = ?$

$$A = P \left( 1 + \frac{r}{n} \right)^{nt} = 12000 \left( 1 + 0.08 \right)^{1 \times 3} = 12000 (1.08)^3 = 15,116.54$$

Therefore, after 3 years, Susan will have L\$15,116.54

- (b) **Given:**  $P = \text{L}\$12,000$ ,  $r = 8\% = 0.08$ ,  $n = 2$  and  $t = 3$ . **Required:**  $A = ?$

$$A = P \left( 1 + \frac{r}{n} \right)^{nt} = 12000 \left( 1 + \frac{0.08}{2} \right)^{2 \times 3} = 12000 (1.04)^6 = 15,183.83.$$

Therefore, after 3 years, Susan will have L\$15,183.83.

- (c) **Given:**  $P = \text{L}\$12,000$ ,  $r = 8\% = 0.08$ ,  $n = 4$  and  $t = 3$ . **Required:**  $A = ?$

$$A = P \left( 1 + \frac{r}{n} \right)^{nt} = 12000 \left( 1 + \frac{0.08}{4} \right)^{4 \times 3} = 12000 (1.01)^8 = 15,218.90.$$

Therefore, after 3 years, Susan will have L\$15,218.90.

### EXERCISES

- How much interest would be earned on an investment of L\$24,500 at 4.5% interest for 2 years?
- For how long would L\$5,000 have to be left in an account paying 4% of simple interest per annum to give a balance of L\$6,000?
- William deposited L\$4,340 in an account. At the end of the three years if he received L\$130.20 interest. What was the rate of interest?
- Jessica deposits L\$2000 in a savings account at a bank, which has a rate of 6%.

- (a) Find the amount,  $A$ , of money in the account after 3 years subject to compound interest.
  - (b) Find the total interest during the three years
5. Richard borrowed L\$500 for four years and agree to pay 6 % compound interest for this period. What amount will he have to pay back?
  6. The value of a computer depreciates at a rate of 20% per annum. A new computer costs L\$1200.
    - (a) What will be its value after 2 years?
    - (b) What will be its value after 6 years?
    - (c) What will be its value after 10 years?
  7. Sarah deposits L\$7500 in a savings account that has a rate of 6%. The interest is compounded monthly.
    - (a) How much money will she have after five years?
    - (b) Find the interest after five years.
  8. Suppose L\$25,000 is deposited in a saving account paying an annual interest rate of 8%. How much money is accumulated in 5 years if interest is compounded:
    - (a) yearly
    - (b) semiannually?
    - (c) quarterly?
    - (d) monthly?

All fixed assets such as machinery, building, furniture etc. gradually diminish in value as they get older and becomes worn out by constant use in business.

### ACTIVITY 8

1. What is depreciation?
2. A computer was bought for L\$1,200. Its value depreciated at the rate of 12% per year. Find its value after two years.
3. What is hire purchase?
4. What are the advantages of hire purchase? Discuss with your friends.

The monetary value of an asset decreases over time due to use, wear and tear or obsolescence. This decrease is measured as depreciation.

**EXAMPLE 25**

A motorcycle was bought for L\$6,250. Its value depreciated at the rate of 9% per annum. Find its value after two years.

**Solution**

Present value of motorcycle after 0 years = L\$6,250

Depreciated rate: 8% of L\$6,250 =  $0.08 \times \text{L\$}6,250 = \text{L\$}500$ .

Value of the motorcycle after 1 year =  $\text{L\$}6,250 - \text{L\$}500 = \text{L\$}5,750$ .

Interest: 8% of L\$5,750 =  $0.08 \times \text{L\$}5,750 = \text{L\$}460$

Value of the motorcycle after 2 years =  $\text{L\$}5,750 - \text{L\$}460 = \text{L\$}5,290$ .

When the value of an item decreases with increase of time, then we say that the value of the item depreciates year by year. So, depreciation is the process of deducting the total cost of something bought for business.

**Key Point**

- If the rate of depreciation is  $r$  (as a decimal) per year and the present value of the asset is  $V$ , then the depreciated value,  $A$ , at the end of  $t$  years is given by

$$A = V(1 - r)^t$$

**EXAMPLE 26**

In 2022, the value of a new car is L\$1,350,000. It depreciates 5% each year. How much will the car be worth in 2027?

**Solution**

**Given:**  $V = \text{L\$}1,350,000$ ,  $r = 5\% = 0.05$  and  $t = 2027 - 2022 = 5$  years.

**Required:**  $A = ?$

$$A = V(1 - r)^t = 135000(1 - 0.05)^5 = 135000(0.95)^5 = 1,044,604.27$$

Therefore, the value of the car in 2027 will be L\$1,044,604.27.

**EXAMPLE 27**

A printing company bought a machine for L\$250,000. The rate of depreciation is 10%. Find the depreciated value of the machine after 3 years. Also, find the amount of depreciation.

**Solution**

**Given:**  $V = \text{L\$}250,000$ ,  $r = 10\% = 0.1$  and  $t = 3$  years.

**Required:**  $A = ?$  and the amount of depreciation.

$$A = V(1 - r)^t = 25000(1 - 0.1)^3 = 135000(0.9)^3 = 182,250$$

Therefore, the value of the machine will be L\$182,250.

$$\text{Amount of depreciation} = V - A$$

$$= 250,000 - 182,250 = 67,750.$$

Therefore, the amount of depreciation after 3 years is L\$67,750.

While a type of credit called personal contract purchase is perhaps the most popular option, the likes of leasing and *hire purchase* have been around for a while. Hire purchase is an arrangement for buying something, where the buyer makes an initial down payment and pays the balance plus interest in installments.

### EXAMPLE 28

A pair of shoes costs L\$2,000 when paid in cash. But there is an option to buy it under the hire purchase scheme, which is that one has to pay a deposit of 40% of cash price and 8 weekly installments of L\$156.25. Calculate the purchase price and the extra amount under this hire purchase scheme.

#### Solution

$$\text{Initial deposit} = \frac{40}{100} \times 2000 = 800.$$

$$\text{Total amount paid in Installments} = 156.25 \times 8 = \text{L\$1,250}.$$

$$\begin{aligned} \text{Total amount of the pair of shoes under hire purchase scheme} &= 800 + 1,250 \\ &= \text{L\$2050}. \end{aligned}$$

$$\text{Extra money paid} = 2050 - 2,000 = \text{L\$50}.$$

### EXERCISES

1. The present cost of an article is L\$20,000. What would be its cost after 2 years if the rate of inflation is 30%?
2. A laptop computer depreciates by 25% per year. If it costs L\$2,000, what will it cost after 2 years?
3. A man purchased a car in L\$500,000. In the first year of purchase, it depreciated 18%. What will be the value of the car after one year?
4. A car bought for Birr 1,800,000 depreciates by 10% a year. After, how many years can one buy the car at about half its original price?
5. The value of a new car in 2022 is L\$1,500,000. It depreciates 8% each year. How much will the car be worth in 2030?

6. Under a scheme, the purchase price of a refrigerator was L\$7,000, if 15% of the total amount is to be paid as deposit and the rest is to be paid with interest rate of 10% per annum in 2 years in 2 equal installments. Calculate the amount of each installment.

### KEY TERMS

- Surds
- Principal
- Hire purchase
- Percentage
- Depreciation
- Present value
- Rationalize the denominator
- Simple Interest
- Compound interest
- Rate of interest

### SUMMARY

- A general surd is an irrational number of the form  $a\sqrt[n]{p}$ , where  $a$  is a rational number,  $\sqrt[n]{p}$  is an irrational number and  $\sqrt[n]{\phantom{x}}$  is called a radical.
- If a positive whole number is not a perfect square, then its square root is called a surd. A surd cannot be written as a fraction, and is an example of an irrational number.
- When simplifying surds, look for square factors of the number inside the square root and then use
  - (i)  $\sqrt{a \times a} = a$
  - (ii)  $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$
  - (iii)  $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$
- The following formulas may be used to simplify expressions involving surds:
  - (i)  $\sqrt{a} \times \sqrt{a} = a$
  - (ii)  $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
  - (iii)  $a\sqrt{d} \times b\sqrt{d} = abd$
- The following formulas may be used to simplify expressions involving surds:
  - (i)  $\sqrt{a} \div \sqrt{a} = 1$
  - (ii)  $\sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} = \sqrt{a \div b}$
  - (iii)  $a\sqrt{b} \div c\sqrt{d} = \frac{a\sqrt{b}}{c\sqrt{d}} = \frac{a}{c} \sqrt{\frac{b}{d}}$

- A fraction in the form of  $\frac{a}{\sqrt{b}}$  is a surd, when denominator  $\sqrt{b}$  is a surd. To convert a surd  $\frac{a}{\sqrt{b}}$  into a rational number, multiply both numerator and denominator by  $\sqrt{b}$ . That is,

$$\frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{a \times \sqrt{b}}{\sqrt{b} \times \sqrt{b}} = \frac{a\sqrt{b}}{b}.$$

- If the denominator of a fraction contains an expression of the form:
  - (i)  $\sqrt{n} + m$ , then rationalize the denominator by multiplying both numerator and denominator by  $\sqrt{n} - m$ .
  - (ii)  $\sqrt{n} - m$ , then rationalize the denominator by multiplying both numerator and denominator by  $\sqrt{n} + m$ .
  - (iii)  $\sqrt{n} + \sqrt{m}$ , then rationalize the denominator by multiplying both numerator and denominator by  $\sqrt{n} - \sqrt{m}$ .
  - (iv)  $\sqrt{n} - \sqrt{m}$ , then rationalize the denominator by multiplying both numerator and denominator by  $\sqrt{n} + \sqrt{m}$ .
- The formula for calculating simple interest is:

$$I = P \times r \times t.$$

where,  $I$  is interest earned,  $P$  is principal (amount invested),  $r$  is rate of interest (as a decimal) and  $t$  = time

The total amount owing can then be calculated by adding  $I$  to  $P$ . That is,

$$A = P + I.$$

- The following are the main formulas used in calculating **simple interest**, **principal**, **time**, **rate** and **amount**.
  - (i) To find **simple interest**, we use the formula:  $I = P \times r \times t$ .
  - (ii) To find the **principal**, we use the formula:  $P = \frac{I}{r \times t}$ .
  - (iii) To find the **time** (number of years) we use the formula:  $t = \frac{I}{P \times r}$ .
  - (iv) To find the **rate of interest**, we use the formula:  $r = \frac{I}{P \times T}$ .
  - (v) To find the **amount**, we use the formula:  $I = A - P$ .

- If a principal  $P$  is invested at an annual rate of  $r$ , compounded annually, then the amount after  $t$  years is given by

$$A = P(1 + r)^t.$$

- If a sum of money called **principal  $P$** , is deposited in a bank, that pays an annual interest rate  $r$  compounded at a regular interval of time  $n$  times per year, the total accumulated amount  $A$  in  $t$  years is given by:

$$A = P \left( 1 + \frac{r}{n} \right)^{nt}.$$

- If the rate of depreciation is  $r$  (as a decimal) per year and the present value of the asset is  $V$ , then the depreciated value,  $A$ , at the end of  $t$  years is given by

$$A = V(1 - r)^t.$$

### EXERCISES

- Determine whether the following real numbers are surds or not.

(a)  $\sqrt{28}$

(c)  $\sqrt{98}$

(e)  $\sqrt{169}$

(b)  $\sqrt{48}$

(d)  $\sqrt{363}$

(f)  $\sqrt{392}$

- Decide if the following numbers are surds.

(a)  $\frac{3\sqrt{8}}{\sqrt{2}}$

(c)  $\sqrt{\frac{24}{75}}$

(e)  $\frac{\sqrt{125}}{\sqrt{12}}$

(b)  $\frac{2\sqrt{2}}{5}$

(d)  $\frac{\sqrt{80}}{\sqrt{20}}$

(f)  $\sqrt{\frac{63}{567}}$

- Simplify the following surds.

(a)  $\sqrt{40}$

(c)  $\sqrt{216}$

(e)  $\sqrt{1125}$

(g)  $\sqrt{243}$

(b)  $\sqrt{800}$

(d)  $\sqrt{256}$

(f)  $\sqrt{5400}$

(h)  $\sqrt{972}$

- Simplify the following surds.

(a)  $2\sqrt{63}$

(b)  $3\sqrt{45}$

(c)  $2\sqrt{108}$

(d)  $4\sqrt{500}$

- Simplify the following surds.

(a)  $\frac{\sqrt{27}}{3}$

(b)  $\frac{\sqrt{72}}{6}$

(c)  $\frac{\sqrt{240}}{200}$

(d)  $\frac{\sqrt{63}}{3}$

- (e)  $\frac{2\sqrt{45}}{3}$       (f)  $\frac{2\sqrt{600}}{30}$       (g)  $\frac{3\sqrt{147}}{28}$       (h)  $\frac{3\sqrt{98}}{7}$
6. Simplify the following.
- (a)  $\sqrt{\frac{15}{16}}$       (b)  $\sqrt{\frac{27}{49}}$       (c)  $\sqrt{\frac{108}{49}}$       (d)  $\sqrt{\frac{80}{125}}$
7. Express the following surds as a square root of a positive integer (or as  $\sqrt{n}$ )
- (a)  $6\sqrt{3}$       (b)  $11\sqrt{2}$       (c)  $17\sqrt{2}$       (d)  $7\sqrt{10}$
8. Simplify each of the following.
- (a)  $\sqrt{3} \times \sqrt{72}$       (c)  $3\sqrt{32} \times 2\sqrt{72}$   
 (b)  $\sqrt{50} \times \sqrt{80}$       (d)  $3\sqrt{12} \times 2\sqrt{24}$
9. Simplify each of the following.
- (a)  $2\sqrt{5}(2\sqrt{3} - \sqrt{2})$       (c)  $\sqrt{5}(\sqrt{80} - \sqrt{20})$   
 (b)  $2\sqrt{3}(\sqrt{48} + \sqrt{18})$       (d)  $3\sqrt{2}(7\sqrt{3} - \sqrt{48})$
10. Simplify the following expressions.
- (a)  $\sqrt{320} \div \sqrt{80}$       (c)  $20\sqrt{80} \div 4\sqrt{5}$   
 (b)  $\sqrt{162} \div \sqrt{54}$       (d)  $12\sqrt{18} \div 3\sqrt{32}$
11. Rationalize the denominator of the following surds.
- (a)  $\frac{12}{\sqrt{6}}$       (b)  $\frac{20}{\sqrt{5}}$       (c)  $\frac{2\sqrt{3}}{\sqrt{2}}$       (d)  $\frac{\sqrt{8}}{\sqrt{12}}$
12. Rationalize the denominator of the following surds.
- (a)  $\frac{3}{2+\sqrt{3}}$       (b)  $\frac{5}{\sqrt{5}-2}$       (c)  $\frac{4}{\sqrt{3}-\sqrt{2}}$       (d)  $\frac{9}{\sqrt{2}+\sqrt{3}}$
13. For how long would L\$2,000 have to be left in an account paying 5% interest per annum to give a balance of L\$3,000?
14. A boy borrows L\$300 from his sister and promises to pay back L\$3.00 a month later. What is this as an annual rate of interest?
15. Calculate the amount accumulated after three years at compound interest in the following cases.
- (a) L\$600 deposited at 9% per annum.

- (b) L\$2000 borrowed at 10% per annum.
16. How much interest would be earned on an investment of L\$20,000 at 5% for 3 years?
17. For how long would L\$8,000 have to be left in an account paying 4% of simple interest per annum to give a balance of L\$10,000?
18. William invests L\$2,500 in an account. After the first year he receives L\$150 interest. What is the rate of interest?
19. Jessica deposits L\$4000 in a savings account in a bank, which has a rate of 8%.
- (a) Find the amount,  $A$ , of money in the account after 5 years subject to compound interest.
- (b) Find the interest.
20. Richard borrowed L\$5,000 for four years and agree to pay 8 % compound interest for this period. What amount will he have to pay back?
21. The value of a computer depreciates at a rate of 10% per annum. A new computer costs L\$1000.
- (a) What will be its value after 3 years?
- (b) What will be its value after 5 years?
22. Sarah deposits L\$8,000 in a savings account that has a rate of 8%. The interest is compounded monthly.
- (a) How much money will she have after 3 years?
- (b) Find the interest after 3 years.
23. Suppose L\$5,000 is deposited in a saving account paying an annual interest rate of 10%. How much money is accumulated in 3 years if interest is compounded:
- (a) yearly (c) quarterly?
- (b) semiannually? (d) monthly?
24. A laptop computer depreciates by 20% per year. If it costs L\$3,000, what will it cost after 3 years?
25. A man purchased a car in L\$800,000. For the first year of purchase, it depreciated 12%. What will be the value of the car after one year?



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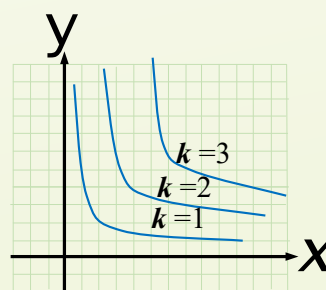
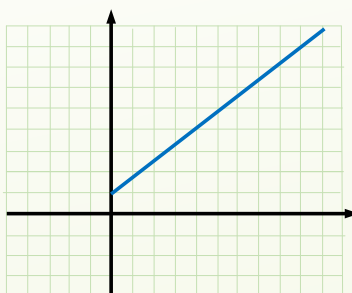
# CHAPTER

# 4

## VARIATION

### Chapter Contents

- 4.1 Direct Variation
- 4.2 Inverse Variation
- 4.3 Joint Variation
- 4.4 Partial Variation
  - Key Terms
  - Summary
  - Exercises



## **Chapter Outcomes**

By the end of this chapter, you will be able to:

- define and solve variation;
- define and solve inverse variation;
- define and solve joint variation;
- define and solve partial variation.

## Introduction

In mathematics, we usually deal with two types of quantities. These are variables and constants. If the value of a quantity remains unaltered under different situations, it is called a **constant**. On the contrary, if the value of a quantity changes under different situations, it is called a **variable**. In this topic, we will learn about direct variation, indirect variation, Joint variation and partial variation.

If two quantities are related in such a way that if an increase in one quantity produces a proportional increase in another quantity or if decrease in one quantity produces a proportional decrease in another quantity, then the two quantities are known as **direct variations**.

### ACTIVITY 1

Think of a number  $n$ , multiply the number  $n$  by 8, divide the result by 2, add 5, and then subtract 4 times the original number  $n$

Side length ( $s$ ) in cm	1	2	3	4	5	6	7	8	9	10
Perimeter ( $P$ ) in cm	4	8	12	16	20	24	28	32	36	40

- Answer the following questions by referring the table given above.
  - What happens to the perimeters, when the length of sides of a square is doubled? Tripled?
  - What will be the lengths of side of a square, if the perimeter of the square is 120 cm? Write a mathematical statement that will relate the two quantities involved?
  - What happens to the length of side, when the perimeter of the square is doubled? Tripled?
  - What will be the perimeter of a square, if the length of side of the square is 40 cm? Write a mathematical statement that will relate the two quantities involved?
- If  $x$  and  $y$  vary directly, as  $x$  decreases, what happens to the value of  $y$ ?
- Which of the following shows direct variation?
  - The cost ( $C$ ) of pen and the money required to buy the pen.
  - The time ( $t$ ) taken to travel and the distance ( $d$ ) covered.
  - The time ( $t$ ) required to perform the given amount of work to the number of persons ( $p$ ) working.
  - The area ( $A$ ) of a rectangle to its width.

4. Write a mathematical equation for the following statements.
- The cost ( $C$ ) of a rice varies directly as its weight ( $W$ ).
  - The circumference ( $C$ ) of a circle varies directly as the length of its radius.
5. Write an equation where  $y$  varies directly as  $x$ .
- $y = 28$  when  $x = 7$
  - $y = 6$  when  $x = 8$
  - $y = 12$  when  $x = 3$
  - $y = 63$  when  $x = 81$

When two quantities are related to each other, their changes may follow a certain pattern.

### EXAMPLE

Suppose the cost of 1 kg of sugar is L\$2, then what would be the cost of 2 kg? 3 kg? 4 kg? 5 kg? and 6 kg?

#### Solution

Cost of 1 kg of sugar is  $1 \times \text{L\$}2$ . Similarly, cost of 2 kg is  $2 \times \text{L\$}2 = \text{L\$}4$ ; cost of 3 kg is  $3 \times \text{L\$}2 = \text{L\$}6$ ; cost of 4 kg is  $4 \times \text{L\$}2 = \text{L\$}8$ ; cost of 5 kg is  $5 \times \text{L\$}2 = \text{L\$}10$ ; and cost of 6 kg is  $6 \times \text{L\$}2 = \text{L\$}12$ .

These pairs of values are shown in the table given below.

Weight ( $W$ ) of sugar (in kg)	1	2	3	4	5	6
Cost ( $C$ ) (in Dollar)	2	4	6	8	10	12

From the table, we observe that, as the weight ( $W$ ) varies the cost ( $C$ ) also varies. Comparing each value of  $C$  to its corresponding value of  $W$ , we notice that all six sets of values result in the same ratio when reduced to its lowest terms:

$$\frac{W}{C} = \frac{2}{1} = \frac{4}{2} = \frac{6}{3} = \frac{8}{4} = \frac{10}{5} = \frac{12}{6} = 2.$$

If a relationship exists between two quantities(variables) so that their ratio is a constant, that relationship between the quantities(variables) is called a **direct variation**.

In every direct variation, we say that one quantity (variable) **varies directly** as the other or that one quantity (variable) is **directly proportional** to the other. The constant ratio is called a **constant of proportionality** (or **constant of variation**).

### DEFINITION

The quantity  $y$  is said to be in **direct proportion** to the quantity  $x$ , written as  $y \propto x$ , if there exists a nonzero constant number  $k$  such that

$$y = kx,$$

where  $k$  is called the **constant of variation** or **constant of proportionality**.

**EXAMPLE**

- (a) If  $y = 4x$ , then  $y$  is directly proportional to  $x$ .
- (b) If  $x = \frac{3}{5}y$ , then  $x$  is directly proportional to  $y$ .

It is important to indicate the order in which the variables are being compared before stating the constant of proportionality (or constant of variation).

**EXAMPLE**

- (a) In comparing  $y$  to  $x$ , if  $\frac{y}{x} = \frac{4}{1}$ , the constant of variation is 4.
- (b) In comparing  $x$  to  $y$ , if  $\frac{x}{y} = \frac{1}{4}$ , the constant of variation is  $\frac{1}{4}$ .

Two quantities  $x$  and  $y$  are said to be ***in direct proportion*** if they increase or decrease together in such a manner that the ratio of their corresponding values remains constant. That is,  $\frac{x}{y} = k$  where  $k$  is a positive number, if  $x$  and  $y$  are in direct variation or vary directly.

**EXAMPLE**

The table, shown below, gives pairs of values for the variables  $x$  and  $y$ .

$x$	1	2	3	10	?
$y$	8	16	24	?	1,600

- (a) Show that one variable varies directly as the other.
- (b) Find the constant of variation by comparing  $y$  to  $x$ .
- (c) Express the relationship between the variables as a formula.
- (d) Find the values missing in the table.

**Solution**

$$\frac{x}{y} = \frac{1}{8}; \frac{x}{y} = \frac{2}{16} = \frac{2 \div 2}{16 \div 2} = \frac{1}{8}; \text{ and } \frac{x}{y} = \frac{3}{24} = \frac{3 \div 3}{24 \div 3} = \frac{1}{8}.$$

Since the given pairs of values have the same ratio,  $x$  and  $y$  vary directly.

Constant of variation =  $\frac{y}{x} = \frac{8}{1} = 8$ . Thus, the constant of variation is 8.

Let us write a proportion that includes both the variables and constant of variation as

$$\frac{x}{y} = \frac{1}{8}. \text{ By applying the cross product, we obtain, } y = 8x.$$

When  $x = 10$ , find  $y$ . Thus,

$$y = 8x = 8(10) = 80$$

When  $y = 1,600$ , find  $x$ . Thus,

$$1,600 = 8x$$

$$x = 1,600 \div 8 = 200$$

Direct variation is the simplest type of variation and in practical life we can find many situations which can be co-related with direct variation.

#### EXAMPLE

The weight of 20 meters of copper wire is 0.9 kg. Find the weight of 170 meters of the same wire.

**W:**

Let us represent the problem in the table as follows:

Wire in meters	20	170
Weight in kg	0.9	$w$

Whenever the length of the wire increases, the weight of the wire also increases. Therefore, the length of the wire is directly proportional to the weight of the wire and the weight of the wire will be greater than 0.9 kg.

$$\frac{20}{170} = \frac{0.9}{w}$$

$$20 \times w = 0.9 \times 170$$

$$w = \frac{0.9 \times 170}{20} = \frac{153}{20} = 7.65$$

Therefore, 170 meters of wire weighs 7.65 kg.

**EXAMPLE**

A worker is paid L\$120 for 8 days work. If he works for 20 days, how much will he be paid?

**Solution**

Let us represent the problem in the table as follows:

Number of days	8	20
Ammount paid	120	$D$

Whenever the days increases, the amount of dollars paid also increases.

Therefore, the number of days is directly proportional to the amount of money he is paid and the amount of money will be greater than L\$120.

$$\frac{8}{20} = \frac{120}{D}$$

$$8 \times D = 20 \times 120$$

$$w = \frac{20 \times 120}{8} = \frac{2400}{8} = 300.$$

Therefore, he will be paid L\$300.

**EXERCISES**

- Which shows direct variation?
  - Speed ( $v$ ) of a vehicle and the distance ( $d$ ) travelled in a fixed time interval.
  - Change in weight and height among individuals.
  - The number ( $n$ ) of books and the amount of money ( $m$ ) required to buy the book.
  - The number of days to the color of cars crossing the road
- If  $A$  varies directly as  $B$  and the value of  $A$  is 60 when  $B$  is 40, what is the equation that describes this direct variation of  $A$  and  $B$ ?
- At light bulb factory, 4 out of every 25 light bulbs are defective. How many light bulbs would we expect to be defective out of 350 light bulbs?
- If 25 meters of cloth costs L\$ 337.50, then
  - what will be the cost of 40 meters of the same type of cloth?
  - what will be the length of the cloth bought for L\$ 810?

5. A baker uses 1800 grams of flour to make 3 loaves of bread. How much flour will he need to make:
  - (a) 2 loaves
  - (b) 5 loaves
  - (c) 9 loaves
  - (d) 4 loaves
6. If 10 liters of petrol costs L\$191.50, then calculate the cost of:
  - (a) 3 liters
  - (b) 7 liters
  - (c) 17 liters
  - (d) 37 liters
7. Henry bought 12 laptop computers for L\$12,600. Find the cost of 8 laptop computers.
8. If the weight of 12 sheets of thick paper is 40 grams, how many sheets of the same paper would weigh 2.5 kg?

If two quantities are related in such a way that if an increase in one quantity produces a proportional decrease in another quantity or if decrease in one quantity produces a proportional increase in another quantity, then the two quantities are known as ***inverse variations***.

### ACTIVITY 2

1. Consider the table values shown below.

$x$	2	4	6	8
$y$	24	12	8	6

- (a) What happens to the value of  $y$ , as  $x$  increases?
  - (b) What happens to the value of  $x$ , as  $y$  increases?
  - (c) Write a mathematical statement to represent the relation between  $x$  and  $y$ ?
  - (d) Find the constant of variation?
2. If  $y$  varies inversely as  $x$  and  $y = 48$  when  $x = 10$ , find  $y$  when  $x = 32$ .
  3. Which shows inverse variation?
    - (a) The number ( $n$ ) of students and the time ( $t$ ) required to clean the class.
    - (b) The time ( $t$ ) taken to travel and the speed ( $v$ ) to cover a fixed distance.
    - (c) The number ( $n$ ) of workers to the time ( $t$ ) taken to finish the job.
    - (d) The number of girls in a class to their height ( $h$ ).
  4. Express each of the following statements into mathematical equation.
    - (a) The number of hours ( $h$ ) required to complete a certain job varies inversely as the number of machine ( $m$ ) used to do the work.



(b) When  $v = 24$ , we get  $t = \frac{144}{24} = 6$  hours.

(c) Since  $d = vt$ , we can find the formula for  $v$  as  $v = \frac{d}{t}$ . Since 1 hour = 60 minutes, 90 minutes is equal to  $1\frac{1}{2}$  hours. Thus, when  $t = 1\frac{1}{2} = \frac{3}{2}$ , we get

$$v = \frac{d}{t} = \frac{144}{\frac{3}{2}} = 144 \times \frac{2}{3} = 96 \text{ km/hour.}$$

### EXAMPLE

Six pipes are required to fill a tank in 1 hour and 20 minutes. How long will it take if only 5 pipes of the same type are used?

#### Solution

Let the desired time to fill the tank be  $t$  minutes and let us represent the problem in the table shown below.

Number of pipes	6	5
Time in minutes	80	$t$

Whenever the number of pipes decreases, the time required to fill the tank increases. Therefore, the number of pipes is inversely proportional to the required time and the time required will be greater than 80 minutes.

$$\frac{6}{5} = \frac{t}{80}$$

$$5 \times t = 6 \times 80$$

$$t = \frac{6 \times 80}{5} = \frac{480}{5} = 96.$$

Therefore, time taken to fill the tank by 5 pipes is 96 minutes or 1 hour and 36 minutes.

### EXAMPLE

If 15 workers can build a wall in 48 hours, how many workers will be required to do the same work in 30 hours?

**Solution**

Let  $w$  be the number of workers employed to build the wall in 30 hours and let us represent the problem by the table given below.

Number of workers	15	$w$
Number of hours	48	30

Whenever the number of hours decreases, the number of workers to build the wall in 30 hours increases. Therefore, the number of workers is inversely proportional to the number of hours and the number of workers will be greater than 15 workers.

$$\frac{15}{w} = \frac{30}{48}$$

$$w \times 30 = 15 \times 48$$

$$w = \frac{15 \times 48}{30} = \frac{720}{30} = 24.$$

Therefore, 24 workers are required to finish the work in 30 hours.

**EXERCISES**

- Which shows direct variation?
  - The number of days to the number of machines to produce a given number of articles.
  - The width of a rectangle and its length when its area is fixed.
  - The money deposited in a bank and the interest earned.
  - The length of the road and the number of cars using the road.
- If  $x$  and  $y$  are inversely proportional and when  $x = 3$ ,  $y = 12$ , then determine whether each of the following is not a possible pair of corresponding values of  $x$  and  $y$ ?
 

(a) $x = 8$ and $y = 4$	(c) $x = 6$ and $y = 8$
(b) $y = 12$ and $x = 4$	(d) $y = 4$ and $x = 9$
- $y$  is inversely proportional to the variable  $x$ . When  $x = 7$ ,  $y = 9$ . Find the formula for  $y$  in terms of  $x$ .

4. It is given that  $y$  is inversely proportional to  $x$ . When  $x = 4$ ,  $y = 20$ . Find
  - (a) the value of  $y$  when  $x = 6$
  - (b) the value of  $x$  when  $y = 45$ .
5. A car covers a distance in 45 minutes with an average speed of 60 km per hour. What is the average speed to cover the same distance in 30 minutes?
6. If two men take 14 days to pick the apples from a tree. How many days will it take seven men to do the same job?
7. A farmer has enough food to feed 20 animals in his cattle for 6 days. How long would the food last if there were 10 more animals in his cattle?
8. A school has 8 periods a day each of 45 minutes duration. How long would each period be, if the school has 9 periods a day, assuming the number of school hours to be the same?

Joint variation is the same as direct variation with two or more quantities. Joint variation describes a situation where one variable depends on two or more other variables and varies directly to the product of each of the other quantities.

### ACTIVITY 3

1. Discuss the relation between direct and joint variation. What is their difference?
2.  $z$  varies jointly with  $x$  and  $y$ . If  $x = 2$ ,  $y = 3$  and  $z = 4$ , write the variation equation and find  $z$  when
$$x = 6 \text{ and } y = 2.$$
3.  $z$  varies inversely with  $x$  and directly with  $y$ . If  $x = 4$ ,  $y = 48$  and  $z = 2$ , write the variation equation and find  $x$  when  $z = 8$  and  $y = 96$ .
4. If  $z$  is jointly varying with  $x$  and  $y$  and  $z = 6$  when  $x = 3$  and  $y = 4$ , find  $z$  when  $x = 7$  and  $y = 4$ . Find the constant of variation.
5. The variable  $x$  is in joint variation with  $y$  and  $z$ . When the values of  $y$  and  $z$  are 4 and 6,  $x$  is 16. What is the value of  $x$  when  $y = 8$  and  $z = 12$ ?

### EXAMPLE

Area  $A$  of a parallelogram varies jointly as the base  $b$  and altitude  $h$  vary. That is,

$$A = kbh$$

**EXAMPLE**

If  $B$  varies jointly as  $A$  and  $C$  and  $B = 36$  when  $A = 24$  and  $C = 6$ , find  $C$  when  $B = 8$  and  $A = 15$ .

**Solution**

From the given information,  $B \propto AC$ , so  $B = kAC$ . Putting  $A = 18$ ,  $B = 36$  and  $C = 6$  we get,

$$36 = k \times 18 \times 6$$

$$36 = k \times 108$$

$$k = \frac{36}{108} = \frac{1}{3}$$

So, we will have the formula:  $B = \frac{1}{3}AC$

Now, when  $A = 15$ ,  $B = 6$ , we get

$$6 = \frac{1}{3} \times 15 \times C$$

$$6 = 5 \times C$$

$$C = \frac{6}{5}$$

**DEFINITION**

If more than two variables are related directly or one variable changes with the change product of two or more variables it is called as joint variation. If  $x$  is in joint variation with  $y$  and  $z$ , written as  $x \propto yz$ , if there exist a constant  $k$  such that:

$$x = kyz.$$

**EXAMPLE**

Find the equation of variation where  $a$  varies jointly as  $b$  and  $c$  and  $a = 36$  when  $b = 3$  and  $c = 4$ .

**Solution**

The equation for the given problem of joint variation is:  $A = kbc$

$$36 = k \times 3 \times 4$$

$$36 = 12k$$

$$k = \frac{36}{12} = 3$$

Therefore, the required equation of variation is:

$$A = 3bc$$

**EXAMPLE**

The variable  $x$  is in joint variation with  $y$  and  $z$ . When the values of  $y$  and  $z$  are 2 and 3,  $x$  is 16. What is the value of  $x$  when  $y = 8$  and  $z = 12$ ?

**Solution**

The equation for the given problem of joint variation is

$$x = kyz,$$

where  $k$  is the constant.

$$16 = k \times 2 \times 3$$

$$16 = 6k$$

$$k = \frac{16}{6} = \frac{8}{3}.$$

So, substituting the value of  $k$  the equation becomes:  $x = \frac{8}{3}yz$ .

Now, when  $y = 8$ , and  $z = 12$ , we get

$$x = \frac{8}{3} \times 8 \times 12 = 256.$$

**EXAMPLE**

The area  $A$  of a triangle varies jointly as the base  $b$  and the altitude  $h$  of the triangle. If  $A = 65 \text{ cm}^2$  when  $b = 10 \text{ cm}$  and  $h = 13 \text{ cm}$ , find the area of a triangle whose base is 8 cm and whose altitude is 11 cm.

**Solution**

$$A = kbh$$

$$65 = k \times 10 \times 13$$

$$65 = 130k$$

$$k = \frac{65}{130} = \frac{1}{2}.$$

Thus, the equation of variation is

$$A = \frac{1}{2}bh.$$

Now, when  $b = 8$ , and  $h = 11$ , we get

$$A = \frac{1}{2} \times 8 \text{ cm} \times 11 \text{ cm} = 44 \text{ cm}^2.$$

**EXERCISES**

1.  $z$  varies jointly with  $x$  and  $y$ . If  $x = 5$ ,  $y = 1$  and  $z = 10$ , write the variation equation and find  $z$  when  $x = 12$  and  $y = 2$ .
2.  $z$  varies jointly with  $x$  and  $y$ . If  $x = 7$ ,  $y = 3$  and  $z = 14$ , write the variation equation and find  $y$  when  $z = 8$  and  $x = 3$ .
3.  $A$  varies jointly with  $B$  and  $C$ .  $B = 8$ ,  $C = 3$  and  $A = 2$ , write the variation equation and find  $b$  when  $A = 12$  and  $B = 16$ .
4. A simple interest varies jointly as principal and time. If L\$2,500 left in an account for 5 years, then the interest is L\$625. How much interest would be earned, if you deposited L\$7,000 for 9 years.
5.  $A$  is in joint variation with  $B$  and square of  $C$ . Where  $A = 144$ ,  $B = 4$  and  $C = 3$ . Then, what is the value of  $A$  when  $B = 6$  and  $C = 4$ ?
6. The area  $A$  of a triangle varies jointly as the base  $b$  and height  $h$ . Area = 12 cm<sup>2</sup> when  $b = 6$  cm and  $h = 4$  cm. Find the base when  $A = 36$  m<sup>2</sup> and  $h = 6$  cm.
7.  $z$  varies jointly as  $x$  and  $y$ . If  $z = 16$  when  $x = 4$  and  $y = 6$ , find the constant of variation and the equation of the variation.
8. Find an equation of variation in which  $y$  varies jointly as  $x$  and  $z$  and  $y = 45$  when  $x = 18$  and  $z = 10$ . Find  $y$  when  $x = 20$  and  $z = 30$ .

A relationship between two variables in which the dependent variable is the sum of a constant number and a constant multiple of the independent variable.

**ACTIVITY 4**

1. What is partial variation? Discuss the difference between joint variation and partial variation.
2. Does the graph of an equation that represent partial variation pass through the origin?
3.  $x$  is partly constant and partly varies as  $y$ .  $x = 15$  when  $y = 3$  and  $x = 23$  when  $y = 5$ . What is the value of  $x$  when  $y = 10$ ?
4.  $m$  is partly constant partly varies as square of  $n$ . When  $n = 2$ ,  $m = 16$  and when  $n = 5$ ,  $m = 58$ .
  - (a) What is the relationship between  $m$  and  $n$ ?
  - (b) Find the value of  $m$  when  $n = 3$ .
5. Give some common formulas that are examples of partial variation.

Partial variation can be written in the form  $y = kx + c$ , where  $x$  and  $y$  are variables and  $k$  and  $c$  are some real constants.  $y = mx + c$  (where  $m$  and  $c$  are constants) is an *equation of straight line which can be an example of partial variation* because, we can see that  $y$  varies partially according to the value of  $x$ .

### DEFINITION

If there is a relationship between two variables where the dependent variable is the sum of a constant multiple of the independent variable and a constant number, then the relationship is called **partial variation**. In other words, a quantity  $y$  is partly constant and partly varies directly as  $x$ , then the law of variation can be expressed as

$$y \propto a + x,$$

That is,

$$y = a + kx,$$

where  $a$  and  $k$  are constants.

### EXAMPLE

$x$  is partly constant and partly varies as  $y$ . If  $x = 45$  when  $y = 10$  and  $x = 87$  when  $y = 24$ . Find

- (a) The formula connecting  $x$  and  $y$ .                      (b)  $x$  when  $y = 18$ .

#### Solution

(a)  $x = a + ky$

$$45 = a + 10k \dots\dots\dots (1)$$

$$87 = a + 24k \dots\dots\dots (2)$$

To solve simultaneously, subtract eq. (1) from eq. (2), we get

$$42 = 14k$$

$$k = \frac{42}{14} = 3$$

To solve for  $a$ , we substitute 3 for  $k$  in eq. (1),

$$45 = a + 10(3)$$

$$45 = a + 30$$

$$a = 45 - 30 = 15$$

Therefore the formula connecting  $x$  and  $y$  is:  $x = 15 + 3y$ .

- (b) Now, let us find  $x$  when  $y = 18$ .

$$x = 15 + 3(18) = 15 + 54 = 69.$$

**EXAMPLE**

$b$  is partly constant and partly varies directly as  $c$  when  $c = 5$ ,  $b = 7$  and when  $c = 7$ ,  $b = 8$ .

- (a) Find the equation connecting  $b$  and  $c$ .  
 (b) Find  $b$  when  $c = 11$

**Solution**

(a)  $b = a + kc$

$$7 = a + 5k \quad \dots\dots\dots (1)$$

$$8 = a + 7k \quad \dots\dots\dots (2)$$

To solve simultaneously, subtract eq. (1) from eq. (2), we get

$$1 = 2k$$

$$k = \frac{1}{2}$$

To solve for  $a$ , we substitute the value of  $k$  in eq. (1),

$$7 = a + \frac{5}{2}$$

$$a = 7 - \frac{5}{2} = \frac{9}{2}.$$

Therefore, the equation connecting  $b$  and  $c$  is:  $b = \frac{9}{2} + \frac{1}{2}c$  or  $b = \frac{1}{2}(9 + c)$

- (b) Now, let us find  $b$  when  $c = 11$ .

$$b = \frac{1}{2}(9 + 11) = \frac{20}{2} = 10.$$

**EXAMPLE**

$m$  is partly constant with  $p$  and partly varies as square of  $q$ . When  $p = 2$  and  $q = 3$ ,  $m = 24$  and when  $p = 3$  and  $q = 2$ ,  $m = 17$ . Find  $m$  when  $p = 4$  and  $q = 5$ .

**Solution**

$$m = ap + bq^2$$

$$24 = 2a + 9b \quad \dots\dots\dots (1)$$

$$17 = 3a + 4b \quad \dots\dots\dots (2)$$

Now, let us solve equation (1) and eq. (2), simultaneously as follows:

$$\begin{cases} 2a + 9b = 24 \mid \times 3 \\ 3a + 4b = 17 \mid \times 2 \end{cases} \Rightarrow - \begin{cases} 6a + 27b = 72 \\ 6a + 8b = 34 \end{cases}$$

$$19b = 38$$

$$b = 2$$

To solve for  $a$ , substitute  $b = 2$  in eq. (1)

$$24 = 2a + 9(2)$$

$$24 = 2a + 18$$

$$2a = 6$$

$$a = 3$$

Thus, we have the formula:  $m = 3p + 2q^2$ .

Now, when  $p = 4$ , and  $q = 5$ , we get

$$m = 3(4) + 2(5)^2 = 12 + 50 = 62.$$

### EXERCISES

- If  $x$  varies jointly as  $m$  and the cube of  $n$ , and  $x = 600$ ,  $m = 5$  and  $n = 4$ , find  $e$  when  $m = 10$  and  $n = 5$ .
- It is given that  $p$  varies directly as  $q$  and is inverse to the square of  $r$ . If  $q = 60$  and  $r = 2$ , then value of  $p$  is 120. Find the value of  $p$  when  $q = 27$  and  $r = 3$ .
- Suppose  $y$  is partly constant and partly varies as  $x$ . When  $y = 12$ ,  $x = 3$  and when  $y = 18$ ,  $x = 6$ . Find
  - The formula connecting  $x$  and  $y$ .
  - The value of  $y$  when  $x = 4$ .
- Suppose  $m$  is partly constant and partly varies as  $n$ . When  $m = 1$ ,  $n = 2$  and when  $m = 4$ ,  $n = 6$ . Find
  - The formula connecting  $m$  and  $n$ .
  - $m$ , when  $n = 9$
- The quantity  $A$  is partly constant and partly varies as  $B$ .
  - Write the formula connecting  $A$  and  $B$ .
  - When  $A = 2$ ,  $B = 10$  and when  $A = 20$ ,  $B = 12$ . Find  $B$  when  $A = 4$ .

### KEY TERMS

- Variation
- Constant of variation
- Direct variation
- Joint variation
- Inverse variation
- Partial variation

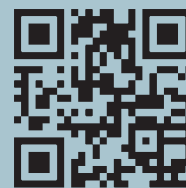
## SUMMARY

- The quantity  $y$  is said to be in **direct proportion** to the quantity  $x$ , written as  $y \propto x$ , if there exists a nonzero constant number  $k$  such that  $y = kx$ , where  $k$  is called the **constant of variation** or **constant of proportionality**.
- The quantity  $y$  is said to be **inverse proportion** to the quantity  $x$ , written as  $y \propto \frac{1}{x}$ , if there exists a constant number  $k$  such that  $y = \frac{k}{x}$ ;  $k \neq 0$ . The number  $k$  is called the **constant of variation**.
- If more than two variables are related directly or one variable changes with the change in the product of two or more variables it is called as joint variation. If  $x$  is in joint variation with  $y$  and  $z$ , written as  $x \propto yz$ , if there exist a constant  $k$  such that  $x = kyz$ .
- A quantity  $y$  is partly constant and partly varies directly as  $x$ , then the law of variation can be expressed as  $y \propto (a + x)$ . That is,  $y = a + kx$ , where  $a$  and  $k$  are constants.

## EXERCISES

1. The commission ( $C$ ) of a salesman varies directly to the amount of his sales ( $S$ ). How will his commission change when his sales are tripled?
2. Amelia is reading a 374-page novel for her English class. It takes her 6 days to read the first 132 pages. If she continues to read at the same rate, how many more days will it take her to finish the novel?
3. A contractor estimates that 3 persons could paint the wall in 8 days. If he uses 12 persons instead of three, how long should they take to complete the job?
4. If 30 persons can reap a field in 17 days, then how many persons should be engaged to reap the same field in 10 days?
5. If  $p$  varies jointly as  $q$  and  $r$  and  $p = 30$  when  $q = 5$  and  $r = 3$ , find  $p$  when  $q = 6$  and  $r = 8$ .
6. The area  $A$  of a triangle varies jointly as its base  $b$  and height  $h$ . If its base is 8 cm and height is 7 cm, it has an area of  $28 \text{ cm}^2$ . Find the base if its height is 5 cm and its area is  $16 \text{ cm}^2$ .
7. The volume ( $V$ ) of a circular cylinder varies jointly to the square of the radius ( $r$ ) of the base and its height ( $h$ ) with  $\pi$  as the constant of variation. If the radius is fixed and  $h$  is doubled, how is  $V$  affected?

8.  $y$  is partly constant and partly varies as  $x$ . When  $x = 4$ ,  $y = 0$  and when  $x = 12$ ,  $y = 40$ .
  - (a) Find the equation connecting  $x$  and  $y$ .
  - (b) Find  $y$  when  $x = 3$ .
9.  $y$  varies partly as  $x$  and partly as the square of  $x$ . When  $x = 2$ ,  $y = 4$  and when  $x = 5$ ,  $y = 65$ .
  - (a) Express  $y$  in terms of  $x$ .
  - (b) Find  $y$  when  $x = 8$ .
10. The cost  $C$  (in L\$) of making a call partly constant and partly varies with the number of seconds,  $n$  spends on the call. A call of 1 minutes 5 seconds attracts L\$8 while a call of 2 minutes attracts L\$29. Calculate the cost of making a call of 3 minutes 10 seconds.



M11CH05

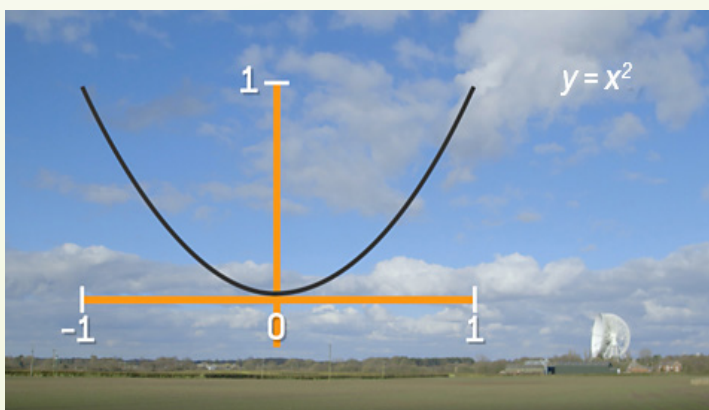
# CHAPTER

# 5

## QUADRATIC FUNCTIONS AND EQUATIONS

### Chapter Contents

- 5.1 Quadratic Functions
- 5.2 Quadratic Equations
- 5.3 Solving Quadratic Equations by Factorization
- 5.4 Quadratic Problems
- 5.5 Solving Quadratic Equations by Completing the Square
- 5.6 Quadratic Graphs
- 5.7 Solving Quadratic Equations Graphically
  - Key Terms
  - Summary
  - Exercises



## Chapter Outcomes

*Learners are able to apply concepts of statistics, ratio, rates and percentages to solve problems on these topics. The students will be able to analyze and interpret or explain practical data and solve problems relating to scale drawing , ratio, rates, and percent.*

By the end of this chapter, you will be able to:

- define and discuss quadratic functions;
- define and solve quadratic equations;
- solve quadratic equation by factorization;
- solve quadratic problems;
- solve quadratic equations by completing the square;
- draw quadratic graphs;
- solve quadratic equations graphically.

## Introduction

A special type of polynomial equation that you will work with a lot is a quadratic equation. Quadratic equations are used to solve many real-life problems such as solving motion problems in physics, area-based problems and in other fields of science and mathematics. In this topic, we will learn about quadratic functions, quadratic equations, quadratic problems and solving quadratic equations by factorization, completing the square and graphically.

The word “*quadratic*” is derived from the word “*quad*” which means square because the variable gets *squared*. In other words, a quadratic function is a polynomial function of degree 2. A quadratic function has a minimum of one term which is of the second degree. It is an algebraic function.

### ACTIVITY 1

1. Do you know that when a rocket is launched, its path is described by the solution of a quadratic function?
2. What is linear function.
3. Explain the difference between linear function and quadratic function.
4. What is the domain?

The single defining feature of quadratic function is that they are of the second degree or of degree two. This means that in all quadratic function, the highest exponent of  $x$  in a nonzero term is equal to two. For example,  $f(x) = 2x^2$ ,  $g(x) = x^2 + 3x - 8$ ,  $y = 7 - x^2 + 3x$  and  $y = 3x^2 - 5x + 7$  are quadratic functions.

### DEFINITION

A quadratic function is a function of the form

$$y = f(x) = ax^2 + bx + c,$$

Where  $a$ ,  $b$  and  $c$  are real numbers with  $a \neq 0$ . The domain of a quadratic function is the set of all real numbers,  $\mathbb{R}$ .

In interval notation, the domain of any quadratic function is  $(-\infty, \infty)$ .

### EXAMPLE 1

The following functions are quadratic functions.

- (a)  $f(x) = 2x^2 + 4x - 5$ , where  $a = 2$ ,  $b = 4$  and  $c = -5$ .
- (b)  $f(x) = 3x^2 - 16$ , where  $a = 3$ ,  $b = 0$  and  $c = -16$ .

**EXAMPLE 2**

(c)  $g(x) = 5x - x^2$ , where  $a = -1$ ,  $b = 5$  and  $c = 0$ .

(d)  $y = x^2$ , where  $a = 1$ ,  $b = 0$  and  $c = 0$ .

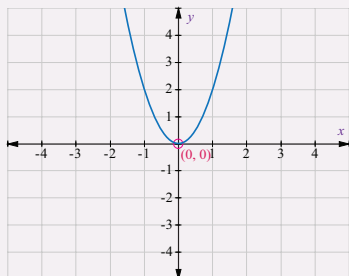


Figure 1.

The most basic quadratic function is  $f(x) = x^2$ , whose graph is shown in Figure 1. Its shape is called a parabola. The point  $(0, 0)$  is called the vertex of the parabola. As shown in the graph, the domain of  $f(x) = x^2$  is the set of all  $x$ -values. That is, the set of all real numbers,  $\mathbb{R}$ . The range of a quadratic function is the set of all  $y$ -values such that  $y \geq 0$ .

**EXAMPLE 3**

Given a quadratic function  $f(x) = x^2 + x - 6$ . Find,

- the value of  $y$  when  $x = 3$
- the value of  $y$  when  $x = 2$
- $f(-4)$
- $f(0)$

**Solution**

$$(a) f(3) = 3^2 + 3 - 6 = 9 + 3 - 6 = 12 - 6 = 6.$$

$$(b) f(2) = 2^2 + 2 - 6 = 4 + 2 - 6 = 6 - 6 = 0.$$

$$(c) f(4) = 4^2 + 4 - 6 = 16 + 4 - 6 = 20 - 6 = 14.$$

$$(d) f(20) = 0^2 + 0 - 6 = 0 + 0 - 6 = 0 - 6 = -6.$$

**EXAMPLE 4**

Let  $f(x) = 3x^2 - bx + 8$  be a quadratic function. Then find the value of  $b$  if  $f(-2) = 24$ .

**Solution**

$$f(-2) = 3(-2)^2 - b(-2) + 8$$

$$24 = 12 + 2b + 8$$

$$24 = 20 + 2b.$$

$$4 = 2b$$

$$b = 2$$

Thus, the value of  $b$  is 2 and the quadratic function is:  $f(x) = 3x^2 - 2x + 8$

**EXERCISES**

1. Determine whether each of the following functions is quadratic or not.

(a)  $f(x) = 2x^3 - 2x + 5$

(c)  $y = 5x - 3$

(b)  $y - 3 = 2x^2$

(d)  $h(x) = 5 - 3x^2 + 4x$

2. For each of the following quadratic functions, identify a, b and c?

	Quadratic function	(a)	(b)	(c)
(a)	$f(x) = x^2 + 2x - 3$			
(b)	$g(x) = -2x^2 - 4x + 9$			
(c)	$y = x^2 - x + 2$			
(d)	$f(x) = 9 - 4x^2$			

3. Let  $f(x) = -x^2 + 4x + 7$  be a quadratic function. Then,

(a) find  $x = \frac{b}{2a}$ .

(b) evaluate  $f\left(\frac{b}{2a}\right)$ .

4. Given a quadratic function  $f(x) = -2x^2 + x + 3$ . Find,

(a) the value of  $y$  when  $x = -1$

(c)  $f(0)$

(b) the value of  $y$  when  $x = 2$

(d)  $f(-3)$

5. Let  $f(x) = ax^2 + 2x - 3$  be a quadratic function. Then find the value of  $a$ , when  $f(-3) = 27$ .

The quadratic equations are second-degree equations in  $x$  that have *maximum* two solutions for  $x$ . These two solutions for  $x$  are also called the roots of the quadratic equations. We shall learn more about the roots of a quadratic equation in the next sub-topic.

A quadratic equation in one variable also called a second-degree polynomial equation in one variable. A second-degree equation in one variable is an equation in which the variable is raised to the second power. For example,  $2x^2 + x - 6 = 0$ ,  $3y^2 = 7y - 4$  and  $3x^2 + 13x + 12 = 0$  are quadratic equations.

## ACTIVITY 2

1. What is equation?
2. What is the first condition for an equation to be a quadratic equation?
3. What is the standard form of a quadratic equation?
4. Find the zeros of each of the following quadratic functions.
  - (a)  $f(x) = x^2 + 1$
  - (b)  $g(x) = 2x^2 - 6x$
  - (c)  $f(x) = x^2 - x - 2$
  - (d)  $f(x) = x^2 + 5x + 6$
5. What is the domain and range of quadratic function?

## DEFINITION

An equation that can be written in the form

$$ax^2 + bx + c = 0,$$

where  $a$ ,  $b$  and  $c$  are real numbers with  $a \neq 0$ , is called a quadratic equation in  $x$ . The given form is called standard form.

## EXAMPLE 5

The following equations are examples of quadratic equation in the form:  $ax^2 + bx + c = 0$ .

- (a)  $2x^2 - 2x - 5 = 0$ , with  $a = 2$ ,  $b = -2$  and  $c = -5$ .
- (b)  $3x^2 + 6x = 0$ , with  $a = 3$ ,  $b = 6$  and  $c = 0$ .
- (c)  $5x^2 - 25 = 0$ , with  $a = 5$ ,  $b = 0$  and  $c = -25$ .

To ensure the presence of the  $x^2$  term, the number  $a$ , in the general expression  $ax^2 + bx + c$  cannot be zero. However,  $b$  or  $c$  may be zero. For example,  $4x^2 + 3x = 0$ ,  $2x^2 - 3 = 0$  and  $6x^2 = 0$  are also quadratic equations.

Frequently, quadratic equations occur in nonstandard form but where necessary they can be rearranged into standard form.

## EXAMPLE 6

- (a)  $2x^2 + 5x = 8$  can be rewritten as  $2x^2 + 5x - 8 = 0$ .
- (b)  $3x^2 = 9x - 8$  can be rewritten as  $3x^2 - 9x + 8 = 0$ .

**EXAMPLE 7**

Which of the following are quadratic equations in  $x$ ?

(a)  $x^2 + x^4 - 2x = 7$

(b)  $x^2 + 3x = 7$

(c)  $x^2 + 5x = 0$

(d)  $x^2 = 9$

(e)  $\frac{1}{3}x^2 - x + 2 = 0$

(f)  $\frac{3}{x^2} - 2x + 4 = 0$

**Solution**

All the above equations are quadratic equations except (a) and (f).

**EXAMPLE 8**

For each of the following quadratic equations identify  $a$ ,  $b$  and  $c$ .

(a)  $2x(3x - 5) = 0$

(b)  $(3 - 2x)(x + 4) = 0$

**Solution**

To identify  $a$ ,  $b$  and  $c$ , first multiply the expressions to eliminate the bracket.

(a)  $2x(3x) - 2x(5) = 0$

$$6x^2 - 10x = 0.$$

So,  $6x^2 - 10x = 0$  is a quadratic equation, with  $a = 6$ ,  $b = -10$  and  $c = 0$ .

(b)  $(3 - 2x)(x + 4) = 0$

$$3(x + 4) - 2x(x + 4) = 0$$

$$3x + 12 - 2x^2 - 8x = 0$$

$$-2x^2 - 5x + 12 = 0.$$

So,  $-2x^2 - 5x + 12 = 0$  is a quadratic equation, with  $a = -2$ ,  $b = -5$  and  $c = 12$ .

**EXERCISES**

1. Find the zeros of each of the following quadratic functions.

(a)  $f(x) = x^2 - 1$

(c)  $f(x) = x^2 + x - 2$

(b)  $g(x) = 3x^2 - 9x$

(d)  $f(x) = x^2 + x - 6$

2. Rewrite the following expression into standard form of quadratic equations.

(a)  $2x^2 = 6$

(c)  $5x + 6 = x^2$

(b)  $x^2 = 2 - x$

(d)  $2x^2 - 5 = 3x^2 + 7x + 7$

3. Which of the following are quadratic equations in  $x$ ?

(a)  $x^3 + x^2 = 7 + 2x$

(b)  $x^2 + 3x = 7 - x^2$

(c)  $x^2 + 6x = 9$

(d)  $\frac{2}{x} + x^2 = 5$

(e)  $\frac{2}{3}x^2 + 8 = 1 - 5x$

(f)  $x^2 + 5 = 0$

(g)  $2x^2 + 5x - 3 = 0$

(h)  $x^4 + 2 = x^2$

4. For each of the following quadratic equations identify  $a$ ,  $b$  and  $c$ .

(a)  $x(2x + 3) = 0$

(b)  $(2x + 3)(x + 2) = 0$

(c)  $(x + 3)(2 - x) = 0$

(d)  $(2x - 5)(2x + 5) = 0$

To solve a quadratic equation,  $ax^2 + bx + c = 0$ , we must find values of the unknown variable  $x$  which make the left-hand and right-hand sides equal. Such values are known as **solutions** or **roots** of the quadratic equation.

### ACTIVITY 3

1. What is factorization?

2. Explain the difference between solving quadratic equations in comparison to solving linear equations.

3. Factorize each of the following expression.

(a)  $18x - 3x^2$

(b)  $y^2 - 49$

(c)  $x^2 + 7x + 12$

(d)  $6x^2 - x - 12$

4. Factorize each of the following quadratic equations.

(a)  $2x^2 - 4x = 0$

(b)  $x^2 + 5x + 6 = 0$

(c)  $x - x^2 + 12 = 0$

(d)  $2x^2 + x - 6 = 0$

Factored quadratic equation can be solved using the zero-product principle.

### Zero-Product principle

If the product two numbers is zero, then one of these factors must be zero. That is, for any real numbers  $p$  and  $q$ , if  $p \times q = 0$ , then either  $p = 0$  or  $q = 0$ .

To solve the quadratic equation  $ax^2 + bx + c = 0$ , we factorize the expression  $ax^2 + bx + c$  into linear factors, whenever possible, and solve the linear equation using the fact that  $p \times q = 0$ .

### 1. Solving Quadratic Equations of the form $ax^2 + bx = 0$ by Factorization

The quadratic expression of the form  $ax^2 + bx = 0$  can be factorized as:

$$ax^2 + bx = 0$$

$$x(ax + b) = 0$$

$$x = 0 \text{ or } ax + b = 0$$

$$x = 0 \text{ or } x = -\frac{b}{a}.$$

#### EXAMPLE 10

Solve each of the following quadratic equations.

(a)  $2x^2 + 6x = 0$

(b)  $14x - 7x^2 = 0$

#### Solution

(a) The common factor of  $2x^2$  and  $6x$  is  $2x$ . Thus,

$$2x^2 + 6x = 0$$

$$2x(x + 3) = 0$$

$$2x = 0 \text{ or } x + 3 = 0$$

$$x = 0 \text{ or } x = -3$$

(b) The common factor of  $7x^2$  and  $14x$  is  $7x$ . Thus,

$$14x - 7x^2 = 0$$

$$7x(2 - x) = 0$$

$$7x = 0 \text{ or } 2 - x = 0$$

$$x = 0 \text{ or } x = 2.$$

### 2. Solving Quadratic Equations of the form $a^2x^2 - b^2 = 0$ by Factorization.

Using the difference of two squares formula:  $a^2 - b^2 = (a + b)(a - b)$ , the quadratic equation of the form  $x^2 - k^2 = 0$  can be factorized as  $(x + k)(x - k) = 0$  which implies that  $x = -k$  or  $x = k$ .

#### EXAMPLE 9

Solve each of the following quadratic equations.

(a)  $4x^2 - 16 = 0$

(b)  $25 - y^2 = 0$

#### Solution

(a)  $4x^2 - 16 = 0$

$$(2x)^2 - 4^2 = 0$$

$$(2x + 4)(2x - 4) = 0$$

$$2x + 4 = 0 \text{ or } 2x - 4 = 0$$

$$2x = -4 \text{ or } 2x = 4$$

$$x = -2 \text{ or } x = 2.$$

(b)  $25 - y^2 = 0$

$$5^2 - y^2 = 0$$

$$(5 + y)(5 - y) = 0$$

$$5 + y = 0 \text{ or } 5 - y = 0$$

$$y = -5 \text{ or } y = 5.$$

### 3. Solving any Quadratic Equation by Factorization

To factorize the trinomial of the form  $ax^2 + bx + c = 0$ , where  $a$ ,  $b$  and  $c$  are integers and  $a \neq 0$ , we consider two cases: when  $a = 1$  and when  $a \neq 1$ .

**Case 1:** When  $a = 1$

If  $a = 1$  the general quadratic equation,  $ax^2 + bx + c = 0$  is reducible to the equation of the form  $x^2 + bx + c = 0$ .

To factorize an expression of the form  $x^2 + bx + c$ , we look for two numbers: whose **sum** is  $b$  and whose **product** is  $c$ .

To find the two numbers whose sum is  $b$  and whose product is  $c$ , find two factors of  $c$  which adds up to  $b$ .

#### EXAMPLE 11

Solve each of the following quadratic equations.

(a)  $x^2 + 5x + 6 = 0$

(b)  $x^2 - 2x - 8 = 0$

#### Solution

- (a) First, we find two numbers  $a$  and  $b$  such that  $a + b = 5$ , the coefficient of  $x$ ,  $a \times b = 6$ , the constant term. The factors of 6 are (in pairs): 1 and 6, 2 and 3,  $-1$  and  $-6$ ,  $-2$  and  $-3$ . We take 2 and 3, as their sum is 5 and their product is 6. Thus, the middle term of  $x^2 + 5x + 6 = 0$  can be split into two as follows:

$$x^2 + \underline{2x} + \underline{3x} + 6 = 0$$

$$x(x + 2) + 3(x + 2) = 0$$

$$(x + 3)(x + 2) = 0$$

$$x + 3 = 0 \text{ or } x + 2 = 0$$

$$x = -3 \text{ or } x = -2.$$

- (b) Factors of  $-8$  are, in pairs,  $-1$  and  $8$ ;  $1$  and  $-8$ ;  $2$  and  $-4$ ;  $4$  and  $-2$ . But the required numbers are  $-4$  and  $2$  whose sum is  $-2$  and whose product is  $-8$ . Thus, the middle term of  $x^2 - 2x - 8 = 0$  can be split into two as follows:

$$\underline{x^2} + \underline{2x} - \underline{4x} - 8 = 0$$

$$x(x + 2) - 4(x + 2) = 0$$

$$(x - 4)(x + 2) = 0$$

$$x - 4 = 0 \text{ or } x + 2 = 0$$

$$x = 4 \text{ or } x = -2.$$

In general, in a quadratic equation of the form  $x^2 + bx + c = 0$ , if  $p$  and  $q$  are numbers satisfying the conditions,  $p + q = b$  and  $p \times q = c$ , then

$$x^2 + bx + c = 0$$

$$x^2 + (p + q)x + pq = 0$$

$$\underline{x^2} + \underline{px} + \underline{qx} + \underline{pq} = 0$$

$$x(x + p) + q(x + p) = 0$$

$$(x + p)(x + q) = 0$$

$$x + p = 0 \text{ or } x + q = 0$$

$$x = -p \text{ or } x = -q.$$

In particular, let  $p = q$ , then the expression  $x^2 + bx + c$  is reducible to the equation of the form  $x^2 + 2px + p^2 = 0$ . Then it can be written as:

$$(x + p)(x + p) = 0$$

$$(x + p)^2 = 0.$$

### EXAMPLE 12

Factorize each of the following.

(a)  $x^2 + 6x + 9 = 0$

(b)  $x^2 - 10x - 25 = 0$

#### Solution

- (a) Factors of  $9$  are  $\pm 1, \pm 3, \pm 9$ . But  $3 + 3 = 6$  and  $3 \times 3 = 9$ . Thus,

$x^2 + 6x + 9 = 0$  can be expressed as:

$$x^2 + 6x + 9 = 0$$

$$x^2 + 3x + 3x + 9 = 0$$

$$\underline{x^2} + \underline{3x} + \underline{3x} + \underline{9} = 0$$

$$x(x + 3) + 3(x + 3) = 0$$

$$(x + 3)(x + 3) = 0.$$

$$(x + 3)^2 = 0$$

$$x + 3 = 0$$

$$x = -3.$$

(b) Factors of 25 are  $\pm 1$ ,  $\pm 5$  and  $\pm 25$ . But,  $-5 + (-5) = -10$  and  $-5 \times -5 = 25$ .

$$x^2 - 10x - 25 = 0$$

$$x^2 - 5x - 5x - 25 = 0$$

$$\underline{x^2} - \underline{5x} - \underline{5x} - \underline{25} = 0$$

$$x(x - 5) - 5(x - 5) = 0$$

$$(x - 5)(x - 5) = 0$$

$$(x - 5)^2 = 0$$

$$x - 5 = 0$$

$$x = 5.$$

**Case 2:** When  $a \neq 1$

If  $a \neq 1$ , the general quadratic expression  $x^2 + bx + c = 0$  can be expressed

as  $ax^2 + bx + c = 0$

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0.$$

To factorize a general quadratic expression,  $ax^2 + bx + c = 0$ ,  $a \neq 1$  it is recommended to follow the following three steps.

**Step 1:** Find the *product* of  $a \times c$ ,

**Step 2:** Find pairs of *factors* of  $a \times c$

If  $a \times c$  is *positive*, find a pair whose *sum* is  $b$ .

If  $a \times c$  is *negative*, find a pair whose *difference* is  $b$ .

**Step 3:** Write  $b$  in terms of the sum of the pair and then factorize again.

### EXAMPLE 13

Solve each of the following quadratic equations.

(a)  $2x^2 + 9x + 10 = 0$

(b)  $6x^2 + 35x - 6 = 0$

**Solution**

(a) The product  $a \times c$  is 20. The factors 20 are (in pairs), 1 and 20; 2 and 10; 4 and 5. We need only the two numbers 4 and 5 whose sum is 9. Thus,

$$2x^2 + 9x + 10 = 0$$

$$2x^2 + 4x + 5x + 10 = 0$$

$$2x^2 + 4x + 5x + 10 = 0$$

$$2x(x + 2) + 5(x + 2) = 0$$

$$(2x + 5)(x + 2) = 0$$

$$2x + 5 = 0 \text{ or } x + 2 = 0$$

$$2x = -5 \text{ or } x = -2$$

$$x = -\frac{5}{2} \text{ or } x = -2.$$

- (b) The product  $a \times c$  is  $-36$ . The factors of  $-36$  are (in pairs)  $-1$  and  $36$ ;  $1$  and  $-36$ ;  $2$  and  $-18$ ;  $-2$  and  $18$ ;  $-3$  and  $12$ ;  $3$  and  $-12$ ;  $4$  and  $-9$ ;  $-4$  and  $9$ ; and  $6$  and  $-6$ . We need only the two numbers  $-1$  and  $36$  whose sum is  $35$ . Thus,

$$6x^2 + 35x - 6 = 0$$

$$6x^2 + 4x + 5x + 10 = 0$$

$$6x^2 + 36x - x - 6 = 0$$

$$6x(x + 6) - 1(x + 6) = 0$$

$$(6x - 1)(x + 6) = 0$$

$$6x - 1 = 0 \text{ or } x + 6 = 0$$

$$6x = 1 \text{ or } x = -6$$

$$x = \frac{1}{6} \text{ or } x = -6.$$

### EXERCISES

1. Factorize each of the following expressions.

(a)  $12x + 4x^2$

(c)  $x^2 - x - 6$

(b)  $\frac{1}{2}y^2 - 25$

(d)  $2x^2 + 7x - 15$

2. Factorize each of the following quadratic equations.

(a)  $6x^2 - 18x = 0$

(e)  $x^2 - x - 6 = 0$

(b)  $2x^2 - 12x = 0$

(f)  $6 - x - 2x^2 = 0$

(c)  $\frac{4}{9}x^2 - 49 = 0$

(g)  $2x^2 - x - 6 = 0$

(d)  $\frac{9}{16} - \frac{1}{4}y^2 = 0$

(h)  $2x^2 + 11x + 12 = 0$

3. Find the roots of the following quadratic equation using factorization method.

(a)  $\frac{4}{9}y^2 - \frac{25}{49} = 0$

(c)  $6x^2 - 18x = 0$

(b)  $\frac{1}{9}x^2 - 16 = 0$

(d)  $63x - 7x^2$

4. Find the roots of the following quadratic equations using factorization method.

(a)  $x^2 + 3x + 2 = 0$

(e)  $x^2 - 2x - 8 = 0$

(b)  $x^2 + 4x + 3 = 0$

(f)  $x^2 + 11x + 30 = 0$

(c)  $x^2 - 5x + 6 = 0$

(g)  $x^2 + 4x - 21 = 0$

(d)  $x^2 + 7x + 12 = 0$

(h)  $x^2 - x - 20 = 0$

5. Find the roots of the following quadratic equations using factorization method.

(a)  $2x^2 + \frac{5}{3}x - 2 = 0$

(d)  $2x^2 - 3x - 5 = 0$

(b)  $\frac{2}{5}x^2 - x - \frac{3}{5} = 0$

(e)  $6x^2 + 7x = 3$

(c)  $21x^2 - 2x + \frac{1}{21} = 0$

(f)  $2x^2 + 9x - 18 = 0$

(g)  $3x^2 + 5x - 2 = 0$

(h)  $6x^2 + 5x - 6 = 0$

Many physical and mathematical problems are in the form of quadratic equations. In mathematics, quadratic equations are used to solve a variety of real-life problems. In this sub-topic, we will learn about real-life applications of quadratic equation and we will understand the importance of quadratic equations using some practical examples.

#### ACTIVITY 4

1. What is the quadratic equation and what is it used for?
2. What makes a problem quadratic?
3. Did you know that there are many uses of quadratic equations in daily life?
4. What are quadratic equations used for in real life?
5. Suppose you are throwing a baseball 10 m above the ground. The baseball will reach its peak height and fall back to the ground. At time  $t$ , the height  $h$  from the ground is given by  $h = -2t^2 + 8t + 10$ . Calculate the time taken by the baseball to reach the ground.

To solve word problems, we follow the following steps.

**Step 1:** Read the problem carefully and draw diagram whenever possible.

**Step 2:** List all of the unknown numerical quantities involving in the problem and call it any letters such as  $p, q, x, y, z$ , etc.

**Step 3:** Using information given in the wording of the problem, set up a word equation that contains words as well as numbers.

**Step 4:** Put in numbers in place of words whenever possible to set up an algebraic equation containing one variable and known numerical constant.

**Step 5:** Solve the equation for the variable.

### EXAMPLE 14

There is a hall whose length is five times the width. The area of the floor is  $45 \text{ m}^2$ . Find the length and width of the hall.

#### Solution

Let the length of the rectangle be  $l$  and the width of the rectangle be  $w$ .

**Given:**  $l = 5w$ ,  $w > 0$  and  $A = lw = 45 \text{ m}^2$ . Thus,

$$A = lw = w(5w) = 5w^2$$

$$45 = 5w^2$$

$$w^2 = 9$$

$$w^2 - 9 = 0$$

$$(w + 3)(w - 3) = 0$$

$w = -3$  or  $w = 3$ . but  $w > 0$ , we will only take  $w = 3$ .

Therefore, the width is 3 m and height is  $5(3) = 15 \text{ m}$ .

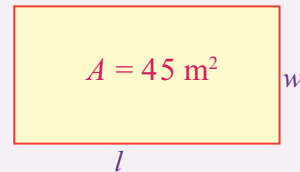


Figure 2.

### EXAMPLE 15

The three sides of a right angled-triangle are  $x, x + 1$  and 5. Find  $x$  and the area, if the longest side is 5.

#### Solution

The longest side will be the hypotenuse. Thus, using the Pythagoras Theorem, we can write:

$$BC^2 + CA^2 = AB^2$$

$$(x + 1)^2 + x^2 = 5^2$$

$$x^2 + 2x + 1 + x^2 = 25$$

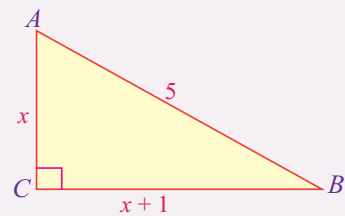


Figure 3.

$$2x^2 + 2x + 1 = 25$$

$$2x^2 + 2x - 24 = 0$$

$$x^2 + x - 12 = 0$$

$$(x + 4)(x - 3) = 0$$

$$x + 4 = 0 \text{ or } x - 3 = 0$$

$$x = -4 \text{ or } x = 3.$$

We can only take  $x = 3$  because the length cannot be negative. So,  $x = 3$ .

$$\text{Area: } A = \frac{1}{2} lb = \frac{1}{2} \times 3 \times 4 = 6.$$

### EXAMPLE 16

A ball is thrown upwards from a rooftop, 80 m above the ground. It will reach a maximum vertical height and fall back to the ground. The height of the ball from the ground at time  $t$  is  $h$  and is given by  $h = -16t^2 + 64t + 80$ . Find:

- The height reached by the ball after 1 second.
- The time it will take before hitting the ground.

#### Solution

- The given equation is  $h = -16t^2 + 64t + 80$ .

To find  $h$  after 1 second, we substitute  $t = 1$ .

$$\begin{aligned} h &= -16(1)^2 + 64(1) + 80 \\ &= -16 + 64 + 80 \\ &= 128 \text{ m.} \end{aligned}$$

- When the ball hits the ground,  $h = 0$ .

$$-16t^2 + 64t + 80 = 0$$

Dividing the equation by  $-16$ , gives

$$t^2 - 4t - 5 = 0$$

$$(t - 5)(t + 1) = 0$$

$$t - 5 = 0 \text{ or } t + 1 = 0$$

$$t = 5 \text{ or } t = -1.$$

The time  $t$  cannot be negative, so, the time is 5 seconds.

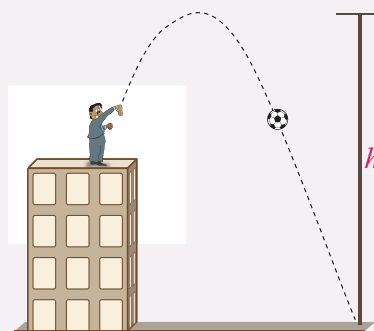


Figure 4.

**EXAMPLE 17**

A rectangular garden has an area of 14 square meters and a perimeter of 18 meters. Find the dimensions of the rectangular garden.

**Solution**

Let the length of the rectangle be  $\ell$  and the width of the rectangle is  $w$ .

$$P = 2(\ell + w) \Rightarrow 18 = 2(\ell + w) \Rightarrow \ell + w = 9 \Rightarrow \ell = 9 - w.$$

$$A = \ell w \Rightarrow \ell w = 14 \Rightarrow (9 - w)w = 14 \Rightarrow 9w - w^2 = 14 \Rightarrow w^2 - 9w + 14 = 0$$

$$\Rightarrow w^2 - 2w - 7w + 14 = 0 \Rightarrow w(w - 2) - 7(w - 2) = 0$$

$$\Rightarrow (w - 2)(w - 7) = 0 \Rightarrow w - 2 = 0 \text{ or } w - 7 = 0 \Rightarrow w = 2 \text{ or } w = 7$$

Thus,  $\ell = 9 - 2 = 7$  or  $\ell = 9 - 7 = 2$ .

Therefore, the dimension is 2 m by 7 m or 7 m by 2 m.

**EXAMPLE 18**

Find two consecutive even integers whose product is 168.

**Solution**

Let the first even integer be  $x$ . Then, its consecutive integer is  $x + 2$ .

$$x(x + 2) = 168 \Rightarrow x^2 + 2x = 168 \Rightarrow x^2 + 2x - 168 = 0$$

$$\Rightarrow x^2 - 12x + 14x - 168 = 0 \Rightarrow x(x - 12) + 14(x - 12) = 0$$

$$\Rightarrow (x - 12)(x + 14) = 0 \Rightarrow x - 12 = 0 \text{ or } x + 14 = 0$$

$$\Rightarrow x = 12 \text{ or } x = -14.$$

Thus, if  $x = 12$ , then  $x + 2 = 12 + 2 = 14$ ; and if  $x = -14$ , then  $x + 2 = -14 + 2 = -12$ .

Therefore, the two consecutive integers are 12 and 14 or -14 and -12.

**EXERCISES**

1. The product of two consecutive integers is 132. What is the degree of the equation you used above to find the solution? Find the two integers.
2. The length of a rectangle is greater than its width by 3 cm. If its area is 10 sq.cm, find the perimeter.
3. The product of two consecutive odd integers is 195. Find the integers.
4. Find the base and height of a triangle whose base is 4 cm more than six times its height and has an area of 456 square centimeters.

5. A rectangular tablecloth has an area 120 square centimeter. The width is 2 centimeters shorter than the length. What are the length and width of the tablecloth?
6. An arrow is shot from the ground into the air at an initial speed of 108 m/s. Use the formula  $h = -16t^2 + v_0t$  to determine when the arrow will be 180 meters from the ground.
7. A man throw a ball into the air with a velocity of 96 m/s. Use the formula  $h = -16t^2 + v_0t$  to determine when the height of the ball will be 48 meters.
8. A rectangular garden has an area of  $14 \text{ m}^2$  and a perimeter of 18 meters. Find the dimensions of the rectangular garden.

In all cases, it is not convenient to solve a quadratic equation by factorization method. Thus, to solve the given quadratic equation, we use an alternative method, called **completing the square method**. In this sub-topic, we will learn a method for solving any kind of quadratic equation.

### ACTIVITY 5

1. What is the missing constant term in the perfect square that starts with:
 

(a) $x^2 + 4x$	(c) $x^2 + 10x$
(b) $x^2 - 6x$	(d) $x^2 - 2x$
2. Write each of the following quadratic expression in the form  $(x + k)^2$ .
 

(a) $x^2 + 2x + 1$	(c) $x^2 - 8x + 16$
(b) $x^2 - 6x + 9$	(d) $4x^2 + 4x + 1$

If  $x^2 + 8x$  is the beginning of a perfect square expression, what should be the constant term?

Let us assume that the expression can be factored as the perfect square  $(x + k)^2$  where the value of constant  $k$  is still unknown. This expression is expanded as  $x^2 + 2kx + k^2$ , which tells us two things:

1. The coefficient of  $x$ , which we know to be 8, should be equal to  $2k$ . This means that  $k = 4$ .
2. The constant number we need to add is equal to  $k^2$ , which is  $4^2 = 16$ .

In general, to solve a quadratic equation using completing the square method, we follow the following steps.

**Step 1:** Write the given quadratic equation in the standard form.

**Step 2:** Make the coefficient of  $x^2$  equal to 1, if it is not.

**Step 3:** Shift the constant term to right hand side (R.H.S) of the equation.

**Step 4:** Add  $\left(\frac{1}{2} \text{ of the coefficient of } x\right)^2$  to both sides of the equation.

**Step 5:** Express the left-hand side (L.H.S) as the perfect square of a suitable binomial expression and simplify the right-hand side (R.H.S).

**Step 6:** Take square root of both sides.

**Step 7:** Obtain the values of  $x$  by shifting the constant term from L.H.S to R.H.S.

**EXAMPLE 19**

Solve the following equations using completing the square method.

(a)  $x^2 + 4x + 3 = 0$

(c)  $x^2 - 6x + 8 = 0$

(b)  $x^2 + 10x - 24 = 0$

**Solution**

(a)  $x^2 + 4x + 3 = 0$

$x^2 + 4x = -3$  ..... (shift 3 to right hand side).

$x^2 + 4x + 4 = -3 + 4$  .....(add  $\left(\frac{4}{2}\right)^2 = 4$  to both sides, where 4 is the coefficient of  $x$ ).

$(x + 2)^2 = 1$  .....(express as perfect square).

$x + 2 = \pm\sqrt{1}$

$x = -2 \pm 1$

$x = -3$  or  $x = -1$

(b)  $x^2 + 10x - 24 = 0$

$x^2 + 10x = 24$  ..... (shift  $-24$  to right hand side).

$x^2 + 10x + 25 = 24 + 25$  .....(add  $\left(\frac{10}{2}\right)^2 = 25$  to both sides, where 10 is the coefficient of  $x$ ).

$(x + 5)^2 = 49$ .....(express as perfect square).

$x + 5 = \pm\sqrt{49}$

$x = -5 \pm 7$

$x = -12$  or  $x = 2$

$$(c) \quad x^2 - 6x + 8 = 0$$

$$x^2 - 6x = -8 \quad \dots\dots\dots (\text{shift } 8 \text{ to right hand side}).$$

$$x^2 - 6x + 9 = -8 + 9 \quad \dots\dots (\text{add } \left(\frac{-6}{2}\right)^2 = 9 \text{ to both sides, where } -6 \text{ is the coefficient of } x).$$

$$(x - 3)^2 = 1 \quad \dots\dots\dots (\text{express as perfect square}).$$

$$x - 3 = \pm\sqrt{1}$$

$$x = 3 \pm 1$$

$$x = 2 \text{ or } x = 4$$

### EXAMPLE 20

Solve the following equations using completing the square method.

$$(a) \quad 3x^2 + 6x + 9 = 0$$

$$(b) \quad 2x^2 - 8x - 4 = 0$$

$$(c) \quad -6x^2 + 6x + 36 = 0$$

#### Solution

$$(a) \quad 3x^2 + 6x + 9 = 0$$

$$x^2 + 2x + 3 = 0 \quad \dots\dots\dots (\text{dividing both sides by } 3, \text{ to make the coefficient of } x \text{ equal to } 1)$$

$$x^2 + 2x = -3 \quad \dots\dots\dots (\text{shift } 3 \text{ to right hand side}).$$

$$x^2 + 2x + 1 = -3 + 1 \quad \dots\dots (\text{add } \left(\frac{2}{2}\right)^2 = 1 \text{ to both sides}).$$

$$(x + 2)^2 = -2 \quad \dots\dots\dots (\text{express as perfect square}).$$

$$x + 2 = \pm\sqrt{-2}$$

Since the square of a real number cannot be negative. The given equation has no solution.

$$(b) \quad 2x^2 - 8x - 4 = 0$$

$$x^2 - 4x - 2 = 0 \quad \dots\dots\dots (\text{dividing both sides by } 2).$$

$$x^2 - 4x = 2 \quad \dots\dots\dots (\text{shift } -2 \text{ to right hand side}).$$

$$x^2 - 4x + 4 = 2 + 4 \quad \dots\dots (\text{add } \left(\frac{2}{2}\right)^2 = 4 \text{ to both sides}).$$

$$(x - 2)^2 = 6 \quad \dots\dots\dots (\text{express as perfect square}).$$

$$x - 2 = \pm\sqrt{6}$$

$$x = 2 \pm \sqrt{6}$$

$$x = 2 - \sqrt{6} \text{ or } x = 2 + \sqrt{6}.$$

(c)  $-6x^2 + 6x + 36 = 0$

$$x^2 - x - 6 = 0 \quad \dots\dots\dots(\text{dividing both sides by } -6).$$

$$x^2 - x = 6 \quad \dots\dots\dots(\text{shift } -3 \text{ to right hand side}).$$

$$x^2 - x + \left(-\frac{1}{2}\right)^2 = 6 + \left(-\frac{1}{2}\right)^2$$

$$x^2 - x + \frac{1}{4} = 6 + \frac{1}{4}$$

$$\left(x - \frac{1}{2}\right)^2 = \frac{25}{4}$$

$$x - \frac{1}{2} = \pm \sqrt{\frac{25}{4}} = \pm \frac{5}{2}$$

$$x = \frac{1}{2} \pm \frac{5}{2}$$

$$x = \frac{1}{2} - \frac{5}{2} \text{ or } x = \frac{1}{2} + \frac{5}{2}$$

$$x = -2 \text{ or } x = 3.$$

**EXERCISES**

1. Find the roots of the following quadratic equations using completing the square method.

(a)  $x^2 - 5x + 1 = 0$

(e)  $x^2 + 4x - 15 = 0$

(b)  $x^2 + 8x + 4 = 0$

(f)  $x^2 - 6x + 4 = 0$

(c)  $x^2 - 6x = 5$

(g)  $x^2 + 2x = 3$

(d)  $x^2 - 3x - 10 = 0$

(h)  $x^2 - 2x - 35 = 0$

2. Find the roots of the following quadratic equations by completing the square.

(a)  $2x^2 - 7x + 3 = 0$

(e)  $2x^2 - 7x - 15 = 0$

(b)  $2x^2 + x - 4 = 0$

(f)  $2y^2 + y = 3$

(c)  $2x^2 - 5x + 3 = 0$

(g)  $5x^2 + 9x = 3$

(d)  $6x^2 - x - 2 = 0$

(h)  $3x^2 + 6x - 4 = 0$

Unlike the graphs of linear functions, the graphs of quadratic functions do not look like straight line. Specifically, the graphs of quadratic functions are U-shaped curve called **parabolas**. In this sub-topic, we will learn about quadratic graphs and their features when the coefficient of  $x^2$  is positive or negative.

### ACTIVITY 6

1. What is the difference between the graphs of linear and quadratic functions?
2. Define the vertex of a parabola.
3. What is  $y$ -intercept?  $x$ -intercept?
4. What is line of symmetry?
5. Is there any difference between zeros or roots of a quadratic function and the  $x$ -intercepts of the graph of quadratic function?

Graphs of quadratic function have the following shapes of a parabola. The orientation of graph can be either open upwards or open downwards depending upon the coefficient of  $x^2$  in the given quadratic function.

As you know that the general form of quadratic function is  $y = ax^2 + bx + c$ . If coefficient of  $x^2$  is greater than zero, that is,  $a > 0$ , then the graph will open upward as shown in Figure 5. The point  $V$  is the turning point or vertex of the graph and it is the minimum point of the graph.

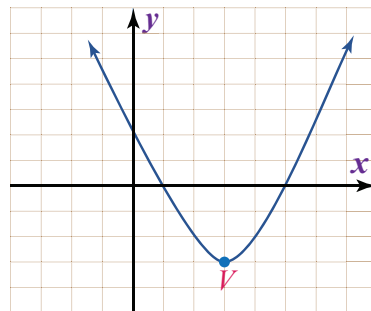


Figure 5.

### EXAMPLE 21

Draw the graph of  $y = x^2 + 2x - 3$ .

#### Solution

Plot the graph of given quadratic function  $y = x^2 + 2x - 3$  by taking some value for  $x$  as shown in the table below.

$x$	-4	-3	-2	-1	0	1	2	3	4
$y$	5	0	-3	-4	-3	0	5	12	21

Since, the coefficient of  $x^2$  is positive, the graph will be an upward opening as shown in figure 6. The point  $V(-1, -4)$  is the turning point or the vertex of the graph and it is the minimum point of the graph.

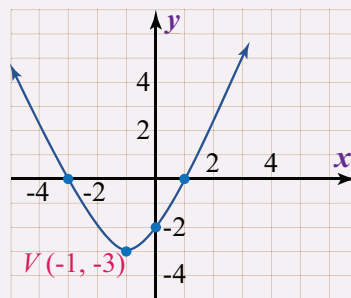


Figure 6.

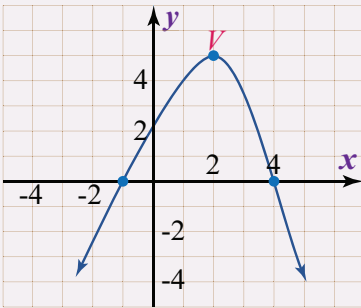


Figure 7.

If coefficient of  $x^2$  is less than zero, that is,  $a < 0$ , then the graph will open downward as shown in Figure 7. The point  $V$  is the turning point or vertex of the graph and it is the maximum point of the graph.

**EXAMPLE 22**

Draw the graph of  $y = -x^2 + 2x + 3$ .

**Solution**

Plot the graph of the given quadratic function

$y = -x^2 + x + 3$  by taking some value for  $x$  as shown in the table below.

$x$	-4	-3	-2	-1	0	1	2	3	4
$y$	-21	-12	-5	0	3	4	3	0	-5

Since, the coefficient of  $x^2$  is negative, the graph will be a downward opening as shown in figure 8. The point  $V(1, 3)$  is the turning point or the vertex of the graph and it is the maximum point of the graph.

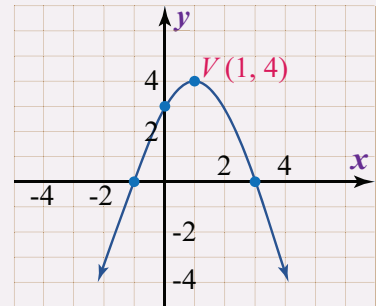


Figure 8.

When graphing parabolas, we want to include certain special points in the graph.

- The ***y-intercept*** is the point where the graph intersects the *y-axis*. The ***x-intercepts*** are the points where the graph intersects the *x-axis*.
- The ***vertex*** is the point that defines the *minimum* or *maximum* of the graph.
- The ***line of symmetry*** (also called the *axis of symmetry*) is the vertical line through the vertex, about which the parabola is symmetric.

For any parabola, we will find the vertex and *y-intercept*. In addition, if the *x-intercepts* exist, then we have to determine those as well.

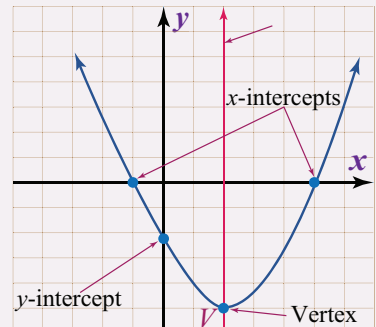


Figure 9.

Given a quadratic function  $f(x) = ax^2 + bx + c$ , find the *y-intercept* by evaluating the function when  $x = 0$ . In general,  $f(0) = a(0)^2 + b(0) + c = c$  and we have *y-intercept*  $(0, c)$ .

Next, the  $x$ -intercepts, if they exist, can be found by setting  $f(x) = 0$ . Doing this, we have  $ax^2 + bx + c = 0$ . If  $p$  and  $q$  are zeros of the quadratic function  $f(x)$ , then the points  $(p, 0)$  and  $(q, 0)$  are  $x$ -intercepts.

Using the fact that a parabola is symmetric, we can determine the vertical line of symmetry using the  $x$ -intercepts. To do this, we find the  $x$ -value by  $x = -\frac{b}{2a}$ . Therefore, the line of symmetry is the vertical line  $x = -\frac{b}{2a}$ . We can use the line of symmetry to find the vertex. So, the vertex is  $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ .

### EXAMPLE 23

Draw the graph of  $f(x) = -x^2 - 2x + 3$ .

#### Solution

**Step 1:** Determine the  $y$ -intercept. To do this, set  $x = 0$  and find  $f(0)$ .

$$f(0) = -(0)^2 - 2(0) + 3 = 0 + 0 + 3 = 3.$$

Therefore, the  $y$ -intercept is  $(0, 3)$ .

**Step 2:** Determine the  $x$ -intercepts if any. To do this, set  $f(x) = 0$  and solve for  $x$ .

$$f(x) = -x^2 - 2x + 3$$

$$0 = -x^2 - 2x + 3$$

$$x^2 + 2x - 3 = 0$$

$$(x + 3)(x - 1) = 0$$

$$x + 3 = 0 \text{ or } x - 1 = 0$$

$$x = -3 \text{ or } x = 1.$$

Here, where  $f(x) = 0$ , we obtain two solutions. Hence, there are two  $x$ -intercepts,  $(-3, 0)$  and  $(1, 0)$ .

**Step 3:** Determine the vertex. One way to do this is to first use  $x = -\frac{b}{2a}$  to find the  $x$ -value of the vertex and then substitute this value in the function to find the corresponding  $y$ -value. In the equation,  $f(x) = -x^2 - 2x + 3$ , we have  $a = -1$  and  $b = -2$ . Thus,

$$x = -\frac{b}{2a} = \frac{-(-2)}{2(-1)} = \frac{2}{-2} = -1.$$

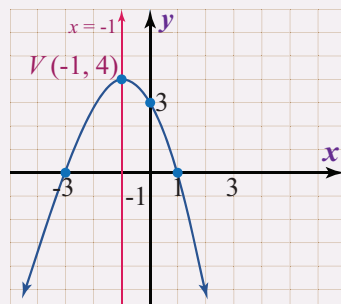


Figure 10.

Substitute -1 into the original function to find the corresponding  $y$ -value.

$$f(-1) = -(-1)^2 - 2(-1) + 3 = -1 + 2 + 3 = 4.$$

Therefore, the vertex is  $(-1, 4)$ .

Using the above information, we can draw the graph of  $f(x) = -x^2 - 2x + 3$  as shown in Figure 10.

### EXERCISES

- Find the vertex and line of symmetry for each of the following graphs of quadratic functions.
 

(a) $y = x^2 - 6x + 8 = 0$	(c) $y = 4x^2 - 32x + 62$
(b) $f(x) = x^2 - 6x + 5 = 0$	(d) $2x^2 + 12x + 16$
- Find the maximum or minimum value of each of the following quadratic functions.
 

(a) $y = x^2 - 8x + 12$	(c) $f(x) = 2x^2 + 4x + 5$
(b) $f(x) = x^2 + 2x - 8$	(d) $y = 3x^2 + 8x + 4$
- Draw the graphs of each of the following quadratic functions.
 

(a) $f(x) = 2x^2 - 4x + 8$	(c) $y = -x^2 - 6x + 1$
(b) $f(x) = -2x^2 - 12x + 3$	(d) $y = x^2 - 4x - 5$
- Determine whether each of the following functions open upward or downward?
 

(a) $y = x^2 - 9x + 20$	(d) $y = -2x^2 + 5x + 12$
(b) $f(x) = x^2 - 12x + 32$	(e) $y = -6x^2 + 13x - 6$
(c) $y = x^2 + 3x + 4$	(f) $y = 2x - x^2 + 6$
- Determine the  $x$ -intercepts, if they exist and the  $y$ -intercept.
 

(a) $y = x^2 + 4x + 12$	(c) $f(x) = 2x^2 + 5x - 3$
(b) $y = x^2 - 13x + 12$	(d) $f(x) = 3x^2 - 4x - 4$

Graphing a quadratic equation is a matter of finding its vertex, direction and  $x$ -intercepts and  $y$ -intercepts. In this sub-topic we will see how to solve quadratic equation graphically.

## ACTIVITY 7

- Graph the equation  $y = 2x - 3$  by plotting points.
- Let  $y = 2x^2 - x - 3$  be a quadratic equation. Then, find
  - $f(-2)$
  - $f(-1)$
  - $f(0)$
  - $f(5)$
- Let  $f(x) = \frac{1}{2}x^2 + \frac{5}{6}x - 3$  be a quadratic equation. Then, evaluate
  - $x = -\frac{b}{2a}$
  - $f\left(-\frac{b}{2a}\right)$
- When you are solving a quadratic equation graphically, what do you look for on the graph?

Recall that the solutions of a quadratic equation are called the **roots** of the equation. The roots of a quadratic equation are the ***x*-intercepts** of the graph.

## EXAMPLE 24

Draw the graphs of  $y = x^2 - x - 2$  using a table of values. Use the graph to find the following:

- Roots of  $y = x^2 - x - 2$
- Vertex of the graph
- Line of symmetry of graph

**Solution**

- First, we make a table of points by taking some suitable values of  $x$ .

$x$	-4	-3	-2	-1	0	1	2	3	4
$y$	18	10	4	0	-2	-2	0	4	10

By plotting and joining the points of the table, we will get a curve as shown in the figure 11.

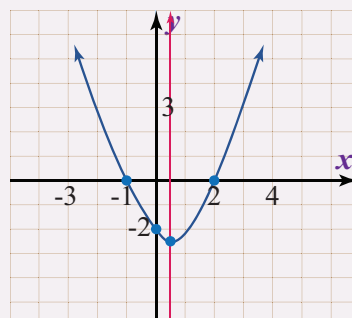


Figure 11.

- In the given equation,  $a = 1$  and  $b = -1$ . So, to find the vertex, we use the formula:

$$x = -\frac{b}{2a} = \frac{-(-1)}{2(1)} = \frac{1}{2} \quad \text{and} \quad f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2} - 2 = \frac{1}{4} - \frac{1}{2} - 2 = -\frac{9}{4}.$$

Thus, the vertex is  $\left(\frac{1}{2}, -\frac{9}{4}\right)$ .

- The line of symmetry is the vertical line  $x = \frac{1}{2}$ .

**EXAMPLE 25**

1. Using the graph of the quadratic equation represented by the graph shown in Figure 12.
  - (a) Find the number of roots and their values.
  - (b) Find the  $x$ -intercepts of the graph.
  - (c) Find the vertex of the graph.
  - (d) Find line of symmetry.
  - (e) Find the general equation of the quadratic function represented by the given graph.
2. Draw the graphs of  $y = 2x^2 - 8$  a table of values. Also, evaluate the following from the graph:
  - (a) Number of real roots.
  - (b) The vertex of the graph.
  - (c) Minimum value of the function.

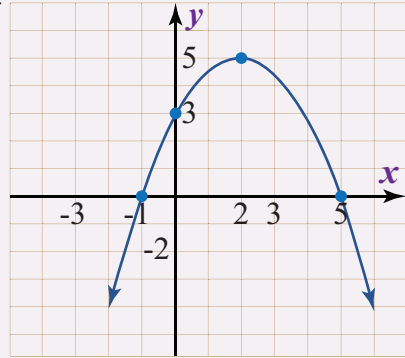


Figure 12.

You can determine the number and nature of roots of a quadratic equation by looking at its graph. The  $x$ -intercepts of the graph represent the roots of a quadratic equation. If the graph of equation cuts the  $x$ -axis at two distinct points, then the given equation will have two distinct real roots.

The graph of quadratic equation with two distinct real roots is as shown in Figure 13.

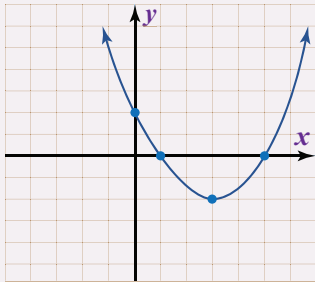


Figure 13.

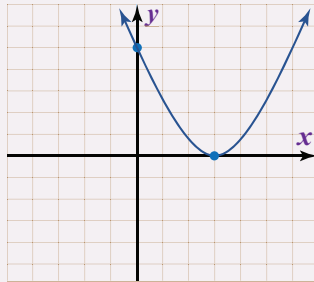


Figure 14.

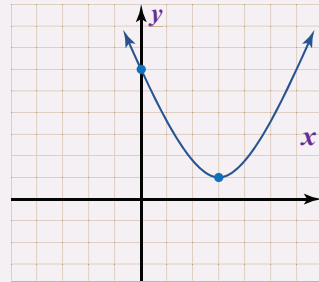


Figure 15.

If the graph of an equation touches the  $x$  axis at one point, then the given equation will have one real root. The graph of a quadratic equation with one real roots is as shown in Figure 14.

If the graph of quadratic equation never cuts or touches the  $x$  axis, then there will be no real root for that equation. The graph of a quadratic equation without any real root is shown in Figure 15.

**EXAMPLE 26**

Use the graph to find the zeros of the following quadratic functions.

(a)  $f(x) = x^2 + 2x - 3$

(b)  $f(x) = -x^2 + 2x + 8$

**Solution**

- (a) First, we make a table of points by taking some suitable values of  $x$ .

$x$	-4	-3	-2	-1	0	1	2	3	4
$y$	5	0	-3	-4	-3	0	5	12	21

By plotting and joining the points of the table, we will get a curve as shown in the figure 16.

As shown in the graph the graph intersects the  $x$ -axis at a  $(-3, 0)$  and  $(-1, 0)$ .

Therefore, the zeros of the given quadratic function are  $x = -3$  and  $x = -1$ .

- (b) First, we make a table of points by taking some suitable values of  $x$ .

$x$	-4	-3	-2	-1	0	1	2	3	4	5
$y$	-16	11	0	5	8	9	8	3	0	-7

By plotting and joining the points of the table, we will get a curve as shown in the figure 16.

As shown in the graph the graph intersects the  $x$ -axis at a  $(-2, 0)$  and  $(4, 0)$ .

Therefore, the zeros of the given quadratic function are  $x = -2$  and  $x = 4$ .

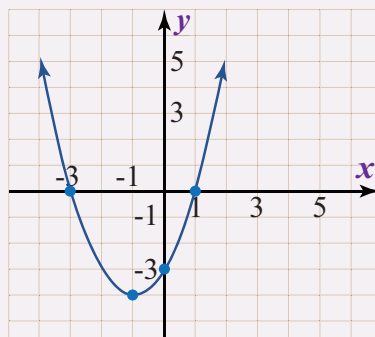


Figure 16.

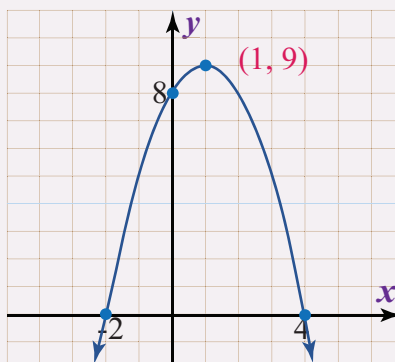


Figure 17.

**EXERCISES**

- Use the graph to find the zeros of the following quadratic functions.
  - $y = -2x^2 + 8x$
  - $f(x) = 2x^2 - 8x$
  - $y = x^2 + 2x - 15$
  - $-x^2 + 4x + 12$
- Draw the graph of each of the following quadratic functions and use the graph to find the zeros. Create a table of values if necessary.
  - $y = x^2 + 3x - 10$
  - $f(x) = x^2 + 8x$
  - $f(x) = 2x^2 + 16x$
  - $y = 3x^2 + 4x - 4$

3. Draw the graph of each of the following quadratic functions
- $y = x^2 + 4x + 5$
  - $-2x^2 - 4x + 6$
  - $f(x) = x^2 - 6x + 9$
  - $f(x) = -4x^2 + 4$
4. Draw the graphs of  $y = 3x - x^2 + 4$  using a table of values. Use the graph to find the roots of  $3x - x^2 + 4 = 0$ .

### KEY TERMS

- Axis of symmetry
- Completing the square
- Factorization
- Line of symmetry
- Parabola
- Quadratic function
- Quadratic equation
- Roots
- Solution
- Vertex
- $x$ -intercepts
- $y$ -intercept

### SUMMARY

- A quadratic function is a function of the form  $y = f(x) = ax^2 + bx + c$ , where  $a$ ,  $b$  and  $c$  are real numbers with  $a \neq 0$ .
- The domain of a quadratic function is the set of all real numbers,  $\mathbb{R}$ .
- In interval notation, the domain of any quadratic function is  $(-\infty, \infty)$ .
- An equation that can be written in the form  $ax^2 + bx + c = 0$ , where  $a$ ,  $b$  and  $c$  are real numbers with  $a \neq 0$ , is called a quadratic equation in  $x$ . The given form is called standard form.
- If the product two numbers is zero, then one of these factors must be zero. That is, for any real numbers  $p$  and  $q$ , if  $p \times q = 0$ , then either  $p = 0$  or  $q = 0$ .
- To factorize a general quadratic expression,  $x^2 + bx + c = 0$ ,  $a \neq 1$  it is recommended to follow the following three steps.

**Step 1:** Find the product of  $a \times c$ ,

**Step 2:** Find pairs of factors of  $a \times c$

If  $a \times c$  is positive, find a pair whose sum is  $b$ .

If  $a \times c$  is negative, find a pair whose difference is  $b$ .

**Step 3:** Write  $b$  in terms of the sum of the pair and then factorize again.

- To solve word problems, we follow the following steps.

**Step 1:** Read the problem carefully and draw diagram whenever possible.

**Step 2:** List all of the unknown numerical quantities involving in the problem and call it any letters such as  $p, q, y, z$  etc.

**Step 3:** Using information given in the wording of the problem, set up a word equation that contains words as well as numbers.

**Step 4:** Put in numbers in place of words whenever possible to set up an algebraic equation containing one variable and known numerical constant.

**Step 5:** Solve the equation for the variable.

- To solve a quadratic equation using completing the square method, we follow the following steps.

**Step 1:** Write the given quadratic equation in the standard form.

**Step 2:** Make the coefficient of  $x^2$  equal to 1, if it is not.

**Step 3:** Shift the constant term to right hand side (R.H.S) of the equation.

**Step 4:** Add  $\left(\frac{1}{2} \text{ of the coefficient of } x\right)^2$  to both sides of the equation.

**Step 5:** Express the left-hand side (L.H.S) as the perfect square of a suitable binomial expression and simplify the right-hand side (R.H.S).

**Step 6:** Take square root of both sides.

**Step 7:** Obtain the values of  $x$  by shifting the constant term from L.H.S to R.H.S.

- Features of the graph of quadratic function.
  - (i) The orientation of a parabola is that it either opens up or opens down.
  - (ii) The vertex is the lowest point or the highest point on the graph.
  - (iii) The axis of symmetry is the vertical line that goes through the vertex. If  $h$  is the  $x$ -coordinate of the vertex, the equation for the axis of symmetry is  $x = h$ .
  - (iv) The maximum or minimum value of a parabola is the  $y$ -coordinate of the vertex.
  - (v) The  $y$ -intercept is the point at which the parabola crosses the  $y$ -axis.

- (vi) The  $x$ -intercepts are the points at which the parabola crosses the  $x$ -axis. The domain of a quadratic function is all real numbers,  $(-\infty, \infty)$ .
- (vii) The range of a quadratic function starts or ends with the value of the  $y$ -coordinate of the vertex.
- The graph of a quadratic function  $f(x) = ax + bx + c$  has a minimum turning point with  $a > 0$  and a maximum turning point when  $a < 0$ . The turning point lies on the line of symmetry.

**EXERCISES**

- Given a quadratic function  $f(x) = -x^2 - x + 6$ . Then,
  - Determine  $a$ ,  $b$  and  $c$
  - Find the value of  $y$  when  $x = -1$
  - The value of  $y$  when  $x = 2$
  - $f(-3)$ ,  $f(-2)$ ,  $f(0)$ ,  $f(3)$
- Find the zeros of each of the following quadratic functions.
 

(a) $f(x) = x^2 - 9$	(c) $f(x) = x^2 + x - 12$
(b) $g(x) = 2x^2 - 8x$	(d) $f(x) = x^2 - x - 10$
- For each of the following quadratic equations identify  $a$ ,  $b$  and  $c$ .
 

(a) $x(2 - x) = 0$	(c) $(x - 3)(x + 3) = 0$
(b) $(x + 3)(3x - 2) = 0$	(d) $(7 - 2x)(2x + 7) = 0$
- Factorize each of the following quadratic equations.
 

(a) $8x^2 - 24x = 0$	(c) $-2x^2 - 5x + 12 = 0$
(b) $x^2 - x = 0$	(d) $3x^2 + 14x + 15 = 0$
- Find the roots of the following quadratic equations using factorization method.
 

(a) $x^2 + x - 12 = 0$	(e) $-2x^2 + x + 6 = 0$
(b) $x^2 + 8x + 7 = 0$	(f) $-3x^2 - 10x + 8 = 0$
(c) $x^2 + 5x + 6 = 0$	(g) $6x^2 - x - 12 = 0$
(d) $x^2 - 7x + 12 = 0$	(h) $7x^2 + 15x + 2 = 0$
- Find the roots of the following quadratic equation by completing the square.
 

(a) $4x^2 - 12x + 5 = 0$	(c) $x^2 + 4x - 5 = 0$
(b) $x^2 - 6x + 1 = 0$	(d) $9x^2 - 12x - 21 = 0$

7. Find the vertex and line of symmetry of each of the following graphs of the quadratic functions.
- (a)  $y = x^2 - 4x$  (c)  $y = 2x^2 - 3x - 9$   
(b)  $f(x) = x^2 - x - 6$  (d)  $g(x) = 2x^2 + 12x$
8. Find the maximum or minimum value of the following quadratic functions.
- (a)  $y = x^2 - 4x + 4$  (c)  $f(x) = -x^2 - x + 6$   
(b)  $f(x) = x^2 + 2x - 1$  (d)  $y = 3x^2 + 10x - 8$
9. Draw the graphs of the following quadratic functions.
- (a)  $f(x) = 2x^2 + 7x + 6$   
(b)  $f(x) = x - 2x^2 + 15$
10. Determine whether each of the following functions open upward or downward?
- (a)  $y = x^2 - x + 12$  (c)  $y = x^2 + 8x + 15$   
(b)  $f(x) = -x^2 - x + 2$  (d)  $y = -2x^2 + 11x - 12$
11. Determine the  $x$ -intercepts, if they exist and the  $y$ -intercept.
- (a)  $y = x^2 + 6x + 9$  (c)  $f(x) = 2x^2 + x - 6$   
(b)  $y = x^2 + 2x - 15$  (d)  $f(x) = 3x^2 - 10x + 8$
12. Use the graph to find the zeros of the following quadratic functions.
- (a)  $y = -2x^2 + 32x$  (c)  $y = x^2 - 2x - 15$   
(b)  $f(x) = 4x^2 - 20x$  (d)  $-x^2 - x + 12$
13. The three sides of a right triangle form three consecutive even integers. Find the lengths of the triangle.
14. The hypotenuse of a right triangle  $ABC$  is 5 cm long. One leg is 1 cm less than the other. Find the lengths of the legs.
15. The product of two consecutive odd natural numbers is 15. Find the two even numbers.

# CHAPTER



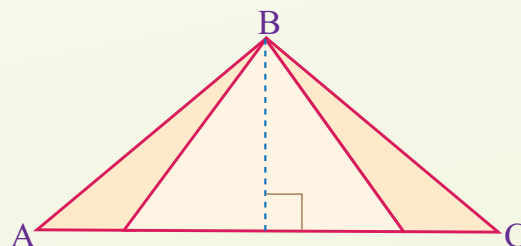
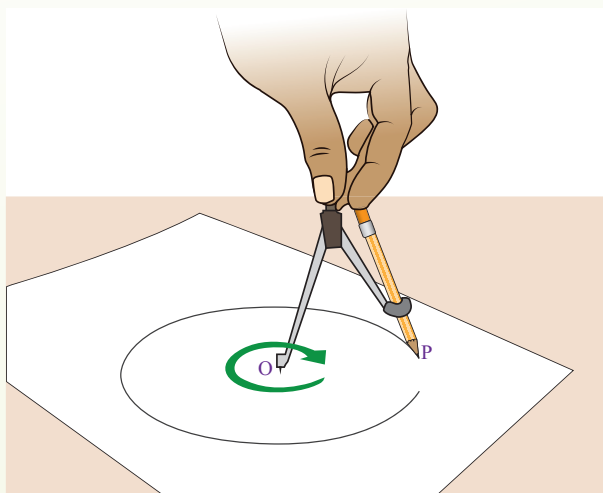
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# 6

## MENSURATION

### Chapter Contents

- 6.1 The Circle as a Locus
- 6.2 Circle Theorems
- 6.3 Tangents to a Circle
- 6.4 Alternate Segments
- 6.5 Perimeter of Plane Shapes
- 6.6 Area of Rectangles and Squares
- 6.7 Area of Parallelograms
- 6.8 Area of Triangles
- 6.9 Circles Arcs and Sectors
  - Key Terms
  - Summary
  - Exercises



## **Chapter Outcomes**

Upon the completion of this chapter learners will be able to:

- discuss and construct the circle as a locus;
- state and use circle theorems;
- identify and construct tangent to a circle;
- identify and construct alternate segment;
- calculate perimeter of plane shapes;
- find the area of rectangle and square;
- find the area of parallelograms;
- find the area of triangles;
- discuss and construct circles;
- calculate the arcs and sectors.

## Introduction

A *circle* is a closed shape formed by tracing a point that moves in a plane such that its distance from a given point is constant. The word circle is derived from the Greek word “*Kirkos*”, meaning *hoop* or *ring*. *Circles* are the most symmetric plane figures and they possess remarkable geometric properties. In this topic, we shall explore the circle as a locus, circle theorems, tangent to a circle and alternate segment.

The knowledge of a circle is very important in solving many real-life problems. The importance of circle can be seen from the designs featured in the artefacts of ancient civilizations as well as its use in modern architecture all over the world. In this topic, we will also learn about perimeter of rectangle and square; area of parallelogram; rectangle and square; and arcs and sectors.

In geometry, a *locus* (plural *loci*) is a set of points which satisfies a given condition or situation for a shape or a figure. The word *locus* is derived from the word *location*. Before the 20<sup>th</sup> century, geometric shapes were considered as an entity or place where points can be located or can be moved. But in modern Mathematics the entities are considered as the set of points that satisfy the given condition. In this sub-topic, we will explore the circle as a locus.

### ACTIVITY 1

1. What is locus of a point?
2. List objects in your surrounding which are circular in shape.
3. Drawing practice:

Take a compass and fix a pencil in it. Put its pointed leg on a point on a sheet of a paper. Open the other leg to some fixed distance. Keeping the pointed leg on the same point, rotate the other leg through one revolution as shown in Figure 1.

Based on your drawing try to answer the following questions.

- (a) What can you say about the shape of the figure traced by the pencil on paper?
  - (b) Choose at least five points on your drawing and using your ruler measure the distance between these points and the point O. What can you say about these distances?
  - (c) Define a circle. How did we get a circle?
4. Describe the various parts of circle.

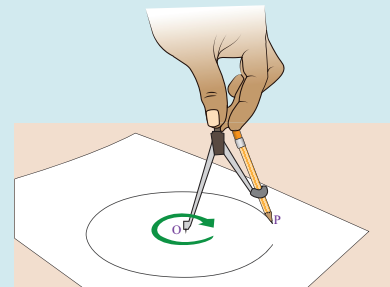


Figure 1.

The locus of points defines a shape in geometry.

### DEFINITION

A path consisting of a set of points whose location is governed by some definite rule called **locus**. Locus in Latin means “position” or “location”.

### EXAMPLE 1

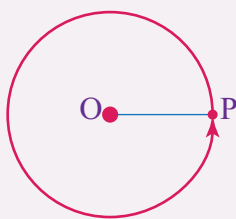


Figure 2.

Let a point P move in such a way that it is always at a constant distance,  $r$ , from a fixed-point O, as shown in Figure 2.

The locus of a point P is a circle which has its center O and has radius equal to  $r$  units.

A circle is the locus of all the points which are equidistant from the center. So, a circle as a locus can be defined as follows:

### DEFINITION

With respect to the locus of the points or loci, the circle is defined as the set of all points equidistant from a fixed point, where the fixed point is the center of the circle and the distance of the sets of points from the center is the radius of the circle. Let us say, O is the center of the circle and  $r$  is the radius of the circle, that is, the distance from point O to the set of all points or the locus of the points.

To be able to understand the properties of a circle and use them in different applications, you must be aware of the basic terms related to a circle.

### DEFINITION

- The distance from the center of a circle to any point on its boundary is called **radius** of circle.
- A line segment passing through the center that connects two points on the boundary is called **diameter** of circle.
- Any line segment touching the circle at two different points on its boundary is called **chord** of a circle.

Figure 3 shows various parts of a circle:

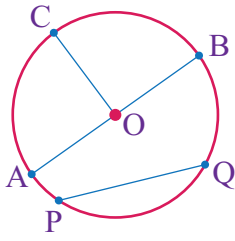


Figure 3.

Point O is the center of the circle. A circle is usually named by its center.

The line segment OC (or  $\overline{OC}$ ) is radius of the circle.

The line segment AB (or  $\overline{AB}$ ) is diameter of the circle.

The line segment PQ (or  $\overline{PQ}$ ) is chord of a circle.

### DEFINITION

The set of points on a circle (any part of a circle) contained in one of the two half-planes determined by the line through any two distinct points of a circle are called an **arc** of a circle.

In circle O of figure 4, the part of the circle shown in broken line from point A to point D is **arc AD**, symbolized by  $\widehat{AD}$ .

An **arc** is any piece of the circle between two points on the circle. As shown in figure 4, arc ABD is denoted by  $\widehat{ABD}$  and arc ACB is denoted by  $\widehat{ACB}$ .

Arcs of a circle can be classified as **semi-circle**, **major arc** and **minor arc**.

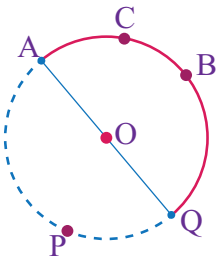


Figure 4.

- (i) **Semi-circle** is half of a circle whose end points are the end points of a diameter of the circle O and its degree measure is  $180^\circ$ .
- (ii) In figure 5,  $\overline{AB}$  is diameter of the circle O. So,  $\widehat{ACB}$  and  $\widehat{AEB}$  are semi-circles.
- (iii) **Major arc** is the part of a circle which is *greater than* a semi-circle. In figure 5, points C and D are not the end points of the diameter. So,  $\widehat{CAD}$  or  $\widehat{CED}$  is **major arc** of circle O.
- (iv) **Minor arc** is the part of a circle which is *less than* a semi-circle. In figure 5, points C and D are not the end points of the diameter. So,  $\widehat{ABD}$  is **minor arcs** of the circle O. Also,  $\widehat{AC}$  and  $\widehat{AED}$  are minor arcs.

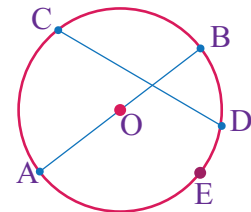


Figure 5.

### EXERCISES

1. What do you call the longest chord of a circle?
2. How many radii does a circle have?

3. How many circles can be drawn through three noncollinear points.
4. Determine whether each of the following statements is true or false.
  - (a) A circle can be drawn through three points on a plane.
  - (b) A line segment joining two points on a circle is a diameter.
  - (c) Any chord of a circle is a diameter of a circle.
  - (d) The length of a diameter is twice of the length of a radius.
5. Use Figure 6 to answer the following questions.

- (a) Name the radii of the circle.
- (b) Name the diameters of the circle.
- (c) Name the chords of the circle.
- (d) Which two terms can be used to describe  $\overline{PR}$  ?
- (e) Name five line segments that have the same length.
- (f) Which term best describes  $\overline{OC}$  ?

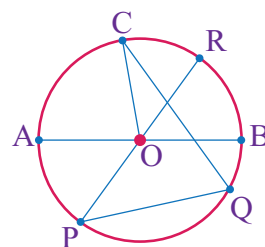


Figure 6.

### ACTIVITY 2

1. Define perpendicular bisector of a chord.
2. What is a central angle of a circle.
3. Show that the perpendicular bisector of a chord passes through the center of a circle.

In this sub-topic, we will discuss some important circle theorems. We will also learn to apply these theorems to solve different types of mathematical problems involving circle.

#### Theorem 1:

The perpendicular from the center of a circle to a chord bisects the chord. In other words, the perpendicular bisector of a chord passes through the center of the circle.

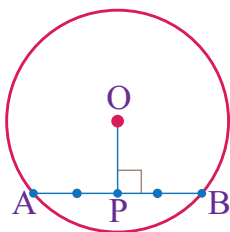


Figure 7.

Figure 7 shows a chord  $\overline{AB}$  bisected by a perpendicular  $\overline{OP}$  from the center  $O$ .

As shown in the figure, chord  $\overline{OP}$  is perpendicular bisector of  $\overline{AB}$ .

Symbolically,  $\overline{OP} \perp \overline{AB}$  and  $\overline{AP} = \overline{PB}$ .

**EXAMPLE 2**

In the figure 8, O is the center of the circle, RS is the diameter of the circle.  $PR = RQ = 15$  cm,  $OR = 8$  cm. Find the radius of the circle.

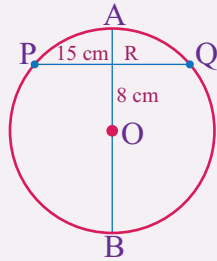


Figure 8.

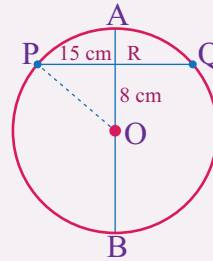


Figure 9.

**Solution**

Join O with P as shown in the figure 9.

OP is the radius of the circle and diameter AB bisects the chord PQ.

Therefore,  $\overline{AB} \perp \overline{PQ}$ .

$\Delta OPR$  is right angled-triangle. So, by Pythagoras Theorem, we have

$$OP^2 = PR^2 + RO^2$$

$$OP = \sqrt{PR^2 + RO^2} = \sqrt{15^2 + 8^2} = \sqrt{225 + 64} = 17 \text{ cm.}$$

Therefore, the radius of the circle is 17 cm.

**EXAMPLE 3**

$\overline{AB}$  and  $\overline{CD}$  are two parallel chords on the opposite sides of the center of a circle and the distance between  $\overline{AB}$  and  $\overline{CD}$  is 21 cm. If  $AB = 24$  cm and  $CD = 18$  cm, find the radius of the circle.

**Solution**

Let O be the center of the circle and r be its radius, as shown in the figure 10.

Draw  $\overline{OE} \perp \overline{AB}$  and  $\overline{OF} \perp \overline{CD}$ . Now,  $AE = EB$  and

$CF = FD$ .  $AE = 12$  cm  $CF = 9$  cm

Let  $OE = x$ , therefore  $OF = 21 - x$ .

In right-angled triangle OAE by Pythagoras theorem, we have

$$x^2 + 12^2 = r^2 \dots\dots\dots (1)$$

In right angled-triangle OCF by Pythagoras theorem, we have

$$(21 - x)^2 + 9^2 = r^2 \dots\dots\dots (2)$$

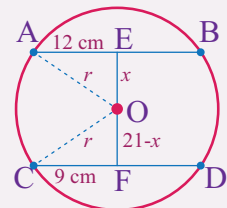


Figure 10.

Equating eq. (1) and eq. (2), we get

$$x^2 + 12^2 = (21 - x)^2 + 9^2$$

$$x^2 + 144 = 441 - 42x + 81$$

$$144 = 522 - 42x$$

$$42x = 378$$

$$x = 9.$$

Putting  $x = 5$  in equation (1), we get

$$9^2 + 12^2 = r^2$$

$$81 + 144 = r^2$$

$$r = \sqrt{225} = 15\text{cm}.$$

Therefore, the radius of the circle is 15 cm.

### Theorem 2:

Equal chords of a circle are equidistant from the center.

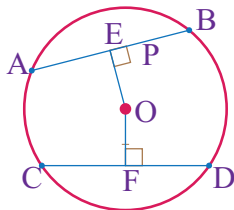


Figure 11.

Figure 11 shows chords  $\overline{AB}$  and  $\overline{CD}$  equidistant from the center of circle  $O$ . That is,  $\overline{OP} = \overline{OQ}$ .

As shown in figure 11, if  $AB = CD$  then  $OP = OQ$  and vice versa. The converse of the theorem is also true. That is, two or more chords in a circle that are equidistant from the circle are always equal in length.

### EXAMPLE 4

In Figure 12,  $O$  is the center of the circle,  $AB = CD$ ,

If  $OE = 2x + 3$  and  $OF = 4x - 5$ . Find the value of  $x$ .

#### Solution

Since, equal chords of a circle are equidistant from the center.

Thus,

$$OE = OF$$

$$2x + 3 = 4x - 5$$

$$2x - 4x = -5 - 3$$

$$-2x = -8$$

$$x = 4$$

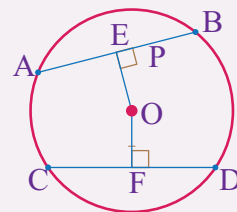


Figure 12.

**Theorem 3:**

When two circles intersect each other, then the line joining their centers bisects their common chord at right angles.

In Figure 13,  $O_1$  and  $O_2$  are the centers and  $\overline{AB}$  is the common chord of the two circles. Also,  $\overline{O_1O_2} \perp \overline{AB}$  and  $AP = BP$ .

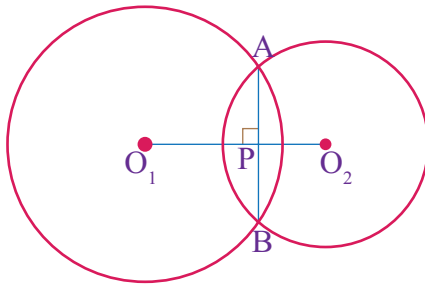


Figure 13.

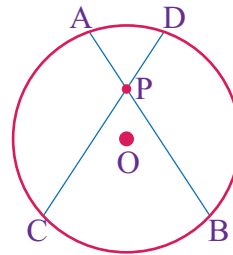


Figure 14.

**Theorem 4:**

Products of intercepts of two intersecting chords are equal.

In Figure 14,  $AP \times PB = CP \times PD$ .

**EXAMPLE 5**

In the figure 15, PQ and RS are two chords intersecting at point P. Find the value of  $b$ .

**Solution**

We know that products of intercepts of two intersecting chords are equal. Thus,

$$AP \times PB = CP \times PD$$

$$12 \times 4 = b \times 8$$

$$8b = 48$$

$$b = 6.$$

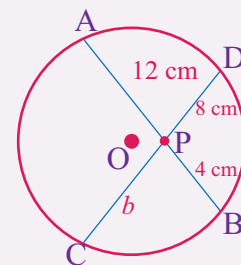


Figure 15.

When an angle is on a circle, the vertex is on the circumference of the circle. One type of angle on a circle is the inscribed angle, from the previous section.

**DEFINITION**

A **central angle** of a circle is an angle whose vertex is the center of the circle and whose sides are radii of the circle.

In figure 16,  $\angle AOB$  is the central angle of the circle  $O$ .

In figure 16  $\widehat{APB}$  is said to be *intercepted* by  $\angle AOB$  and  $\angle AOB$  is said to be *subtended* by  $\widehat{APB}$ . The measure of the central  $\angle AOB$  is equal to the measure of the intercepted  $\widehat{APB}$ .

That is,  $m(\angle AOB) = m(\widehat{APB})$ .

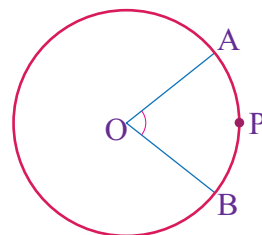


Figure 16.

### Theorem 5:

The angle subtended by an arc at the center of a circle is double the size of the angle subtended by the same arc at the circle.

Theorem 5 is illustrated by figure 17, where the central angle  $\angle AOC$  is subtended by  $\widehat{AC}$  and the angle  $\angle ABC$  is subtended by  $\widehat{AC}$ . Thus,  $m(\angle AOC) = 2m(\angle ABC)$ .

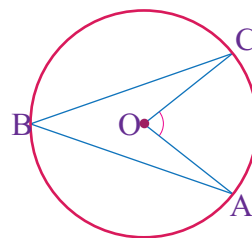


Figure 17.

### EXAMPLE 6

In the figure 18,  $m(\angle ABC) = 35^\circ$  and  $O$  is the center of the circle.

Find the value of  $\alpha$ .

#### Solution.

By Theorem 5 we have  $m(\angle AOC) = 2m(\angle ABC)$ . So,  
 $\alpha = 2m(\angle ABC) = 2 \times 35^\circ = 70^\circ$  and  $m(\angle AOC) = \alpha = 70^\circ$ .

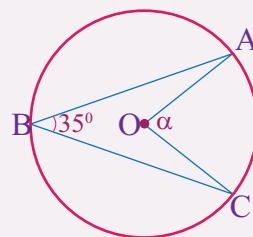


Figure 18.

### DEFINITION

An *inscribed angle* is an angle whose vertex lies on the circle and whose sides are chords of the circle.

In figure 19,  $\angle BAC$  and  $\angle BDC$  are inscribed angles of circle with center  $O$ .  $\angle BAC$  and  $\angle BDC$  inscribed  $BC$  and  $\widehat{BC}$  subtends  $\angle BAC$  and  $\angle BDC$ .

The relationship between the measure of the central angle and inscribed angle by the same arc is given in theorem 6.

### Theorem 6:

The measure of an inscribed angle is equal to *half* the measure of its intercepted arc.

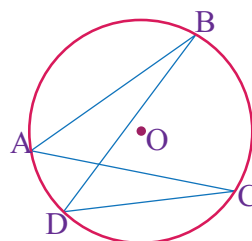


Figure 19.

Theorem 6 is illustrated in Figure 20. Note that  $\angle APB$  is the inscribed angle and it is intercepted by  $\widehat{AB}$ . Thus,

$$m(\angle APB) = \frac{1}{2}m(\widehat{AB})$$

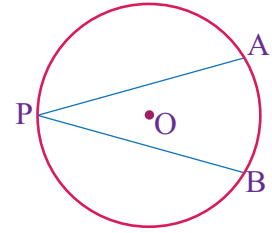


Figure 20.

**EXAMPLE 7**

In figure 21, O is the center of the circle. If  $m(\angle ACF) = 25^\circ$  and  $m(\angle AFC) = 75^\circ$ , then find  $m(\angle BAC)$  and  $m(\angle BOC)$ .

**Solution**

In  $\triangle ACF$ , we have

$$m(\angle CAF) + m(\angle AFC) + m(\angle ACF) = 180^\circ$$

$$m(\angle CAF) + 75^\circ + 25^\circ = 180^\circ$$

$$m(\angle CAF) + 100^\circ = 180^\circ$$

$$m(\angle CAF) = 180^\circ - 100^\circ = 80^\circ$$

By Theorem 6, we have

$$m(\angle CAF) = \frac{1}{2}m(\widehat{BC})$$

$$80^\circ = \frac{1}{2}m(\widehat{BC})$$

$$m(\widehat{BC}) = 2 \times 80^\circ = 160^\circ$$

$$m(\angle BAC) = \frac{1}{2}m(\widehat{BC}) = \frac{1}{2} \times 160^\circ = 80^\circ$$

$$m(\angle BOC) = m(\widehat{BC}) = 160^\circ \text{ because } \angle BOC \text{ is central angle.}$$

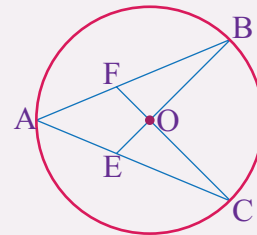


Figure 21.

**Theorem 7:**

If two inscribed angles intercept the same arc, then these angles are congruent.

Theorem 7 is illustrated in Figure 22. Note that  $\angle APB$  and  $\angle AQB$  both intercept  $\widehat{AB}$ . Because  $m(\angle APB) = \frac{1}{2}m(\widehat{AB})$  and  $m(\angle AQB) = \frac{1}{2}m(\widehat{AB})$ . Therefore,  $\angle APB \cong \angle AQB$ .

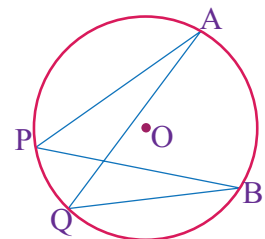


Figure 22.

**Theorem 8:**

An angle inscribed in a semi-circle is a right angle.

Theorem 8 is illustrated in Figure 23, where  $\angle ABC$  is inscribed in the semicircle  $\widehat{ABC}$ . Note that  $\angle ABC$  also intercepts semicircle  $\widehat{ADC}$ . That is,

$$m(\angle ABC) = \frac{1}{2}m(\widehat{ADC}) = \frac{1}{2} \times 180^\circ = 90^\circ.$$

Therefore,  $\angle ABC$  is a right angle.

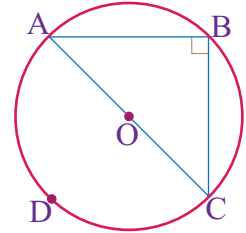


Figure 23.

**Theorem 9:**

The measure of an angle formed by two chords intersecting inside a circle is **half** the sum of the measure of the arcs subtending the angle and its vertical opposite angle.

In Figure 24, the two chords  $\overline{AB}$  and  $\overline{CD}$  intersecting at point P inside circle O.  $\angle DPB$  intercepts  $\widehat{DB}$  and  $\angle APC$  intercepts  $\widehat{AC}$ . Thus, according to Theorem 9, we have

$$m(\angle DPB) = \frac{1}{2}m(\widehat{AC} + \widehat{DB}).$$

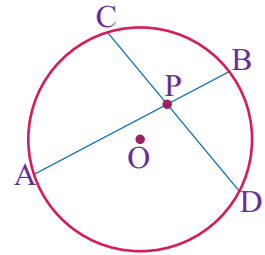


Figure 24.

**EXAMPLE 8**

In Figure 25,  $\widehat{AC} = 52^\circ$  and  $\widehat{DB} = 78^\circ$ . Find the measure of  $\angle DPB$ .

**Solution**

$$m(\angle DPB) = \frac{1}{2}m(\widehat{AC} + \widehat{DB})$$

$$m(\angle DPB) = \frac{1}{2}(52^\circ + 78^\circ) = \frac{1}{2} \times 130^\circ = 65^\circ.$$

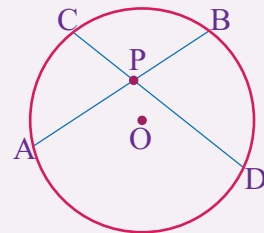


Figure 25.

**EXERCISES**

- In the figure 26,  $OP = 12$  cm,  $AB = 32$  cm and  $OP \perp AB$ . Calculate the value of  $x$ , if O is the center of the circle
- In figure 27, CD and AB are two chords intersecting at P. If  $CP = 3$  cm,  $PD = 30$  cm and  $PB = 12$  cm, then find the value of AP.

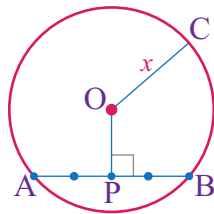


Figure 26.

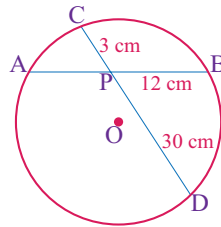


Figure 27.

- A chord of length 24 cm is at a distance of 5 cm from the center of a circle. What is the length of the chord of the same circle which is at a distance of 12 cm from the center of the given circle?
- In figure 28, O is the center of the circle, chord AB is equal to chord CD and  $\overline{OE}$  and  $\overline{OF}$  are perpendiculars to chords. If  $FD = 12$  cm and  $OE = 9$  cm, find the length of  $\overline{AE}$ .

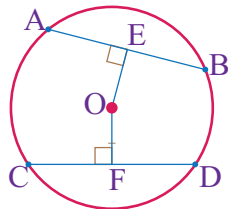


Figure 28.

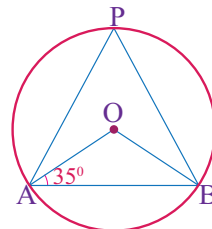


Figure 29.

- In the figure 29, O is the center of circle and  $\angle OAB = 35^\circ$ , find  $\angle APB$ .
- In the figure 30, O is the center of the circle,  $\angle AOB = 125^\circ$ ,  $\angle BCE = 28^\circ$ . Find the measure of  $\angle CEB$ .

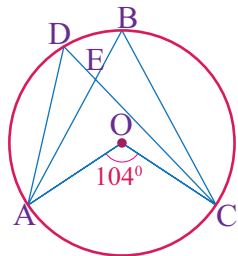


Figure 30.

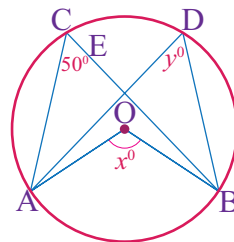


Figure 31.

- In the figure 31, if O is the center of the circle, find the value of  $x$  and  $y$ .
- In the figure 32, AC is the largest chord of the circle with center O and  $\angle BAO = 40^\circ$ . Find the measure of  $\angle ACB$ .

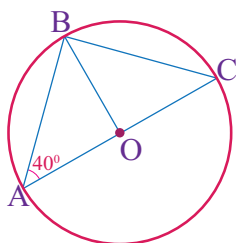


Figure 32.

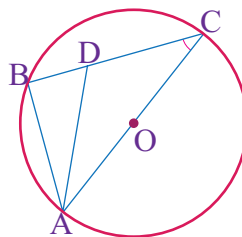


Figure 33.

9. In the figure 33, AC is the diameter of the circle with center O. If  $AD = CD$  and  $\angle ACD = 25^\circ$ , find  $\angle BAD$
10. If the chord of a circle is equal to its radius, find the angle measure subtended by the chord in the alternate segment of the circle.

The word “tangent” means “to touch”. The Latin word for the same is “tangere”. In general, we can say that the line that intersect the circle exactly at one point on its circumference and never enters the circle’s interior is a tangent. In this sub-topic, we will learn about the concept of tangent to a circle and various theorems related to tangents.

### ACTIVITY 3

1. Define tangent of a circle and point of tangency?
2. Give at least three properties of of tangent to a circle.
3. What does it mean when the line are tangent.
4. How do you draw a tangent?

In geometry, a tangent is a line drawn from an external point and passes through a point on the circle.

### DEFINITION

- (a) If a line intersects a circle at exactly one point, then the line is called a tangent line of the circle.
- (b) The point at which the tangent intersects the circle is called the point of tangency (or point of contact)
- (c) If a line intersects a circle at two points, then the line is called a **secant line** of the circle.

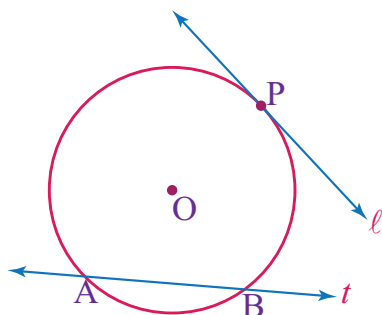


Figure 34.

Figure 34 shows the line  $l$  is the tangent of the circle O and the point P is point of tangency. From the figure 34, it can be concluded that there is only

one tangent to a circle through a point which lies on the circle. The line  $\ell$  is the secant line of the circle  $O$  and it intersects the circle at points  $A$  and  $B$ .

If a line intersects a circle at two points, then the line is called a **secant line** of the circle.

Some of the important theorems based on tangent properties of a circle are given below with their respective examples.

**Theorem 1**

The tangent to the circle is perpendicular to the radius of the circle at the point of contact.

In Figure 35, radius  $OA$  is perpendicular to tangent  $\ell$ .

That is,  $\overline{OP} \perp \ell$ .

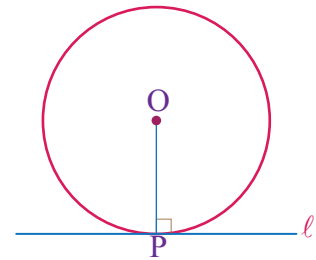


Figure 35.

**EXAMPLE 9**

In the figure 36,  $\overline{AB}$  is a tangent to a circle with center  $O$  and radius  $6$  cm at point  $B$ . It meets the line  $OA$  such that  $OA = 10$  cm. What is the length of  $AB$ ?

**Solution**

We know that the tangent is perpendicular to the radius at the point of contact. As shown in Figure 36,  $\triangle ABO$  is right angled-triangle. Thus, by Pythagoras theorem, we have

$$AB^2 + BO^2 = AO^2$$

$$AB^2 + 6^2 = 10^2$$

$$AB^2 + 36 = 100$$

$$AB^2 = 100 - 36 = 64$$

$$AB = \sqrt{64} = 8 \text{ cm.}$$

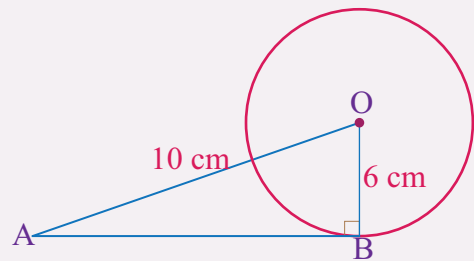


Figure 36.

**Theorem 2**

If two tangents are drawn from an external point of the circle, then they are of equal lengths.

In Figure 37,  $P$  is the external point from where the two tangents  $\overline{PA}$  and  $\overline{PB}$  are drawn. Here,  $PA$  is equal to  $PB$ . That is,  $PA = PB$ .

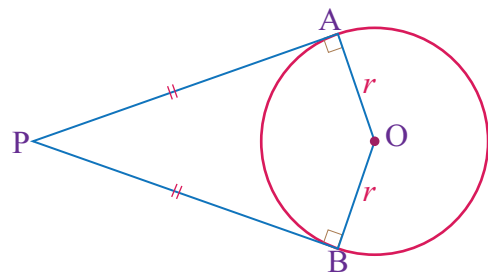


Figure 37.

**EXAMPLE 10**

In the figure 38,  $\overline{PA}$  and  $\overline{PB}$  are the tangents of the circle with center  $O$ . Find the value of  $x$ .

**Solution**

The length of two tangents from a common external point to a circle is equal. Thus,

$$PA = PB$$

$$x^2 - 2x = 12 - x$$

$$x^2 - 2x + x - 12 = 0$$

$$x^2 - x - 12 = 0$$

$$(x + 3)(x - 4) = 0$$

$$x + 3 = 0 \text{ or } x - 4 = 0$$

$$x = -3 \text{ or } x = 4.$$

Therefore, the value of  $x$  is 4 cm.

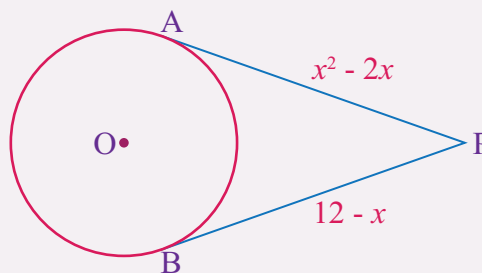


Figure 38.

**Theorem 3**

If from a point outside a circle, a secant and a tangent are drawn, then the tangent is the **mean proportional** between the secant and its external segment.

In figure 39,  $\overline{PB}$  is a secant line and  $\overline{PA}$  is a tangent line. Thus,

$$PA^2 = PB \times PC.$$

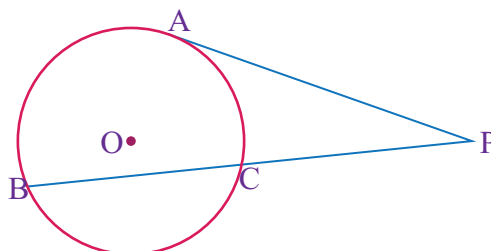


Figure 39.

**EXAMPLE 11**

In figure 40,  $\overline{PB}$  is a secant line and  $\overline{PC}$  is the tangent line to the circle  $O$ . Find the length of the tangent  $\overline{PC}$ .

**Solution**

By Theorem 3, we have

$$PC^2 = PB \times PA$$

$$PC^2 = (14 + 18) \times 18 = 32 \times 18 = 576$$

$$PC = \sqrt{576} = 24 \text{ cm.}$$

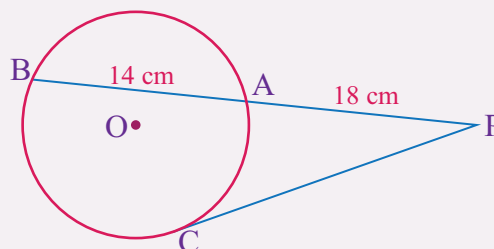


Figure 40.

EXERCISES

1. In the given figure 41, O is the center of circle, AB is tangent to circle at point B and  $\angle BAO = 30^\circ$ , find  $\angle AOB$ .

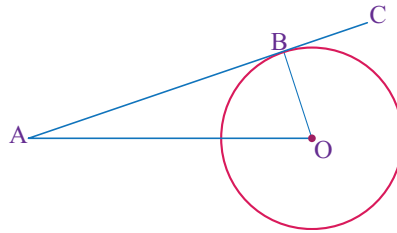


Figure 41.

2. In the given figure 42, the radius of a circle is 9 cm and AB is a tangent drawn from point A. If  $AO = 41$  cm and O is the center of circle, find the length of the tangent

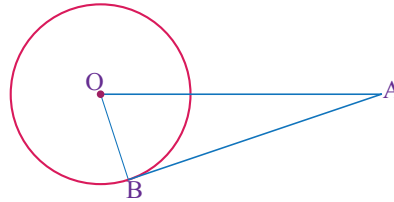


Figure 42.

3. In the figure 43,  $\overline{PA}$  and  $\overline{PB}$  are the tangents of the circle with center O. Find the value of  $x$ .

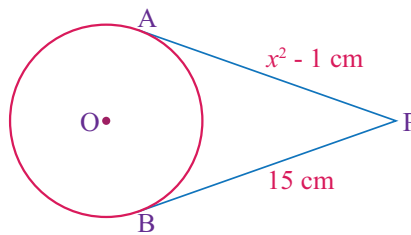


Figure 43.

4. In figure 44, PC is a secant line and PA is tangent of line circle. Find the length of the tangent of the circle.

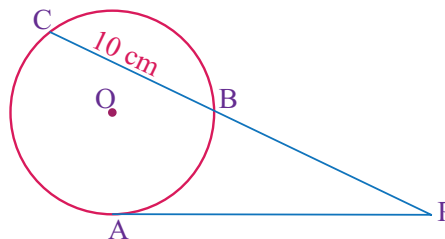


Figure 44.

5. In figure 45, PB is tangent of the circle. If  $PB = 12$  cm and  $PA = 8$  cm, then find the radius of the circle.

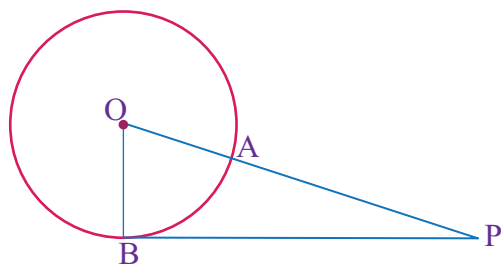


Figure 45.

There exist several geometric properties and theorems about circles. We have studied circle theorems and tangent to a circle. In this sub-topic, we will learn about *alternate segment*.

#### ACTIVITY 4

1. What is segment of a circle?
2. What is chord of a circle?
3. Construct a circle and draw tangent line to a circle. Through the point of contact, draw a chord that should be inclined to the tangent to form an angle.
  - (a) Identify the chord which makes the angle with the tangent and locate the angle in the alternate segment of the chord.
  - (b) How many angles are formed between the tangent line and the chord of the circle?
  - (c) Identify the major segment and the minor segment in the circle.

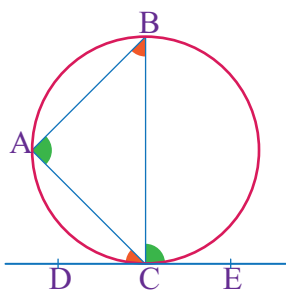


Figure 46.

The segment of a circle is the region between a chord and the corresponding arc of the circle. When a chord is drawn, it creates a major segment and a minor segment in the circle.

Let us observe the figure 46 in which  $\overline{DE}$  is the tangent and  $\overline{BC}$  is a chord.  $\angle BCE$  is made by the tangent and chord  $BC$ . This is equal to  $\angle BAC$  in the alternate segment.  $\angle ACD$  is made by the tangent  $DE$  and chord  $AC$ . This is equal to  $\angle ABC$  in the alternate segment. Here,  $\angle BCE = \angle BAC$  and  $\angle ACD = \angle ABC$ . This is what the alternate segment theorem states.

#### Theorem 1

For any circle, the angle between a tangent and a chord through the point of contact of the tangent is equal to the angle made by the chord in the alternate segment.

**EXAMPLE 12**

A tangent  $\overline{PR}$  is drawn at point B on the circle, as shown in figure 47. A and C are pointing such that  $\overline{BA} = \overline{BC}$ . If  $\angle QBC = 55^\circ$ , what is the measure of  $\angle BCA$ ?

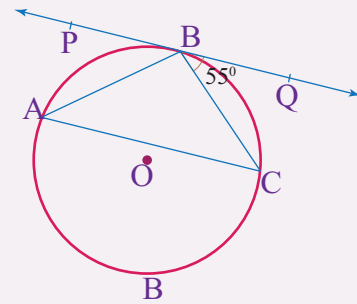


Figure 47.

**Solution**

$\angle QBC = \angle BAC = 55^\circ$  ... (by alternate segment theorem.)

$\overline{BA} = \overline{BC}$  ..... (given.)

$\therefore \triangle ABC$  is isosceles triangle.

$\angle BAC = \angle BCA = 55^\circ$  ..... (opposite angles of an isosceles triangle are equal.)

**EXAMPLE 13**

Given that BC is the chord that makes  $65^\circ$  with the tangent  $\overline{PQ}$ , as shown in figure 48. Find measures of  $\angle ABC$  and  $\angle QAB$ .

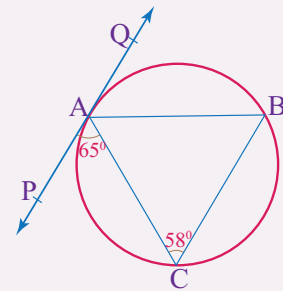


Figure 48.

**Solution**

$\angle ABC = \angle PAC = 65^\circ$  ..... (angle in the alternate segment.)

$\angle QAB = \angle ACB = 58^\circ$  ..... (angle in the alternate segment.)

**EXAMPLE 14**

In figure 49, EF is tangent to the circle O. If BC is diameter of the circle and  $\angle ACB = 30^\circ$ , find the measure of angles  $x$  and  $y$ .

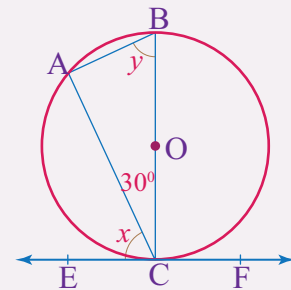


Figure 49.

**Solution**

$\angle CAB = 90^\circ$  ..... (angle in a semi-circle.)

$\angle ACB = 30^\circ$  ..... (given)

$\angle ACB + \angle CAB + \angle ABC = 180^\circ$  ..... (Sum of angles in a triangle =  $180^\circ$ .)

$$30^\circ + 90^\circ + y = 180^\circ$$

$$120^\circ + y = 180^\circ$$

$$y = 60^\circ$$

$x = y = 60^\circ$  ..... (by alternate segment theorem.)

**EXERCISES**

1. In figure 50,  $\overline{ED}$  is tangent to the circle O. If  $\angle ABC = 65^\circ$ , find the measure  $\angle ACB$ .
2. In figure 51,  $\angle CBD = 55^\circ$  and  $\angle ABC = 65^\circ$ . What is the measure of  $\angle ACB$ .

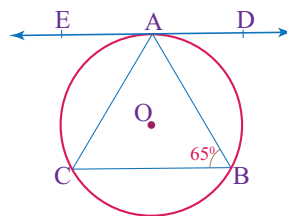


Figure 50.

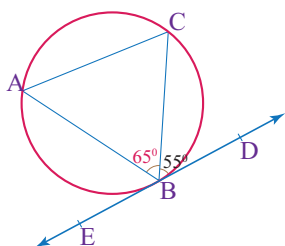


Figure 51.

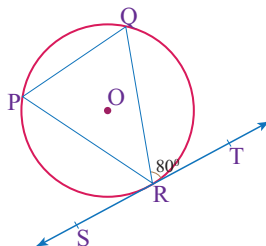


Figure 52.

3. In figure 52, line ST is tangent to the circle O. If  $\angle QRT = 80^\circ$ . Find the measure of  $\angle RPQ$ .
4. In the given figure 53,  $\overline{FD}$  and  $\overline{DE}$  are two tangents of a circle. If  $\angle BCE = 75^\circ$  and  $\angle ABC = 65^\circ$ , then find measures of  $\angle BAF$  and  $\angle ACD$ .

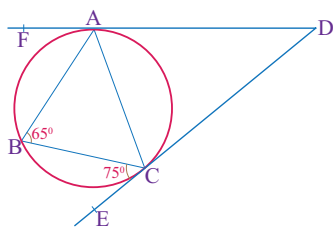


Figure 53.

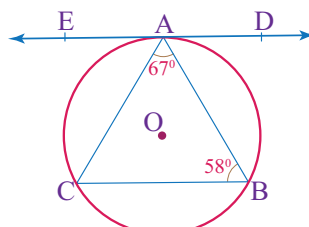


Figure 54.

5. In the given figure 54,  $\overline{DE}$  is a tangent at point A. If  $\angle BAC = 67^\circ$  and  $\angle ABC = 58^\circ$ , then find measures of  $\angle CAE$  and  $\angle BAD$ .

In geometry, perimeter can be defined as the path or the boundary that surrounds a plane shape. The word perimeter has been derived from the Greek word “*peri*” meaning **around** and “*metron*” which means **measure**. Measurement of length is applicable in many real-life situations. For example, farmers have to fence their fields to protect their crops from animals. The required length of the fence can be calculated accurately if one is familiar with the concept of perimeter. In this sub-topic, you will discuss the length and properties of the plane geometrical figures and you

## ACTIVITY 5

1. What is plane figures?
2. Explain two dimensional geometrical figures.
3. Give some examples that represent plane shapes.
4. In group, state the properties of square, rectangle, parallelogram, rhombus and trapezium.
5. Elizabeth is out for a run. She decides to run around a park in a town. The length of the side of the park are 200 meters, 350 meters, 185 meters and 225 meters. How many meters will Elizabeth run?

Perimeter is the distance around the edge of a plane shapes. The perimeter of all plane shapes can be determined by adding the lengths of their sides.

### Triangle

Recall that triangle is a three-sided polygon and it can be classified into different types based on the measure of its sides and angles. The perimeter of a triangle is defined as the total length of its boundary.

#### EXAMPLE 15

In figure 55, ABC is a triangle with side length 5 cm, 7 cm and 10 cm. Find its perimeter.

##### Solution

$$\begin{aligned}\text{Perimeter (P)} &= AB + BC + CA \\ &= 5 \text{ cm} + 7 \text{ cm} + 10 \text{ cm} \\ &= 22 \text{ cm}.\end{aligned}$$

Therefore, the perimeter of triangle ABC is 22 cm.

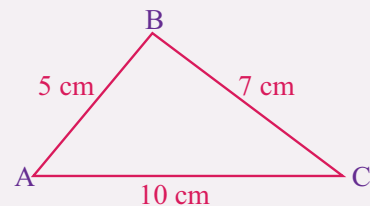


Figure 55.

#### EXAMPLE 16

Find the length of the missing side of a triangular-shaped road sign whose perimeter is 54 m and the two sides are 19 m each.

##### Solution

Let the length of the missing side be  $b$ . Then,

perimeter of a triangle = Sum of lengths of three sides

$$54 = 19 + 19 + b$$

$$54 = 38 + b$$

$$b = 54 - 38 = 16 \text{ m.}$$

Therefore, the length of the missing side is 16 m.

In general, the perimeter of a triangle with length of sides  $a$ ,  $b$  and  $c$  units is given by the formula:

$$P = a + b + c.$$

### EXAMPLE 17

Find the perimeter of  $\triangle ABC$  having side length  $a = 6$  cm,  $b = 8$  cm and  $c = 10$  cm.

#### Solution

$$P = a + b + c = 6 \text{ cm} + 8 \text{ cm} + 10 \text{ cm} = 24 \text{ cm.}$$

Therefore, the perimeter of the triangle is 24 cm.

## Rectangle

Rectangles are four-sided polygons. The perimeter of a rectangle is the sum of the length of its sides. Hence, we can find the perimeter of a rectangle by adding the lengths of all four sides of a rectangle.

### EXAMPLE 18

In figure 56, ABCD is a rectangle whose side lengths are given.

#### Solution

The perimeter of the given rectangle is:

$$P = 9 \text{ cm} + 19 \text{ cm} + 9 \text{ cm} + 19 \text{ cm}$$

$$= 56 \text{ cm.}$$

Therefore, the perimeter of the given rectangle is 56 cm.



Figure 56.

Since opposite sides of a rectangle are always equal, we need to find the dimensions of only two sides to find the perimeter of a rectangle. Thus, the perimeter of a rectangle with length and width is twice of the sum of length and width.

**EXAMPLE 19**

A rectangular playground is 30 m long and 19 m. Find its perimeter.

**Solution**

We know that the perimeter of a rectangle =  $2 \times$  (sum of lengths of adjacent sides)  
 $= 2 \times (30 \text{ m} + 19 \text{ m}) = 2 \times 49 \text{ m} = 98 \text{ m}.$

Therefore, the perimeter of the playground is 98 m.

In general, the perimeter (P) of a rectangle whose length is  $\ell$  and width is  $w$  is given by the formula:

$$P = 2 \times (\ell + w).$$

**EXAMPLE 20**

Find the perimeter and area of a rectangular farm if its length and breadth are 280 m and 90 m, respectively.

**Solution**

$$P = 2(\ell + w) = 2 \times (280 + 90) = 2 \times 370 = 740 \text{ m}.$$

Therefore, the perimeter of the rectangular land is 740 m.

**EXAMPLE 21**

The length of a rectangle is 4 less than 3 times its width. If its length is 14 cm, find its perimeter.

**Solution**

Let  $w$  be the width of the rectangle. Then, its length:  $\ell = 3w - 4$ .

Since the length of the rectangle is 14 cm, we have

$$14 = 3w - 4$$

$$3w = 14 + 4 = 18$$

$$w = 6 \text{ cm}.$$

$$P = 2(\ell + w) = 2 \times (14 + 6) = 2 \times 20 = 40 \text{ cm}.$$

Therefore, the perimeter of the rectangular land is 40 cm.

**EXAMPLE 22**

The perimeter of a rectangular pool is 96 meters. If the length of the pool is 16 meters, then find its width.

**Solution**

$$P = 2(\ell + w) = 2\ell + 2w$$

$$96 = 2 \times 16 + 2w$$

$$96 = 32 + 2w$$

$$2w = 96 - 32 = 64$$

$$w = 32.$$

Therefore, the width of the pool is 32 meters.

**Square**

A square is a type of rectangle in which the adjacent sides are equal. In other words, all sides of a square are equal.

The perimeter of a square is the total length of all sides of the square. Hence, we can find the perimeter of a square by adding all its sides. That is, perimeter

$$(P) = s + s + s + s = 4s \text{ units.}$$

The perimeter of a square whose side length is  $s$  units is 4 times the length of its side.

**EXAMPLE 23**

One of the sides of a square is 6 cm. What will be its perimeter?

**Solution**

$$\text{Perimeter} = 4 \times \text{side length} = 4s = 4 \times 6 \text{ cm} = 24 \text{ cm.}$$

Therefore, the perimeter of the given square is 24 cm.

In general, the perimeter of a square whose side length is  $s$  is given by the formula:

$$P = 4s$$

**EXAMPLE 24**

If the perimeter of a square is 20 cm, then what will be the length of its side?

**Solution**

Let the side length be  $s$  cm. Then,

$$P = 4s$$

$$20 \text{ cm} = 4s$$

$$s = 20 \div 4 = 5 \text{ cm.}$$

Therefore, the length of its side is 5 cm.

**EXAMPLE 25**

In figure 57, ABCD is a square. Find the perimeter of square ABCD.

**Solution**

Since ABCD is a square,  $AB = BC = CD = DA$ .

$$4a = a + 18$$

$$4a - a = 18$$

$$3a = 18$$

$$a = 6 \text{ cm}$$

Thus, perimeter of square ABCD =  $4 \times 6 \text{ cm} = 24 \text{ cm}$ .

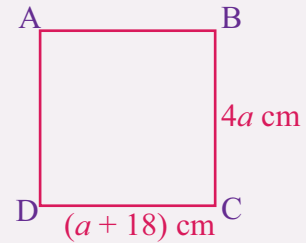


Figure 57.

**Parallelogram**

Recall that a quadrilateral (four-sided polygon) is called a parallelogram if its opposite sides are parallel and are of equal length. The perimeter of a parallelogram is equal to the sum of lengths of all four sides.

**EXAMPLE 26**

In figure 58, ABCD is a parallelogram whose sides are 7 cm, 12 cm, 7 cm and 12 cm. Find the perimeter of the parallelogram.

**Solution**

$$\begin{aligned} \text{The perimeter (P)} &= AB + BC + CD + DA \\ &= 7 \text{ cm} + 12 \text{ cm} + 7 \text{ cm} + 12 \text{ cm} \\ &= 38 \text{ cm} \end{aligned}$$

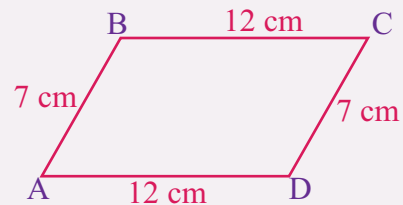


Figure 58.

Therefore, the perimeter of the given parallelogram is 38 cm.

In general, the perimeter of a parallelogram with adjacent sides  $a$  and  $b$  is given by the formula:

$$P = 2 \times (a + b).$$

**EXAMPLE 27**

The perimeter of a parallelogram is equal to 86 cm and one of its sides is 12 cm. Find the length of the other side.

**Solution**

$$P = 2 \times (a + b) = 2a + 2b$$

$$86 = 2 \times 12 + 2b$$

$$86 = 24 + 2b$$

$$2b = 86 - 24 = 62$$

$$b = 31 \text{ cm.}$$

Therefore, the length of the other side is 31 cm.

### EXAMPLE 28

In figure 59, ABCD is a parallelogram. Find the perimeter of parallelogram ABCD.

#### Solution

Since ABCD is a parallelogram,

AB = CD and BC = DA. Thus,

$$3x + 5 = 6x - 10$$

$$6x - 3x = 5 + 10$$

$$3x = 15$$

$$x = 5.$$

Hence, AB =  $4 \times 5 - 5 = 20 - 5 = 15$  cm; BC =  $3 \times 5 + 5 = 15 + 5 = 20$  cm; CD = AB = 15 cm; and BC = DA = 20 cm.

$$P = 2 \times (a + b) = P = 2 \times (15 + 20) = 2 \times 35 = 70 \text{ cm.}$$

Therefore, the perimeter of parallelogram ABCD is 70 cm.

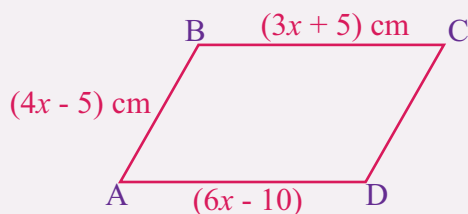


Figure 59.

## Rhombus

A rhombus is a type of parallelogram in which the adjacent sides are equal. In other words, all sides of a rhombus are equal.

The perimeter of a rhombus is the total length of all sides of the rhombus. Hence, we can find the perimeter of a rhombus by adding all its sides. That is, perimeter (P) =  $s + s + s + s = 4s$  units.

### EXAMPLE 29

One of the sides of a rhombus is 7 cm. What will be its perimeter?

#### Solution

Perimeter =  $4 \times \text{side length} = 4s = 4 \times 7 \text{ cm} = 28 \text{ cm}$ .

Therefore, the perimeter of the given rhombus is 28 cm.

In general, the perimeter of a rhombus whose side length is  $s$  is given by the formula:

$$P = 4s$$

**EXAMPLE 30**

If the perimeter of a rhombus is 48 cm, then what will be the length of its side?

**Solution**

Let the side length be  $s$  cm. Then,

$$P = 4s$$

$$48 \text{ cm} = 4s$$

$$s = 48 \div 4 = 12 \text{ cm.}$$

Therefore, the length of its side is 12 cm.

**Trapezium**

Recall that a trapezium is a quadrilateral (four-sided polygon) with exactly one pair of parallel sides. The parallel sides are called the bases and the non-parallel sides are the legs of the trapezium.

The perimeter of a trapezium is the sum of the lengths of its four sides.

**EXAMPLE 31**

In figure 60, ABCD is a trapezium with length of all four sides are given as shown. Find the perimeter of trapezium ABCD.

**Solution**

$$\begin{aligned} \text{The perimeter (P)} &= AB + BC + CD + DA \\ &= 10 \text{ cm} + 9 \text{ cm} + 15 \text{ cm} + 7 \text{ cm} \\ &= 41 \text{ cm.} \end{aligned}$$

Therefore, the perimeter of trapezium ABCD is 41 cm.

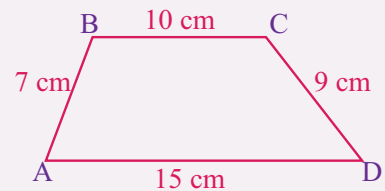


Figure 60.

In general, if the length of the four sides of a trapezium are  $a$ ,  $b$ ,  $c$  and  $d$  units, then, the perimeter of the trapezium is given by the formula:

$$P = a + b + c + d.$$

**EXAMPLE 32**

If the length of parallel sides of a trapezium is 13 cm and 17 cm and the length of non-parallel sides is 7 cm each. Find its perimeter.

**Solution**

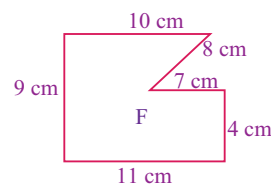
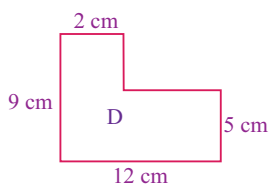
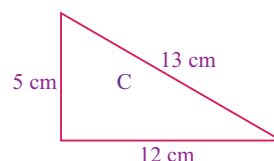
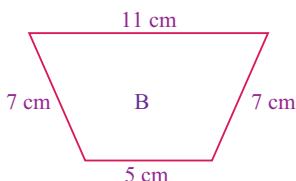
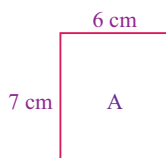
Perimeter of this trapezium is the sum of the lengths of its four sides. Thus,

$$P = 13 \text{ cm} + 17 \text{ cm} + 7 \text{ cm} + 7 \text{ cm} = 44 \text{ cm.}$$

Therefore, the perimeter of this trapezium is 44 cm.

**EXERCISES**

- Find the perimeter of a triangle whose sides are 5 cm, 7 cm and 9 cm.
- The perimeter of a rectangular wire is 198 meters. The same wire is bent into the shape of an equilateral triangle. Find the lengths of each of its sides.
- A rectangle has a width of 5 meters and a length of 125 cm. Find the perimeter of the rectangle both in meters and centimeters.
- If the perimeter of a rectangle is 40 cm and the length of one of its sides is 9 cm. What will be the length of the other side?
- The length of a triangle is 5 less than twice of the width. If the length is 13 cm, then what will be the perimeter of the rectangle?
- The width of a rectangular field is 240 m less than the length. The perimeter of the field is 940 meters. Find the dimensions of the field.
- Find the perimeter of a square with side 17 cm.
- Ella wants to put a lace around a square cloth with length 15 meters. If she bought a lace of length 45 meters, how much more does she need?
- Find the perimeter of a parallelogram if the base and side length are 18 cm and 23 cm.
- Determine the length of another side of a parallelogram whose base is 9 cm and the perimeter is 60 cm.
- Find the perimeter of a rhombus with side 13 cm.
- If the perimeter of a rhombus is 120 cm, what will be the length of sides of the rhombus?
- The parallel sides of a trapezium measures 34 cm and 28 cm. Its other sides are 19 cm and 15 cm. Find the perimeter of the trapezium.
- Find the perimeter of each of the following plane figures.



The area can be defined as the amount of space covered by a flat surface of a particular shape. It is measured in terms of the “*number of square units*”. The area of a polygon is the number of square units inside the polygon. We frequently use area in our daily life. For example, if we want to paint the walls of a room, we have to know the area to be painted so as to buy the adequate amount of paint. In this topic, we will discuss the area of rectangle and square.

### ACTIVITY 6

1. What do we mean by “*square unit*”?
2. Suppose the dimensions of a dining room is 4 m by 4.5 m. If the dimensions of a tile is 20 cm by 30 cm, how many tiles are required to be fitted into the room?
3. A rectangular park with length 24 m and width 16 m is surrounded by a path of 2 m. Find the area of the path.

The area of rectangle is the number of unit squares that can fit into a rectangle.

### DEFINITION

Area occupied by a rectangle within its boundary is called the *area of the rectangle*.

The plane figure shown in figure 61 is a rectangle. So, we can find area of this rectangle by counting the squares inside the rectangle.

There are 24 squares inside the rectangle. Therefore, we can say that the area of this rectangle is 24 square units. We can also find its area by multiplying number of units in its length by the number of units in its width.

As shown in figure 61, there are 6 squares in its length and 4 squares in its width. Thus, 6 units  $\times$  4 units = 24 square units.

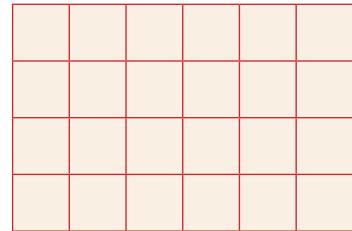


Figure 61.

In general, we can find the area of any rectangle by multiplying its length and its width.

### Theorem 1:

The area (A) of rectangle is the product of its length ( $\ell$ ) and width ( $w$ ). That is,

$$\text{Area} = \text{length} \times \text{width}$$

$$\text{Symbolically, } A = \ell \times w$$

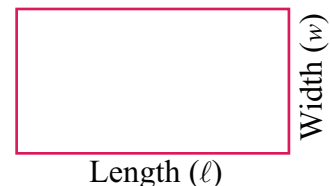


Figure 62.

**EXAMPLE 33**

Find the area of a rectangular farm if its length 240 m and its width is 95 m.

**Solution**

$$\begin{aligned}\text{Area of rectangle} &= \text{length} \times \text{width} \\ &= 240 \text{ m} \times 95 \text{ m} = 22,800 \text{ m}^2.\end{aligned}$$

**Note:**

The term *base* refers to the *length* of a rectangle and the term *height* refers to the *width* of a rectangle. So, we can say that the area of a rectangle = *base*  $\times$  *height* or  $A = b \times h$ .

**EXAMPLE 34**

Find the area and perimeter of each of the following rectangles.

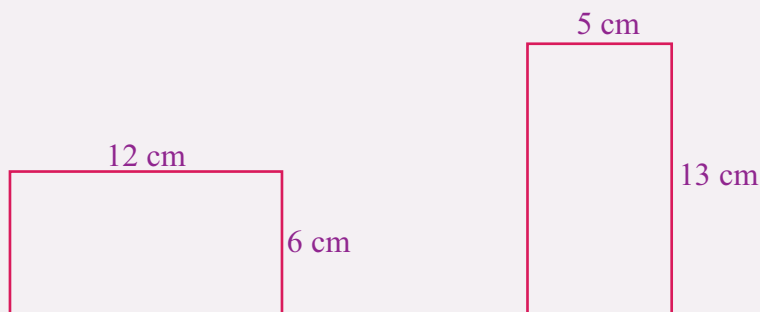


Figure 63.

**Solution**

(a) Area:  $A = \ell \times w = 12 \text{ cm} \times 6 \text{ cm} = 72 \text{ cm}^2$ .

Therefore, area of this rectangle is  $72 \text{ cm}^2$ .

(b) Area:  $A = \ell \times w = 5 \text{ cm} \times 13 \text{ cm} = 65 \text{ cm}^2$ .

Therefore, area of this rectangle is  $65 \text{ cm}^2$ .

**EXAMPLE 35**

Calculate the area of a rectangle with a length of 6 units and a width of 14 units.

**Solution**

$$A = \ell \times w = 6 \text{ units} \times 14 \text{ units} = 84 \text{ square units.}$$

Therefore, the area of the rectangle is 84 square units.

**EXAMPLE 36**

The area of a rectangle is  $300 \text{ cm}^2$  and the length is  $15 \text{ cm}$ . What is the width of the rectangle?

**Solution**

$$A = \ell \times w$$

$$300 \text{ cm}^2 = 15 \text{ cm} \times w$$

$$w = 300 \text{ cm}^2 \div 15 \text{ cm} = 20 \text{ cm}.$$

Therefore, the width of the rectangle is  $20 \text{ cm}$ .

**EXAMPLE 37**

The length of the rectangle exceeds its width by  $2 \text{ cm}$  and area of the rectangle is  $195 \text{ sq.cm}$ . Find the dimensions of the rectangle.

**Solution**

Let  $\ell$  be the length and  $w$  be the width of the rectangle and  $\ell = w + 2$ .

$$\ell \times w = A$$

$$(w + 2) w = 195$$

$$w^2 + 2w - 195 = 0$$

$$(w + 15) (w - 13) = 0$$

$$w + 15 = 0 \text{ or } w - 13 = 0$$

$$w = -15 \text{ or } w = 13$$

Since  $w$  represents width of a rectangle, the negative value is not possible. So, the width is  $13 \text{ cm}$ .

Thus,  $\ell = w + 2 = 13 + 2 = 15 \text{ cm}$ .

Therefore, the dimensions of the rectangle are  $15 \text{ cm}$  by  $13 \text{ cm}$ . Recall that a square is a rectangle having four equal lengths of sides. The area of a square with side length  $a$  is  $a^2$  square units.

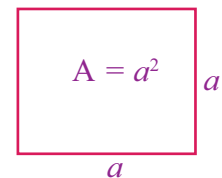


Figure 64.

**EXAMPLE 38**

Find the area of a square plot of land with side  $35 \text{ m}$ .

**Solution**

$$A = a \times a = a^2 = (35 \text{ m})^2 = 1,225 \text{ m}^2.$$

Therefore, area of the square is  $1,225 \text{ m}^2$ .

**EXAMPLE 39**

The area of a square is  $169 \text{ cm}^2$ . What is the length of each side?

**Solution**

$$A = a^2$$

$$169 \text{ cm}^2 = a^2$$

$$a = \sqrt{169} = 13 \text{ cm.}$$

Therefore, the length of each side of the square is 13 cm.

**EXAMPLE 40**

Length of a side of a square field is 275 m. What will be the cost of levelling the field at a rate of L\$10 per square meter?

**Solution**

$$A = a^2 = (275 \text{ m})^2 = 75,625 \text{ m}^2$$

Thus, the cost of levelling the field is  $75,625 \times 1 \text{ m}^2 = 75,625 \times 10 \text{ cents} = \text{L\$}756250$ .

**EXERCISES**

- Find the area of a rectangle of length 42 cm and width 28 cm.
- Find the area of a rectangle whose length is 18 cm and width 9 cm.
- The area of a rectangular plot of land is  $480 \text{ m}^2$ . If its width is 60 m, then find its length.
- The area of a rectangular fence is 600 square meters. If the width of the fence is 15 meters, then find its length.
- Find the area of a square whose side is 8 cm.
- The area of a square is thrice the area of a rectangle. If the area of the square is  $768 \text{ cm}^2$  and length of the rectangle is 32 cm, then find the perimeter of the rectangle.
- Find the cost of flooring a square floor having perimeter 60 m, if the cost of flooring is L\$80 per square meters.
- Find the perimeter of each of the following plane figures.

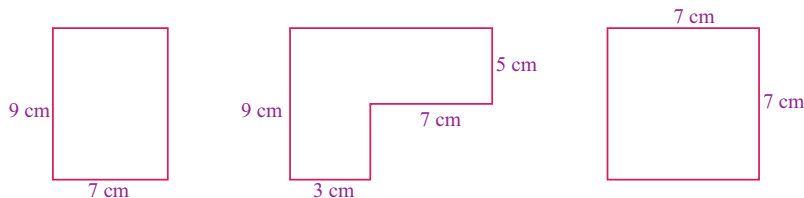


Figure 65.

A

B

C

In geometry, a parallelogram is a special type of quadrilateral which has four sides and the pair of opposite sides are parallel. In a parallelogram, the opposite sides are of equal length. In this sub-topic, we will learn the area of parallelograms.

The area of a parallelogram is the space enclosed within its four sides. Area is equal to the product of length and height of the parallelogram.

### ACTIVITY 7

1. What is the height of a parallelogram?
2. The area of a parallelogram is 72 square units and the base is 9 cm. Find the altitude of the parallelogram.
3. What is the area of a rhombus whose length is 13 cm long and its height is 7 cm long?

### EXAMPLE 41

If the base of a parallelogram is 7 cm and the height is 4 cm, then find its area.

#### Solution

$$\begin{aligned}\text{Area of a parallelogram} &= \text{base} \times \text{height} \\ &= 7 \text{ cm} \times 4 \text{ cm} = 28 \text{ cm}^2.\end{aligned}$$

We can derive the area formula of parallelogram from the formulas of areas of rectangle and triangle.

### Theorem 1:

Suppose  $a$  and  $b$  are the lengths of parallel sides of a parallelogram and  $h$  is the corresponding height of the parallelogram, then based on the length of sides and height of it, the formula for its area is given by:

$$\text{Area} = \text{Base} \times \text{height}$$

$$A = b \times h$$

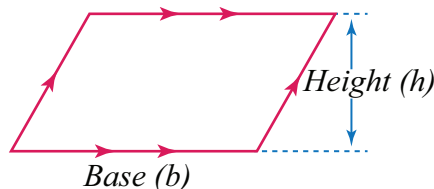


Figure 66.

In figure 66, ABCD is a parallelogram with a base of length  $b$  and with corresponding height of length  $h$ .

$$\text{Area: } A = \text{Base } (b) \times \text{Height } (h)$$

**EXAMPLE 42**

Find the area of each of the following parallelograms.

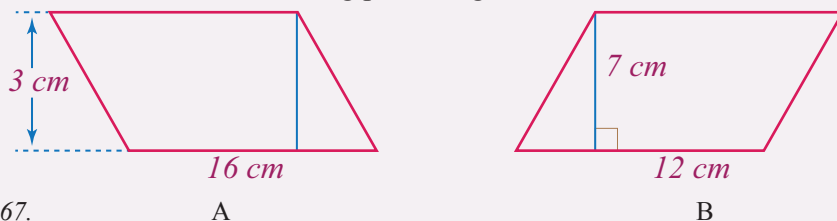


Figure 67.

A

B

**Solution**

(a) Area:  $A = b \times h = 3 \text{ cm} \times 16 \text{ cm} = 48 \text{ cm}^2$ .

Therefore, area of the parallelogram is  $48 \text{ cm}^2$ .

(b) Area:  $A = b \times h = 12 \text{ cm} \times 7 \text{ cm} = 84 \text{ cm}^2$ .

Therefore, area of the parallelogram is  $84 \text{ cm}^2$ .

**EXAMPLE 43**

Find the area of a parallelogram with base of 14 cm and height of 9 cm.

**Solution**

$$A = b \times h = 14 \text{ cm} \times 9 \text{ cm} = 126 \text{ cm}^2.$$

Therefore, area of the parallelogram is  $126 \text{ cm}^2$ .

**EXAMPLE 44**

The base of the parallelogram is thrice its height. If the area is  $192 \text{ cm}^2$ , find the base and height.

**Solution**

Let  $h$  be the height of the parallelogram. Then, the base:  $b = 3h$ . Thus,

$$A = b \times h = 3h \times h = 3h^2$$

$$192 \text{ cm}^2 = 3h^2$$

$$64 \text{ cm}^2 = h^2$$

$$h = \sqrt{64} = 8 \text{ cm}.$$

Therefore, the base of the parallelogram is 24 cm and its height is 8 cm.

**Theorem 2:**

The area  $A$  of a rhombus with a base of length  $b$  and with corresponding height of  $h$  is given by:

$$A = b \times h$$

In figure 68, the quadrilateral ABCD is a rhombus with a base of length  $b$  and height of length  $h$ . Thus, area of the rhombus is:

$$A = b \times h.$$

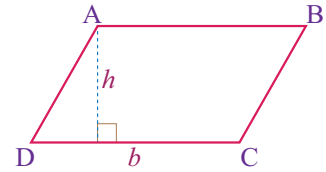


Figure 68.

#### EXAMPLE 45

Find the area of a rhombus whose base is 14 cm and its height is 5 cm.

#### Solution

Area:  $A = bh = 14 \text{ cm} \times 5 \text{ cm} = 70 \text{ cm}^2$ .

Therefore, area of the rhombus is 70 sq. centimeters.

#### EXERCISES

1. Find the area of a parallelogram whose base is 9 cm and height is 6 cm.
2. Find the area of a parallelogram with a base is equal to 13 cm and height is 11 cm.
3. Find the area of a parallelogram whose base is 8 cm and height is 12 cm.
4. The area of a parallelogram is  $450 \text{ cm}^2$ . Its height is twice its base. Find the height and base.
5. The base of the parallelogram is thrice its height. If the area is 243 square units. Then what is the value of its base and height?

The area of a triangle is defined as the total space occupied by the three sides of a triangle. In this sub-topic we will learn about the area formulas for different types of triangles along with some examples.

#### ACTIVITY 8

1. Discuss about the different types of triangles.
2. Derive the area of right-angled triangle using the area of rectangle.
3. What is the area of a triangle with base 6 cm and height 8 cm.

The area of a triangle is the region enclosed within the sides of the triangle. To find the area of a triangle, we apply the method to find the area of rectangles.

In figure 69, there are 24 square units inside. So, the area of the rectangle is 24 square units.

As shown in the figure 68, the rectangle is divided into two equal right-angled triangles. These are the shaded triangle and the unshaded triangle.



Figure 69.

The base of each triangle is 6 units because the length of the rectangle is 6 units. The area of each triangle is half of the area of the rectangle. That is,

$$\text{Area of each triangle} = \frac{1}{2} \times \text{area of the rectangle} = \frac{1}{2} \times 24 = 12 \text{ sq. units.}$$

### Theorem 1:

The area of a right-angled triangle is half of the product of its length ( $\ell$ ) and width ( $w$ ). That is,

$$\text{Area} = \frac{1}{2}(\text{length} \times \text{width}) \text{ or } A = \frac{1}{2} \times \ell \times w = \frac{1}{2} \ell w.$$

We know that the two perpendicular sides of a right-angle triangle are called legs of the right-angled triangle. So, the area of a right-angled triangle with legs of lengths  $a$  units and  $b$  units is given by

$$A = \frac{1}{2} \times a \times b = \frac{1}{2} ab.$$

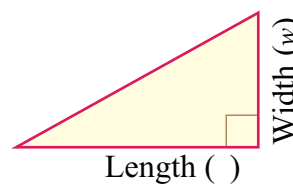


Figure 70.

### EXAMPLE 46

Find the area and perimeter of each of the following right-angled triangles.

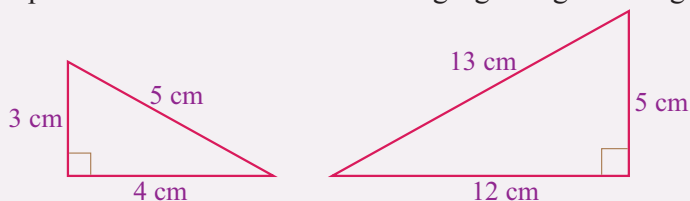


Figure 71.

A

B

### Solution

(a) Area:  $A = \frac{1}{2} \times ab \Leftrightarrow A = \frac{1}{2} \times 3 \text{ cm} \times 4 \text{ cm} \Leftrightarrow A = 6 \text{ cm}^2.$

Perimeter:  $P = 3 \text{ cm} + 5 \text{ cm} + 4 \text{ cm} \Leftrightarrow P = 12 \text{ cm}.$

Therefore, area of the right-angle triangle is 6 sq. centimeters.

Its perimeter is 12 centimeters.

(b) Area:  $A = \frac{1}{2} \times ab = \frac{1}{2} \times 5 \text{ cm} \times 12 \text{ cm} = 30 \text{ cm}^2$ .

Perimeter:  $P = 12 \text{ cm} + 5 \text{ cm} + 13 \text{ cm} = 30 \text{ cm}$ .

Therefore, area of the right-angle triangle is 30 sq. centimeters.

Its perimeter is 30 centimeters.

In Theorem 1 above, we have seen the formula for the area of a right-angled triangle. Now, let us see the area of any triangle having base ( $b$ ) and height ( $h$ ) as shown in figure 72.

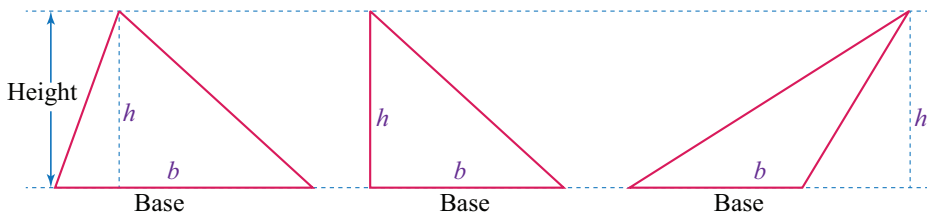


Figure 72.

### Theorem 2:

The area  $A$  of a triangle whose base has length  $b$  and whose corresponding height has length  $h$  is given by

$$A = \frac{1}{2} \times b \times h = \frac{1}{2}bh.$$

### EXAMPLE 47

Find the area of each of the following triangles.

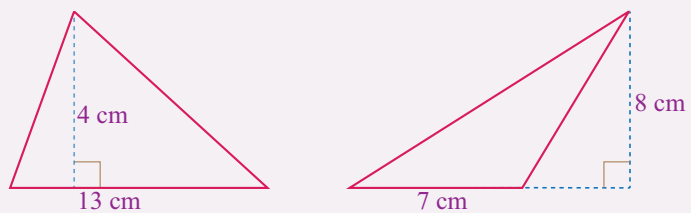


Figure 73.

### Solution

(a) Area:  $A = \frac{1}{2}bh = \frac{1}{2} \times 13 \text{ cm} \times 4 \text{ cm} = 26 \text{ cm}^2$ .

Therefore, area of the triangle is 26 sq. centimeters.

(b) Area:  $A = \frac{1}{2}bh = \frac{1}{2} \times 7 \text{ cm} \times 8 \text{ cm} = 28 \text{ cm}^2$ .

Therefore, area of the triangle is 28 sq. centimeters.

**EXAMPLE 48**

In a triangle whose area is  $72 \text{ cm}^2$ , the base has a length of 8 cm. What is the length of the corresponding height?

**Solution**

$$A = \frac{1}{2} \times b \times h$$

$$72 = \frac{1}{2} \times 8 \times h$$

$$72 = 4 \times h$$

$$h = 72 \div 4 = 18 \text{ cm.}$$

Therefore, the height of the triangle is 18 cm.

**EXERCISES**

1. What is the area of a triangle whose altitude is 8 cm and its base is 9 cm long?
2. Find the area of an isosceles right-angled triangle whose leg measures 12 cm.
3. Area of a triangle is  $45 \text{ cm}^2$ . If base of the triangle is 15 cm long, then what is the length of the height?
4. Find the area of each of the following figures.

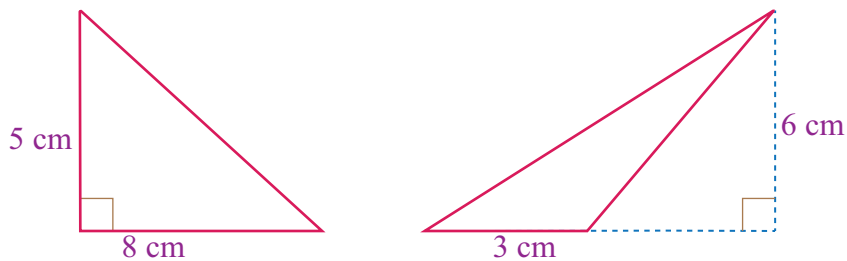


Figure 74.

5. Find the area of the shaded region of the following figure.

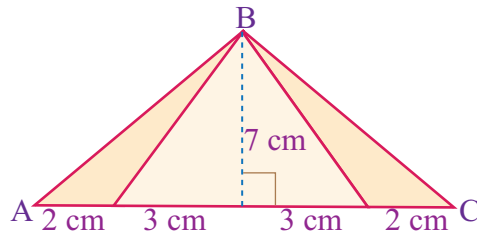


Figure 75.

A circle has always been an important shape among all geometrical figures. There are various concepts and formulas related to a circle. The **arcs** and **sectors** are the most useful of them. In this sub-topic, we shall focus on the area and circumference of a circle, arc length and area and perimeter of a sector of a circle.

### ACTIVITY 9

1. Define a sector of a circle?
2. What is major arc and minor arc of a circle?
3. In the figure 76, identify the minor and major arcs.

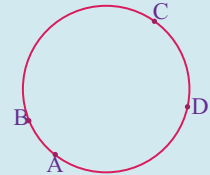


Figure 76.

### Circumference and area of a circle

The ratio of the circumference of a circle to the length of its diameter is a unique positive constant. This constant is represented by the Greek letter  $\pi$  (*pie*).  $\pi$  is the ratio between the circumference  $C$  and the diameter length  $d$  of any circle. Thus,

$$\pi = \frac{C}{d} \text{ in any circle.}$$

In the following theorem, the lengths of the diameter and radius of the circle are represented by  $d$  and  $r$ , respectively and,  $d = 2r$ .

The circumference,  $C$  of a circle is given by the formula

$$C = \pi d \text{ or } C = 2\pi r.$$

### EXAMPLE 49

Find the circumference of a circle by taking  $\pi = \frac{22}{7}$ .

- (a) When the diameter is 84 cm.
- (b) When the diameter is 105 cm.

#### Solution

$$(a) \quad C = \pi d = \frac{22}{7} \times 84 = 22 \times 12 = 264 \text{ cm.}$$

Therefore, the circumference of the circle is 264 cm.

$$(b) \quad C = \pi d = \frac{22}{7} \times 105 = 22 \times 15 = 330 \text{ cm.}$$

Therefore, the circumference of the circle is 330 cm.





**EXAMPLE 55**

If a radius of a circle is 9 cm, then find the arc length of a circle whose central angle is  $120^\circ$ .

**Solution**

$$\ell = \frac{\theta}{180^\circ} \times \pi r = \frac{120^\circ}{180^\circ} \times 9\pi = \frac{2}{3} \times 9\pi = 6\pi \text{ cm.}$$

Therefore, the arc length of the circle is  $6\pi$  cm.

**Perimeter and area of a sector of a circle**

A sector of a circle is the area bounded by two radii and the arc intercepted between them as shown by the shaded region in Figure 78:

The angle of the sector is the angle contained by the two radii of the circle and is the same as the ratio of the area of the sector to the area of the circle. That is,

$$\frac{\text{Area of sector}}{\text{Area of circle}} = \frac{\pi}{360^\circ}.$$

The perimeter,  $P$  and area,  $A$  of a sector, whose central angle  $\theta$  and radius  $r$  is given by:

$$P = 2r + \frac{\theta}{180^\circ} \pi r \text{ or } P = 2r + \ell. \text{ where } \ell \text{ is arc length.}$$

$$A = \frac{\theta}{360^\circ} \pi r^2.$$

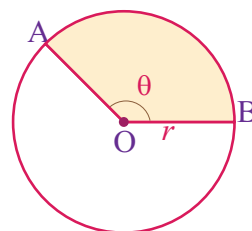


Figure 78.

**EXAMPLE 56**

A sector with perimeter 30 cm has a bounding arc of length 12 cm. Find the length of the radius of the circle.

**Solution**

$$P = 2r + \ell$$

$$30 = 2r + 12$$

$$2r = 30 - 12 = 18$$

$$r = 9 \text{ cm.}$$

Therefore, the length of the radius of the circle is 9 cm.

**EXAMPLE 57**

Find the area of a sector if its central angle is  $150^\circ$  and radius of its circle is 6 cm.

**Solution**

$$A = \frac{\theta}{360^\circ} \pi r^2 = \frac{150^\circ}{360^\circ} \pi (6)^2 = \frac{5}{12} \times 36\pi = 15\pi \text{ cm}^2.$$

Therefore, the area of the sector is  $15\pi \text{ cm}^2$ .

**EXAMPLE 58**

The area of a sector of a circle is  $54\pi \text{ cm}^2$ . If the radius of the circle is 12 cm, what is the degree measure of the central angle of the sector?

**Solution**

$$A = \frac{\theta}{360^\circ} \pi r^2$$

$$54\pi \text{ cm}^2 = \frac{\theta}{360^\circ} \pi (12 \text{ cm})^2$$

$$54\pi \text{ cm}^2 = \frac{\theta}{360^\circ} \times 144\pi \text{ cm}^2$$

$$\theta = \frac{54\pi \text{ cm}^2}{144\pi \text{ cm}^2} \times 360^\circ$$

$$\theta = \frac{3}{8} \times 360^\circ = 135^\circ.$$

**EXERCISES**

- Find the circumference of a circle with each of the given diameters  
(a) 8 cm                      (b) 13 cm                      (c) 15 cm                      (d) 26 cm
- Find the circumference of a circle with diameters 100 mm (Use  $\pi \approx 3.14$ ).
- Find the area of a circle with radius 20cm (Use  $\pi \approx 3.14$ ).
- Let the area of a circle be  $144\pi \text{ cm}^2$ . Then, find its radius, diameter and circumference.
- What would be the length of the arc formed by  $75^\circ$  of a circle having the diameter of 18 cm?
- The length of an arc formed by  $60^\circ$  arc of a circle of radius  $r$  is 12.56 cm. Find the radius of that circle. (Use  $\pi \approx 3.14$ )

7. The diameter of a semicircular shape is 21 cm. What will be the perimeter of this shape?
8. Find the area of a sector if its central angle is  $135^\circ$  and radius of its circle is 8 cm.
9. The area of a sector of a circle is  $27\pi \text{ cm}^2$ . If the radius of the circle is 9cm, what is the degree measure of the central angle of the sector?
10. Find the perimeter of each of the following shaded regions.

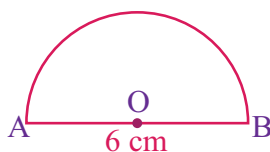
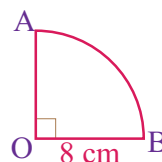


Figure 79.

C



D

## KEY TERMS

- Arc
- Central angle
- Chord
- Diameter
- Inscribed angle
- Locus
- Major arc
- Minor arc
- Parallelogram
- Perpendicular bisector
- Quadrilateral
- Rectangle
- Rhombus
- Radius
- Secant line
- Sector
- Square
- Tangent line

## SUMMARY

- The distance from the center of a circle to any point on its boundary is called *radius* of circle.
- A line segment passing through the center that connects two points on the boundary is called *diameter* of a circle.
- Any line segment touching the circle at two different points on its boundary is called *chord* of a circle.

- The set of points on a circle (any part of a circle) contained in one of the two half-planes determined by the line through any two distinct points of a circle are called an *arc* of a circle.
- The perpendicular from the center of a circle to a chord bisects the chord. In other words, the perpendicular bisector of a chord passes through the center of the circle.
- Equal chords of a circle are equidistant from the center.
- When two circles intersect each other, then the line joining their centers bisects their common chord at right angles.
- Products of intercepts of two intersecting chords are equal.
- A *central angle* of a circle is an angle whose vertex is the center of the circle and whose sides are radii of the circle.
- The angle subtended by an arc at the center of a circle is double the size of the angle subtended by the same arc at the circle.
- An *inscribed angle* is an angle whose vertex lies on the circle and whose sides are chords of the circle.
- The measure of an inscribed angle is equal to half the measure of its intercepted arc.
- If two inscribed angles intercept the same arc, then these angles are congruent.
- The measure of an angle formed by two chords intersecting inside a circle is *half* the sum of the measure of the arcs subtending the angle and its vertical opposite angle.
- If a line intersects a circle at exactly one point, then the line is called a *tangent* line of the circle.
- The point at which the tangent intersects the circle is called the *point of tangency*. (or *point of contact*)
- If a line intersects a circle at two points, then the line is called a *secant line* of the circle.
- Tangent to the circle is perpendicular to the radius of the circle at the point of contact.
- If two tangents are drawn from an external point of the circle, then they are of equal lengths.

- If from a point outside a circle, a secant and a tangent are drawn, then the tangent is the *mean proportional* between the secant and its external segment.
- For any circle, the angle between a tangent and a chord through the point of contact of the tangent is equal to the angle made by the chord in the alternate segment.
- The perimeter of a triangle with length of sides  $a$ ,  $b$  and  $c$  units is given by the formula:

$$P = a + b + c$$

- The perimeter ( $P$ ) of a rectangle whose length is  $\ell$  and width is  $w$  is given by the formula:

$$P = 2 \times (\ell + w)$$

- The perimeter of a square whose side length is  $s$  is given by the formula:

$$P = 4s$$

- The perimeter of a parallelogram with adjacent sides  $a$  and  $b$  is given by the formula:

$$P = 2 \times (a + b)$$

- The perimeter of a rhombus whose side length is  $s$  is given by the formula:  $P = 4s$

- If the length of the four sides of a trapezium are  $a$ ,  $b$ ,  $c$  and  $d$  units, then, the perimeter of the trapezium is given by the formula:  $P = a + b + c + d$

- Area occupied by a rectangle within its boundary is called the *area of the rectangle*.

- The area ( $A$ ) of rectangle is the product of its length ( $\ell$ ) and width ( $w$ ). That is,  $A = \ell \times w$

- Suppose  $a$  and  $b$  are the lengths of parallel sides of a parallelogram and  $h$  is the corresponding height of the parallelogram, then based on the length of sides and height of it, the formula for its area is given by:  $A = b \times h$

- The area  $A$  of a rhombus with a base of length  $b$  and with corresponding height of  $h$  is given by:

$$A = b \times h$$

- Area  $A$  of a triangle whose base has length  $b$  and whose corresponding height has length  $h$  is given by  $A = \frac{1}{2}bh$ .

- The circumference,  $C$  of a circle with radius  $r$  or diameter  $d$  is given by the formula:  $C = \pi d$  or  $C = 2\pi r$ .
- The area  $A$  of a circle whose radius has length  $r$  is given by the formula:  $A = \pi r^2$ .
- The perimeter,  $P$  and area,  $A$  of a sector, whose central angle is  $\theta$  and radius  $r$  is given by:  $P = 2r + \ell$ , where  $\ell$  is arc length.

$$A = \frac{\theta}{360^\circ} \pi r^2.$$

**EXERCISES**

1. From a point  $A$ , the length of the tangent to a circle is 24 cm and the distance of  $A$  from the center is 25 cm. Find the radius of the circle.
2. In the figure 80,  $O$  is the center of circle,  $\angle AOB = 110^\circ$ ,  $\angle BCE = 20^\circ$ . Find the measure of  $\angle CEB$ .
3. In the figure 81, if  $O$  is the center of the circle, find the value of  $x$  and  $y$ .
4. In figure 82,  $OA = 10$  cm and  $PA = 12$  cm, calculate the length of the tangent  $PB$ .
5. In the given figure 83,  $FD$  and  $DE$  are two tangents of a circle. If  $\angle BCE = 65^\circ$  and  $\angle ABC = 55^\circ$ , then find measures of  $\angle BAF$  and  $\angle ACD$ .

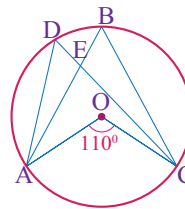


Figure 80.

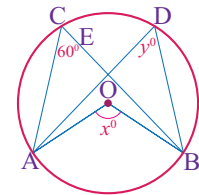


Figure 81.

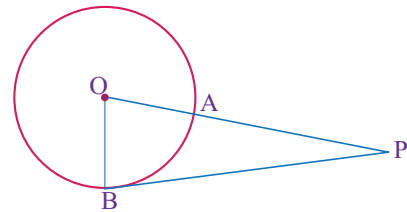


Figure 82.

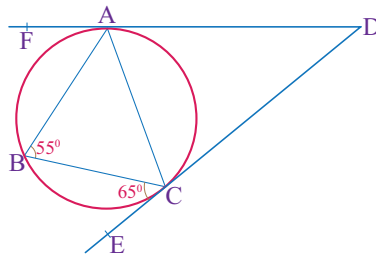


Figure 83.

6. The perimeter of a square is 32cm. What is the area of the square?
7. The footpath of uniform width runs all around a rectangular field 28 meters long and 22 meters wide. If the path occupies 600 m<sup>2</sup> area, find the width of the path.

8. The perimeter of a rectangular wire is 162 meters. The same wire is bent into the shape of an equilateral triangle. Find the lengths of each of its sides.
9. A rectangle has a width of 3 meters and a length of 120 cm. Find the perimeter of the rectangle both in meters and centimeters.
10. The area of a rectangular fence is 300 square meters. If the width of the fence is 15 meters, then find its length.
11. The area of a parallelogram is  $338 \text{ cm}^2$ . If its height is twice its base. Find the height and base.
12. Area of a triangle is  $45 \text{ cm}^2$ . If base of the triangle is 15 cm long, then what is the length of the height?
13. Find the area of a rectangle if its perimeter is 118 m and the difference between its length and breadth is 9 m.
14. The area of a square is thrice the area of a rectangle. If the area of the square is  $768 \text{ cm}^2$  and length of the rectangle is 32 cm, then find the perimeter of the rectangle.
15. Find the cost of flooring a rectangular floor having dimensions of 5 m by 8 m, if the cost of flooring is L\$75/ $\text{m}^2$ .
16. What is the area of a parallelogram whose base is 12 cm and its altitude is 7 cm long.
17. Find the circumference of a circle with diameter 8.4 m. Use  $\pi = \frac{22}{7}$ .
18. If the circumference of a circle is  $12\pi \text{ cm}$ , then find the area of a circle.
19. What would be the length of the arc formed by  $120^\circ$  of a circle having the diameter of 12 cm?
20. Find the perimeter and area of the shaded region of each of the following figures.

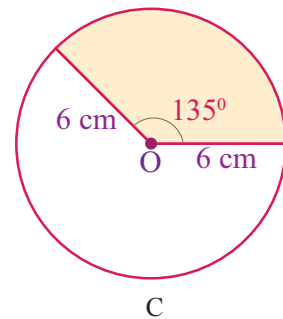
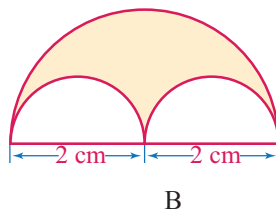
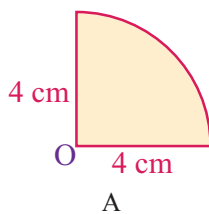


Figure 84.



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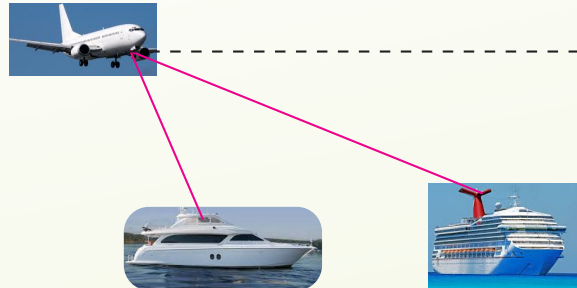
# CHAPTER

# 7

## TRIGONOMETRY

### Chapter Contents

- 7.1 Angle of Slope and Gradient
- 7.2 The Tangent of an Angle
- 7.3 Finding Height and Distance
- 7.4 Problems involving Rotation
- 7.5 Cosine of an Angle
- 7.6 Sine of an Angle
- 7.7 Use of Trigonometry
- 7.8 Trigonometric Ratios of  $30^\circ$ ,  $45^\circ$  and  $60^\circ$
- 7.9 Angle of Elevation and Depression
  - Key Terms
  - Summary
  - Exercise



## Chapter Outcomes

By the end of this chapter, you will be able to:

- define and compute angle of slope and gradient;
- define and compute the tangent of an angle;
- find heights and distances;
- compute problems involving rotation;
- define and compute cosine of an angle;
- define and compute sine of an angle;
- discuss the use of trigonometry;
- calculate the value trigonometric ratios of  $30^\circ$ ,  $45^\circ$  and  $60^\circ$ ;
- angle of elevation and depression.

## Introduction

The word ‘trigonometry’ is formed by two Greek words, “*trigonon*” and “*metron*”, where “*trigonon*” means triangle and “*metron*” means measurement. Trigonometry is a branch of mathematics that deals with measurement of sides and angles of a triangle, and the relationship between them. It plays an important role in various fields such as architecture, astronomy, engineering, etc.

### OPENING PROBLEM

A passenger in an airplane at an altitude of 10 kilometers sees two ships directly to the east of the plane. The angle of depression to the ships are  $60^\circ$  and  $30^\circ$  as shown in figure 1. How far apart are the ships?

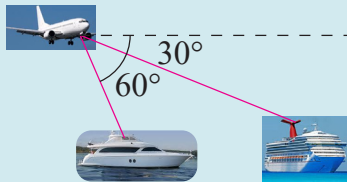


Figure 1.

In mathematics, the slope or gradient of a line is a number that describes both the direction and the *steepness* of the line. In this sub-topic we will learn about slope (also known as gradient).

### ACTIVITY 1

1. What is the slope of  $x$ -axis?
2. What is the slope of  $y$ -axis?
3. What is the slope of a line parallel to the  $x$ -axis? What do you conclude from this result?
4. Find the slope (or gradient) of the line segment joining  $(-3, -1)$  and  $(1, -9)$ .
5. Find the slope of the line with inclination  $135^\circ$ .

The *steepness* or *slope* of a line in the  $xy$ -coordinate system is the ratio of the vertical *rise* (the change in  $y$ -coordinates) to the horizontal run (the change in  $x$ -coordinates) between two points on the line.

The gradient (also known as slope) of a line is defined as:

$$\text{Gradient} = \frac{\text{vertical rise}}{\text{Horizontal run}}$$

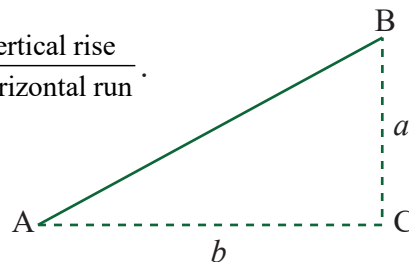


Figure 2.

In figure 2, the gradient (or slope) of the line  $AB$  is given by  $\frac{a}{b}$ .

$$\text{Gradient} = \frac{\text{vertical rise}}{\text{Horizontal run}} = \frac{BC}{AC} = \frac{a}{b}.$$

In general, if  $(x_1, y_1)$  and  $(x_2, y_2)$  are the coordinates of two points, then the vertical rise is  $y_2 - y_1$  and the run is  $x_2 - x_1$ , as shown in figure 3. Thus, the gradient or slope of the line joining the points  $(x_1, y_1)$  and  $(x_2, y_2)$  is given by

$$\begin{aligned} \text{Slope} &= \frac{\text{change in } y\text{-coordinates}}{\text{change in } x\text{-coordinates}} \\ &= \frac{\text{vertical rise}}{\text{horizontal run}} = \frac{y_2 - y_1}{x_2 - x_1}. \end{aligned}$$

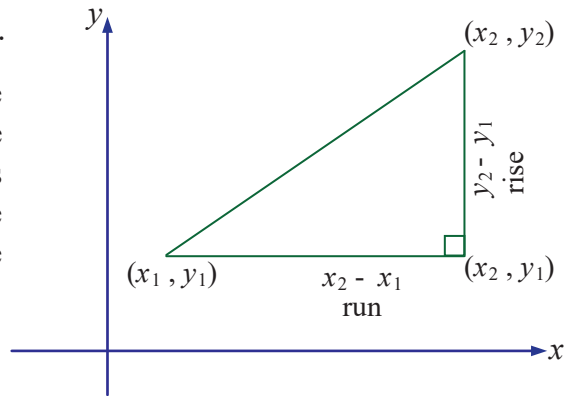


Figure 3.

Slope is often denoted by the letter  $m$ .

### EXAMPLE 1

Find the gradient (slope) of the line joining  $(2, 3)$  and  $(6, -3)$ .

#### Solution

The Slope is:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3 - 3}{6 - 2} = \frac{-6}{4} = -\frac{3}{2}.$$

### DEFINITION

If the two points  $(x_1, y_1)$  and  $(x_2, y_2)$  are two distinct points on a line with  $x_2 - x_1 \neq 0$ , then the **gradient (slope)** of the line joining the two points is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

### EXAMPLE 2

Find the slope of the line joining:

- (a)  $(3, -2)$  and  $(1, -8)$       (b)  $(a, b)$  and  $(b, a)$       (c)  $(-3, 7)$  and  $(5, 7)$

**Solution**

$$(a) \text{ Slope: } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-8 + 2}{1 - 3} = \frac{-6}{-2} = 3.$$

Therefore, the slope of the line passing through points  $(3, -2)$  and  $(1, -8)$  is 3.

$$(b) \text{ Slope: } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{a - b}{b - a} = \frac{-(b - a)}{b - a} = -1.$$

Therefore, the slope of the line passing through points  $(a, b)$  and  $(b, a)$  is  $-1$ .

$$(c) \text{ Slope: } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - 7}{5 + 3} = \frac{0}{8} = 0.$$

Therefore, the slope of the line passing through points  $(-3, 7)$  and  $(5, 7)$  is 0.

From trigonometry, we recall that the tan of an angle  $\theta$  is given by:

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}.$$

Now, since slope is also defined as  $\frac{\text{opposite}}{\text{adjacent}}$ , we have

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = m.$$

This gives us the result:

$$\tan \theta = m.$$

Then, we can find angle  $\theta$  using

$$\theta = \tan^{-1}m.$$

This angle  $\theta$  is called the **inclination** of the line.

The angle measured from positive  $x$ -axis to a line in counter clockwise direction is called the **inclination** of a line. The **tangent** of this angle is called the **slope** of the line as shown in figure 5.

A line making an acute inclination  $\theta$ , with positive direction of the  $x$ -axis has a positive slope. The right triangle PQR, in figure 6, shows that slope of PQ is:

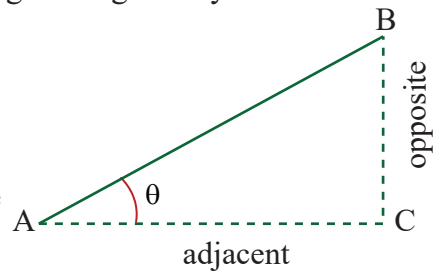


Figure 4.

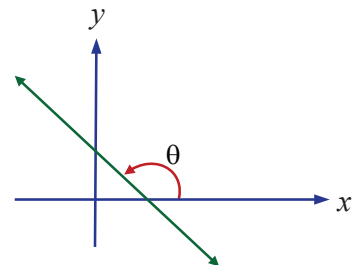


Figure 5.

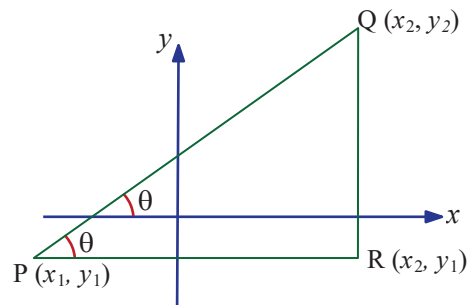


Figure 6.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{opposite side to angle } \theta}{\text{adjacent side to angle } \theta} = \tan \theta.$$

### Remark

In the next sections you will see that there are tabular values which are used to find trigonometric values of an angle  $\theta$ .

Like  $\sin \theta$ ,  $\cos \theta$  or  $\tan \theta$ .

We can also use Scientific calculators to find trigonometric values.

For the purpose of this section we can use this small table of find trigonometric values of some special angles.

Angle	$0^\circ$	$30^\circ$	$45^\circ$	$60^\circ$	$90^\circ$	$120^\circ$	$135^\circ$	$150^\circ$	$180^\circ$
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\tan \theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0

### EXAMPLE 3

If the inclination of a line is  $30^\circ$ , then find its slope.

#### Solution

Slope of the line is:

$$m = \tan \theta = \tan 30^\circ = \frac{1}{\sqrt{3}} \text{ or } \frac{\sqrt{3}}{3}.$$

Similarly, a line with **obtuse** inclination  $\theta$  has negative slope. The right triangle PQR in figure 7 shows that slope of PQ is:

$$\tan \alpha = \frac{\text{opposite}}{\text{adjacent}} = \frac{y_2 - y_1}{x_2 - x_1} = m$$

$$\tan(\pi - \theta) = \frac{y_2 - y_1}{x_2 - x_1} = m; \quad \alpha = \pi - \theta$$

$$\tan(\pi - \theta) = \tan \theta.$$

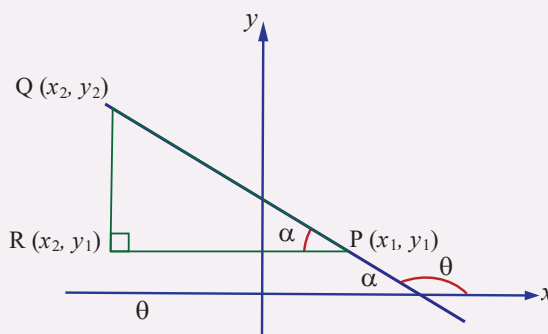


Figure 7.

**EXAMPLE 4**

Find the slope of a line if its inclination is

- (a)  $45^\circ$  (b)  $120^\circ$

**Solution**

- (a) Slope:  $m = \tan \theta = \tan 45^\circ = 1$ .  
 (b) Slope:  $m = \tan \theta = \tan 120^\circ = -\tan 60 = -\sqrt{3}$ .

**EXAMPLE 5**

Find the inclination of a line if its slope is

- (a)  $-1$  (b)  $0$

**Solution**

- |  |  |
|--|--|
| (a) $\tan \theta = m$  | (b) $\tan \theta = m$  |
| $\tan \theta = -1$   | $\tan \theta = 0$  |
| $\theta = \tan^{-1}(-1)$   | $\theta = \tan^{-1}(0) = 0$  |
| Since $90^\circ < \theta \leq 180^\circ$ , we get $\theta = 135^\circ$ . | Since $0^\circ \leq \theta < 90^\circ$ , we get $\theta = 0^\circ$ . |

**EXERCISES**

- Find the slope of the line joining:
 

(a) $(1, -1)$ and $(4, -7)$	(c) $(-2, 3)$ and $(3, -7)$
(b) $(2, 5)$ and $(5, 2)$	(d) $(5, 2)$ and $(5, 9)$
- Find the slope of a line if its inclination is
 

(a) $-45^\circ$	(c) $30^\circ$	(e) $0^\circ$
(b) $60^\circ$	(d) $90^\circ$	(f) $180^\circ$
- What can be said regarding a line if its slope or gradient is:
 

(a) Positive	(b) Zero	(c) Negative
--------------	----------	--------------
- Find angle of the gradient of the line that passes through the points:
 

(a) $(6, -8)$ and $(10, -8)$	(c) $(3, -4)$ and $(-4, 3)$
(b) $(-1, 3)$ and $(3, 5)$	(d) $(-3, -1)$ and $(-3, 7)$
- Find angle of a line if its slope is:
 

(a) $-\sqrt{3}$	(b) $-1$	(c) $-\sqrt{3}$	(d) $\frac{\sqrt{3}}{3}$
-----------------	----------	-----------------	--------------------------

The tangent of an angle is the trigonometric ratio between the adjacent side and the opposite side of a right triangle containing that angle. In this-sub-topic we will learn more about the tangent of an angle. We will learn about the values of the tangent of important angles and we will solve some practice problems.

### ACTIVITY 2

- Consider the right-angled triangle ABC shown in figure 8 to answer the following questions.
  - Name the acute angles of the right-angle triangle ABC.
  - What is the side adjacent to  $\angle A$
  - What is the side opposite to  $\angle A$
  - What is the side adjacent to  $\angle B$ .
  - What do you call the side AB.
- Verify that the values of trigonometric ratios of an angle do not vary with the lengths of the sides of the triangle.

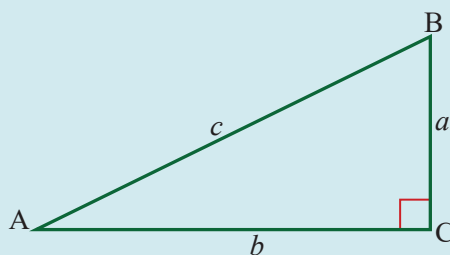


Figure 8.

The tangent of an angle can be defined using a right triangle. The tangent is equal to the length of the side opposite the angle divided by the length of the adjacent side.

In any right triangle, the tangent of an angle is the length of the opposite side (O) divided by the length of the adjacent side (A). It is written simply as “tan”. Using the right triangle ABC which is a right angle at C, shown in figure 8, the tangent of an angle can be defined as follows:

$$\tan(\angle A) = \frac{\text{length of the opposite side (O)}}{\text{length of the adjacent side (A)}} = \frac{BC}{AC} = \frac{a}{b}$$

$$\tan(\angle B) = \frac{\text{length of the opposite side (O)}}{\text{length of the adjacent side (A)}} = \frac{AC}{BC} = \frac{b}{a}$$

### EXAMPLE 6

In a right triangle ABC, shown in figure 9,

- if we have  $\tan(A) = 1.4$  and  $b = 6$ , what is the value of  $a$ ?
- if we have  $a = 13$  and  $\tan(B) = 0.98$ , determine the value of  $b$ ?

(c) if we have  $b = 5.7$  and  $\tan A = 1.8$ , what is the value of  $a$ .

**Solution**

$$(a) \tan(A) = \frac{BC}{AC} = \frac{a}{b}$$

$$1.4 = \frac{a}{6}$$

$$a = 6 \times 1.4 = 8.4$$

$$(b) \tan(B) = \frac{AC}{BC} = \frac{b}{a}$$

$$0.98 = \frac{13}{b}$$

$$0.98 \times b = 13$$

$$b = \frac{13}{0.98} = 13.27$$

$$(c) \tan(A) = \frac{BC}{AC} = \frac{a}{b}$$

$$1.8 = \frac{a}{5.7}$$

$$a = 1.8 \times 5.7 = 10.26$$

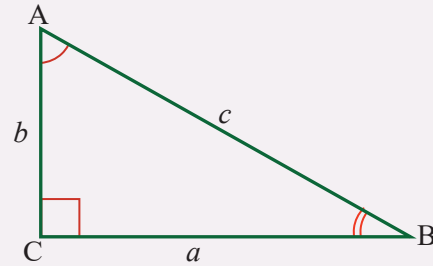


Figure 9.

**EXAMPLE 7**

In figure 10,  $\triangle ABC$  is a right triangle, find  $\tan A$  and  $\tan B$ .

$$\tan A = \frac{\text{opposit side}}{\text{adjacent side}} = \frac{BC}{AC} = \frac{a}{b}$$

$$= \frac{8}{15} \approx 0.533$$

$$\tan B = \frac{\text{opposit side}}{\text{adjacent side}} = \frac{AC}{BC} = \frac{b}{a}$$

$$= \frac{15}{8} \approx 1.875$$

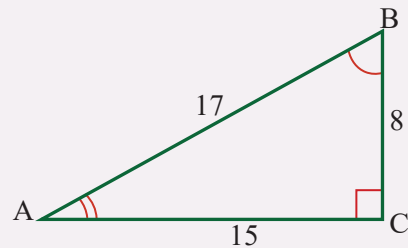


Figure 10.

The circle with center  $(0, 0)$  and radius 1 unit is called the **unit circle**.

Let the terminal side of  $\theta$  intersects the unit circle at point  $P(x, y)$ . Since  $r = 1 = \sqrt{x^2 + y^2}$ , the *sine*, *cosine* and *tangent* function of  $\theta$  are given as follows:

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} = \frac{y}{1}. \text{ So, } \sin \theta = y \text{ or } y = \sin \theta.$$

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r} = \frac{x}{1}. \text{ So, } \cos \theta = x \text{ or } x = \cos \theta.$$

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}. \text{ So, } \tan \theta = \frac{y}{x} \text{ or } \tan \theta = \frac{\sin \theta}{\cos \theta}.$$

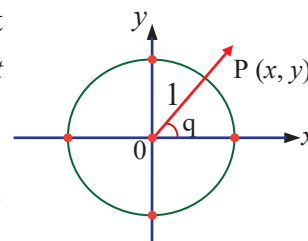


Figure 11.

### Note

The point  $P(x, y)$  is the same as the point  $P(\cos \theta, \sin \theta)$  on the unit circle.

### EXAMPLE 8

Using the unit circle, find the values of tangent of:

- (a)  $0^\circ$                       (b)  $90^\circ$                       (c)  $180^\circ$                       (d)  $270^\circ$

#### Solution

To find the value of the tangent of each angle, consider the figure 12.

- (a) The terminal side of angle  $0^\circ$  intersects the unit circle at the point  $(1, 0)$ .

So,  $(x, y) = (1, 0)$ . Thus,

$$\text{Since } \tan \theta = \frac{y}{x}, \tan 0^\circ = \frac{0}{1} = 0.$$

- (b) The terminal side of angle  $90^\circ$  intersects the unit circle at  $(0, 1)$ .

So,  $(x, y) = (0, 1)$ .

$$\text{Since } \tan \theta = \frac{y}{x}, \tan 90^\circ = \frac{1}{0} \text{ which is undefined.}$$

- (c) The terminal side of angle  $180^\circ$  intersects the unit circle at  $(-1, 0)$ .

So,  $(x, y) = (-1, 0)$ . Thus,

$$\text{Since } \tan \theta = \frac{y}{x}, \tan 180^\circ = \frac{0}{-1} = 0.$$

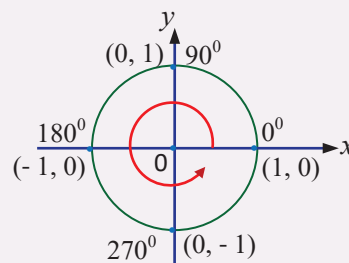


Figure 12.

(d) The terminal side of angle  $270^\circ$  intersects the unit circle at  $(0, -1)$ .

So,  $(x, y) = (0, -1)$ . Thus,

Since  $\tan \theta = \frac{y}{x}$ ,  $\tan 90^\circ = \frac{-1}{0}$  which is undefined.

### EXERCISES

1. In figure 13,  $\triangle ABC$  is a right triangle. If  $\tan \theta = 0.6$ , find the value of  $x$ .

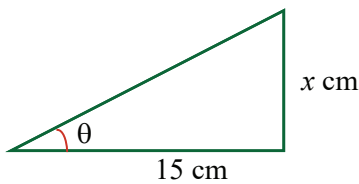


Figure 13.

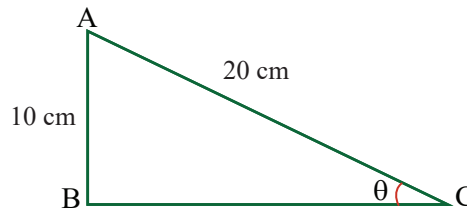


Figure 14.

- In figure 14,  $\triangle ABC$  is a right triangle with  $AB = 10$  cm and  $AC = 20$  cm, find  $\theta$ .
- Calculate the tangent angle of a right triangle whose adjacent side and opposite side are 12 cm and 5 cm respectively?
- Given a right angled-triangle  $ABC$  with a right angle at  $C$ . If  $\angle A = 60^\circ$  and  $AC = 15$  cm, find the length of side  $BC$ .
- Given a right angled-triangle  $ABC$  with a right angle at  $C$ . If  $\angle B = 60^\circ$  and  $AC = 26$  cm, find the length of side  $BC$ .
- In each of the following right-angled triangles, find the unknown parts of the triangle using the tangent of an angle formula.

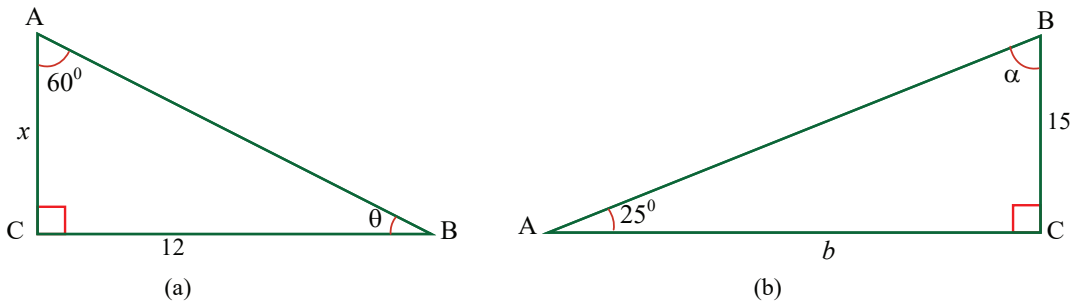


Figure 15.

7. With the help of a unit circle, find the value of:
- |                      |                      |
|----------------------|----------------------|
| (a) $\tan 75^\circ$  | (c) $\tan 210^\circ$ |
| (b) $\tan 150^\circ$ | (d) $\tan 360^\circ$ |
8. Find the value of
- |                      |                       |
|----------------------|-----------------------|
| (a) $\tan 210^\circ$ | (c) $\tan 1350^\circ$ |
| (b) $\tan 135^\circ$ | (d) $\tan 765^\circ$  |

The study of height and distance can be done using trigonometry. Trigonometry ratios are used to solve the problems related to height and distance. This is one of the applications of trigonometry. It helps to find the height of the objects like tree, building, tower, etc. and the distance between the objects. In this sub-topic, we will see different height and distance problems with solutions.

### ACTIVITY 3

- What is the difference between height and distance?
- How do you find the height of objects in mathematics?
- Observing the figure 16, fill in the blank space.
  - The observer to be at point \_\_\_\_\_.
  - The height of the object is shown by line \_\_\_\_\_.
  - The \_\_\_\_\_ of the object from the observer is given by line BC.
- A ladder placed against a wall such that it reaches the top of the wall of height 6 m and the ladder is inclined at an angle of  $60^\circ$ . Find how far the ladder is from the foot of the wall.

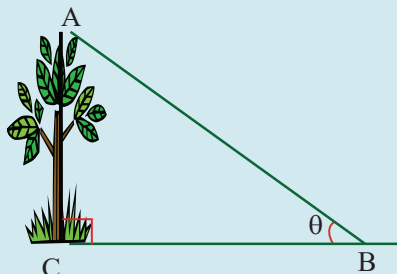


Figure 16.

Height is the measurement of an object in the vertical direction and distance is the measurement of an object from a particular point in the horizontal direction. If we imagine a line connecting the point of observation to the topmost point of the object then the horizontal line, vertical line and the imaginary line will form a right-angled triangle.

While finding height and distances, we have to solve the right-angled triangle. For example, in the figure 17, AB is the height of a tree, BC is the distance between the tree and the observer at C.

Here, if  $\theta$  and BC are known, we can calculate the height of tree AB, or if  $\theta$  and AB are known, we can calculate the distance between the tree and the observer by using trigonometric ratio.

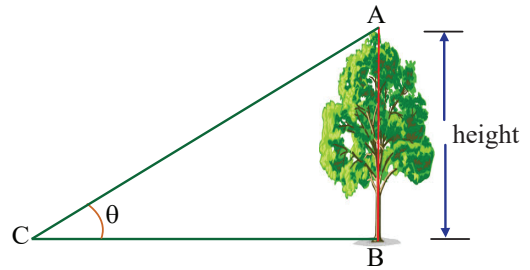


Figure 17.

**EXAMPLE 9**

A man observes the top of a tower 30 m high and finds the elevation of  $60^\circ$ . Find the distance between the man and the foot of the tower.

**Solution**

Let AB be the height of the tower and BC be the distance between the foot of the tower and the man.

In right angled-triangle ABC,  $\angle BCA = 60^\circ$ , we have

$$\tan(\angle C) = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{AB}{BC}$$

$$\tan 60^\circ = \frac{30}{BC}$$

$$BC \times \tan 60^\circ = 30$$

$$BC = \frac{30}{\tan 60^\circ} = \frac{30}{\sqrt{3}} = 10\sqrt{3} = 10 \times 1.732 \\ = 17.32 \text{ m.}$$

Therefore, the distance between the man and the foot of the tower is 17.32 meters.

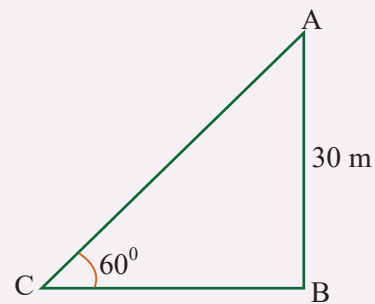


Figure 18.

**EXAMPLE 10**

The angle of inclination from a point on the ground 30 meters away from the base of a water tower to the top of the water tower is  $60^\circ$ . Find the height of the water tower to the nearest meters.

**Solution**

We can represent the problem situation using a right triangle as shown in figure 19. Let the height of the tower be denoted by  $h$ . Then,

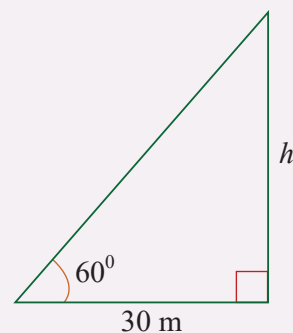


Figure 19.

$$\tan 60^\circ = \frac{h}{30}$$

$$h = 30 \times \tan 60^\circ = 30 \times 1.732 \approx 52 \text{ meters.}$$

Therefore, the height of the tower is 52 meters.

### EXAMPLE 11

A man wants to determine the height of a tower. He measured the angle at A and found that  $\tan A = \frac{3}{4}$ . What is the height of the tower if A is 30 m from the base?

#### Solution

In right angled triangle ABC, we have

$$\tan(\angle A) = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{BC}{AC}$$

$$\frac{3}{4} = \frac{BC}{32}$$

$$BC = \frac{3}{4} \times 32 = 24 \text{ m.}$$

Therefore, the height of the tower is 24 meters.

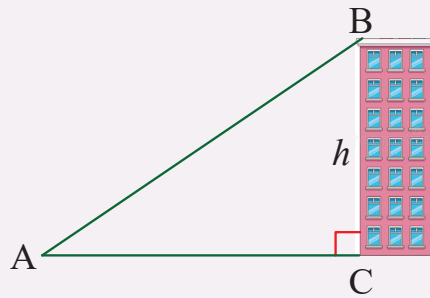


Figure 20.

### EXAMPLE 12

A man 1.5m tall standing at a distance of 50m from a tree observes the angle of elevation of the top of the tree is  $45^\circ$ . Find the height of the tree.

#### Solution

We can represent the problem situation using a right triangle as shown in figure 21. Let CE be the height of the tree; AB be the height of the man and BC is the distance between the foot of the tree and the man.

In right angled-triangle ADE, we have

$$\tan 45^\circ = \frac{ED}{AD} = \frac{ED}{50}$$

$$ED = 50 \times \tan 45^\circ = 50 \times 1 = 50 \text{ m}$$

$$\text{Now, } CE = ED + DC$$

$$= 50 + 1.5 = 51.5 \text{ m}$$

Therefore, the required height of the tree is 51.5 m.

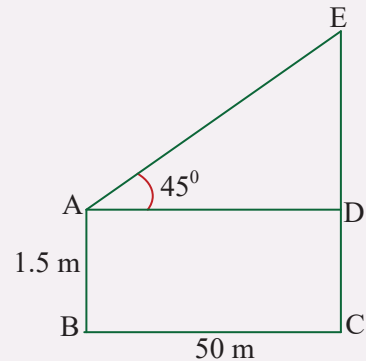


Figure 21.

**EXERCISES**

1. A boy, 1.2m tall, standing at a distance of 20 m from a flag pole observes the angle of elevation of the top of the pole is  $60^\circ$ . Find the height of the pole.
2. The circumference of a circular pond is 176m and a pillar is fixed at the center of the pond. If a person finds the angle of elevation of  $60^\circ$  of the top of the pillar from any point on the bank of the pond, find the height of the pillar above the water pillar.
3. A string of a kite is 100 meters long and it makes an angle of  $60^\circ$  with horizontal. Find the height of the kite, assuming that there is no slack in the string.
4. A man  $\sqrt{3}$  m tall is 72 m away from a tower  $25\sqrt{3}$  m high. Find the angle of elevation of the top of the tower from his eyes.
5. A tower stands vertically on the ground. A point which is 30 m away from the foot of the tower makes an angle of elevation of  $60^\circ$  with the top of the tower. Find the height of the tower.
6. If the angle of elevation of the top of a tower from a distance of 200 m from its foot is  $60^\circ$ , find the height of the tower.

A **rotation** is a type of transformation in a plane that turns every point of a figure through a specified angle and direction about a fixed point. The fixed point is called the center of rotation. In this sub-topic, we will discuss about problems involving rotation.

**ACTIVITY 4**

1. Rotate the following figures about point O through  $90^\circ$  in the clockwise and counterclockwise directions.



(a)



(b)



(c)

Figure 22.

2. Discuss the concept of positive and negative rotation.
3. Which direction represents positive angle?

4. Through how many degrees does the minute hand rotate:
  - (a) in an hour and a half?
  - (b) Forty-five minutes
  - (c) Two hours and fifteen minutes.
5. Reflect a figure in the  $y$ -axis and then reflect it in the  $x$ -axis. Is it the same as rotating a figure by  $180^\circ$ ? Give reason for your answer.

An **angle of rotation** is the measurement of the amount of the angle, by which a figure is rotated counterclockwise about a fixed point, often the center of a circle.

Usually, clockwise rotation is referred as negative rotation. For example, counterclockwise rotation of  $330^\circ$  can also be called as  $-30^\circ$  rotation because  $330^\circ + 30^\circ = 360^\circ$ , which makes a complete rotation. Angle of rotation is also referred as “order of symmetry”.

### EXAMPLE 13

Rotate the quadrilateral, shown in figure 23, about point P through  $90^\circ$  in the clockwise direction



P.

Figure 23.

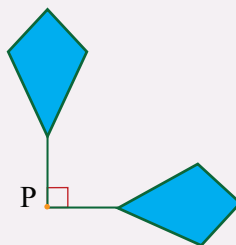


Figure 24.

#### Solution

The given figure can be rotated through  $90^\circ$  clockwise directions as shown in figure 24. Here, the point A is the center of rotation and we can say that angle of rotation is  $90^\circ$ .

Figure 25 represents a pentagon rotating about a point C in the clockwise direction where the angle made by the consecutive position is  $45^\circ$  of the previous position, where point C is the center of rotation and we can say that angle of rotation is  $45^\circ$ .

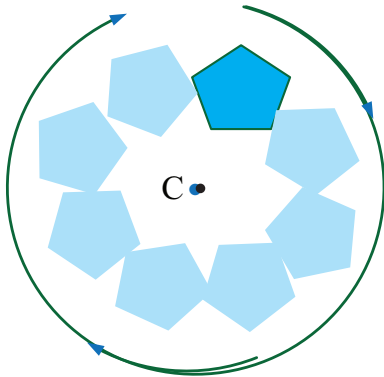


Figure 25.

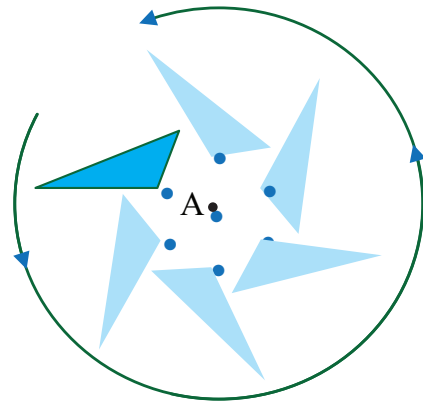


Figure 26.

Figure 26 represents a triangle rotating about point A in the anti-clockwise direction, where the angle made by the consecutive position is  $60^\circ$ , where point A is the center of rotation and we can say that the angle of rotation is  $60^\circ$ .

If we have to rotate a point about the origin, that is., point P ( $x, y$ ) moves under rotation about O through an angle of  $\theta$ , then the new coordinates will be P' ( $x', y'$ );  $OP = OP'$  and  $\angle P'OP = \theta$ ; where positive  $\theta$  is measured anti-clockwise (in the positive direction). Here, point O at the origin is fixed. Figure 27 represents the rotation of point P in the Cartesian Coordinate plane:

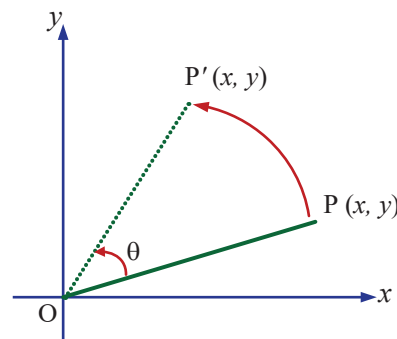


Figure 27.

**Note**

- (i)  $R_\theta$  is denoted as rotation of point about origin O through an angle  $\theta$  in the positive direction (in the anti-clockwise direction).
- (ii)  $R_{-\theta}$  is denoted as rotation of point about the origin O through an angle  $\theta$  in the negative direction (in the clockwise direction).

**EXAMPLE 14**

Find the image of the point (2, 3) about the origin through:

- (a)  $R_{90^\circ}$
- (b)  $R_{-90^\circ}$
- (c)  $R_{180^\circ}$

**Solution**

(a)  $P(x, y) \xrightarrow{R_{90^\circ}} P'(-y, x)$ . Thus,  $(2, 3) \xrightarrow{R_{90^\circ}} P'(-3, 2)$

$$(b) P(x, y) \xrightarrow{R_{-90^\circ}} P'(y, -x). \text{ Thus, } (2, 3) \xrightarrow{R_{-90^\circ}} P'(3, -2)$$

$$(c) P(x, y) \xrightarrow{R_{180^\circ}} P'(-x, -y). \text{ Thus, } (2, 3) \xrightarrow{R_{180^\circ}} P'(-2, -3)$$

In general, we can use the following **rules** to find the image after  $90^\circ$  and  $180^\circ$  clockwise and counter clockwise rotation.

**Rule 1:** The image of any point  $P(x, y)$  under the rotation of  $90^\circ$  in clockwise direction about the origin is  $P'(y, -x)$ .

**Rule 2:** The image of any point  $P(x, y)$  under the rotation of  $90^\circ$  in counter clockwise direction about the origin is  $P'(-y, x)$ .

**Rule 3:** The image of any point  $P(x, y)$  under the rotation of  $180^\circ$  in clockwise and in counter clockwise direction about the origin is  $P'(-x, -y)$ .

#### EXAMPLE 15

When the points  $A(2, 5)$ ,  $B(0, 3)$ ,  $C(-5, -3)$ ,  $D(-1, 2)$ ,  $E(2, -3)$  and  $F(3, 0)$  are rotated about origin  $O$  in clockwise direction about  $90^\circ$ , find the image point.

#### Solution

Using **Rule 1**, we get  $A'(5, -2)$ ,  $B'(3, 0)$ ,  $C'(-3, 5)$ ,  $D'(2, 1)$ ,  $E'(-3, -2)$  and  $F'(0, -3)$ .

#### EXAMPLE 16

When the points  $A(3, -2)$ ,  $B(2, 0)$ ,  $C(-1, -2)$ ,  $D(-2, 1)$ ,  $E(2, 3)$  and  $F(0, -4)$  are rotated about origin  $O$  in counter clockwise direction about  $90^\circ$ , find the image point.

#### Solution

Using **Rule 2**, we get  $A'(2, 3)$ ,  $B'(0, 2)$ ,  $C'(2, -1)$ ,  $D'(-1, -2)$ ,  $E'(-3, 2)$  and  $F'(4, 0)$

#### EXAMPLE 17

When a line having the equation  $2x - 3y = 7$  is rotated about origin  $O$  in the clockwise direction about  $90^\circ$ , find the equation of the image of the given line.

#### Solution

To find the equation of the image of the given line after rotating through  $90^\circ$  clockwise directions about the origin, we use the **Rule 1** given above. Thus, by replacing  $x$  with  $y$  and  $y$  with  $-x$  in the equation, we get the required equation:

$$2(y) - 3(-x) = 7$$

$$2y + 3x = 7 \text{ or } 3x + 2y - 7 = 0.$$

## EXERCISES

- Rotate the given figures about point P through the angles provided.
  - $120^\circ$  in clockwise direction.
  - $90^\circ$  in anti-clockwise direction.

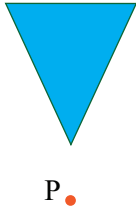


Figure 28.



Figure 29.

- Rotate the given figure 30 about point A.
  - Through  $60^\circ$  anti-clockwise
  - Through  $180^\circ$  clockwise

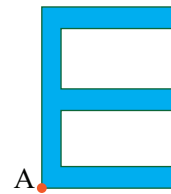


Figure 30.

- Find  $R_{90^\circ}$ ,  $R_{-90^\circ}$  and  $R_{180^\circ}$  and the center of rotation for the given points and represent these on the graph.
  - $(2, -3)$
  - $(-5, 6)$
  - $(-2, -3)$
  - $(4, -3)$
- Find  $R_{90^\circ}$ ,  $R_{-90^\circ}$  and  $R_{180^\circ}$  and the center of rotation for the given equations of lines and represent them on the graph.
  - $5x + 4y = 10$
  - $3x - 2y = 6$
  - $x + y = 8$
  - $3x - 4y = 12$
- Triangle ABC has vertices  $A(-2, 2)$ ,  $B(-10, 6)$ , and  $C(-6, 12)$ . Draw triangle ABC and its image after a rotation of  $90^\circ$  clockwise about the origin.
- A quadrilateral PQRS has vertices  $P(2, 5)$ ,  $Q(1, 1)$ ,  $R(6, 1)$  and  $S(7, 3)$ . Draw the quadrilateral PQRS and its image after a rotation of  $180^\circ$  anti-clockwise about the origin.

In any right triangle the cosine of an angle is the length of the adjacent side divided by the length of the hypotenuse. Cosine is abbreviated by “*cos*”. In this-sub-topic, we will learn about cosine of an angle and the cosine rule.

**ACTIVITY 5**

- Using a ruler, pencil and compass draw right-angle triangle. Identify its acute angles, the opposite side, adjacent side and hypotenuse of the right triangle.
- Identify the opposite, adjacent side and hypotenuse of the following right-angle triangles.

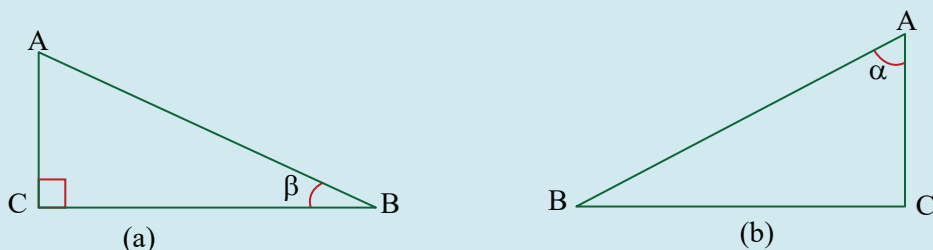


Figure 31.

- Using the unit circle, find the values of cosine of:

(a)  $-90^\circ$                       (b)  $-180^\circ$                       (c)  $-270^\circ$                       (d)  $-360^\circ$

Cosine of an angle is used to solve a right triangle. To solve a triangle means to find the lengths of all its sides and the measures of all its angles.

In figure 32,  $\triangle ABC$  is right triangle. So, the side AC is the side adjacent to angle  $\theta$  and the side AB is the hypotenuse of right  $\triangle ABC$ . In solving a right triangle, we use cosine ratio formula:

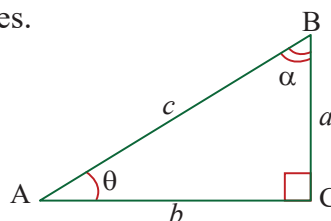


Figure 32.

$$\cos(\angle A) = \frac{\text{side adjacent to } \angle A}{\text{hypotenuse}} = \frac{AC}{AB} = \frac{b}{c}, \text{ or}$$

$$\cos \theta = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}} = \frac{AC}{AB} = \frac{b}{c}. \text{ In short, } \cos \theta = \frac{b}{c}.$$

$$\text{Similarly, } \cos \alpha = \frac{\text{side adjacent to } \alpha}{\text{hypotenuse}} = \frac{BC}{AB} = \frac{a}{c}. \text{ In short, } \cos \alpha = \frac{a}{c}.$$

**EXAMPLE 18**

The length of the hypotenuse of a right triangle 10 cm. If the measure of one of the acute angles of the triangle is  $60^\circ$ , find the length of the side adjacent to this given angle.

**Solution**

Let the length of the side adjacent to the given angle be  $x$ . Then,

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 60^\circ = \frac{x}{10}$$

$$x = 10 \times \cos 60^\circ = 10 \times 0.5 = 5.$$

Therefore, the length of the side adjacent to the given angle is 5 cm.

### Remark

When the angle given is not a special angle, we can use trigonometric table or a scientific calculator to find the values.

Later we will discuss how these tables are prepared.

### EXAMPLE 19

In figure 33,  $\triangle ABC$  is a right triangle with  $m(\angle ABC) = 67^\circ$  and  $BC = 5$  cm. Find the measure of  $a$  and the values of  $b$  and  $c$ .

#### Solution

$$m(\angle A) + m(\angle B) + m(\angle C) = 180^\circ$$

$$m(\angle A) + 67^\circ + 90^\circ = 180^\circ$$

$$m(\angle A) + 157^\circ = 180^\circ$$

$$m(\angle A) = a = 23^\circ$$

$$\cos(\angle B) = \frac{a}{c}$$

$$\cos 67^\circ = \frac{5}{c}$$

$$c \times \cos 67^\circ = 5 \dots\dots\dots \text{(the value of } \cos 67^\circ \text{ can be found from trigonometric table or a scientific calculator)}$$

$$c \times 0.39 = 5$$

$$c = \frac{5}{0.39} = 13 \text{ cm to the nearest whole number.}$$

$$\cos \alpha = \frac{b}{c}$$

$$\cos 23^\circ = \frac{b}{12.8}$$

$$b = 12.8 \times \cos 23^\circ = 12.8 \times 0.92 = 12 \text{ cm to the nearest whole number.}$$

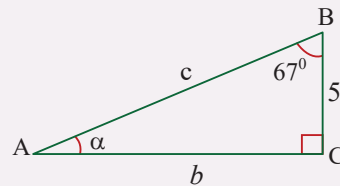


Figure 33.

The circle with center  $(0, 0)$  and radius 1 unit is called the **unit circle**.

Let the terminal side of  $\theta$  intersects the unit circle at point  $P(x, y)$ . Since  $r = 1 = \sqrt{x^2 + y^2}$ , the *cosine* of an angle  $\theta$  is given as follows:

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r} = \frac{x}{1}$$

$$\cos \theta = x \text{ or } x = \cos \theta .$$

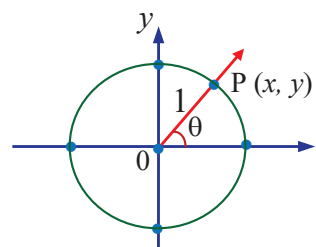


Figure 34.

### Note

$\cos \theta$  is the  $x$ -coordinate of a point  $P(x, y)$  on a unit circle.

### EXAMPLE 20

Using the unit circle, find the values of cosine of:

(a)  $0^\circ$

(b)  $90^\circ$

(c)  $180^\circ$

(d)  $270^\circ$

#### Solution

To find the value of cosine of each angle, consider figure 35.

(a) The terminal side of angle  $0^\circ$  intersects the unit circle at the point  $(1, 0)$ . So,  $(x, y) = (1, 0)$ . Thus, Since  $\cos \theta = x$ ,  $\cos 0^\circ = 1$ .

(b) The terminal side of angle  $90^\circ$  intersects the unit circle at  $(0, 1)$ .

So,  $(x, y) = (0, 1)$ .

Since  $\cos \theta = x$ ,  $\cos 90^\circ = 0$ .

(c) The terminal side of angle  $180^\circ$  intersects the unit circle at  $(-1, 0)$ .

So,  $(x, y) = (-1, 0)$ .

Since  $\cos \theta = x$ ,  $\cos 180^\circ = -1$ .

(d) The terminal side of angle  $270^\circ$  intersects the unit circle at  $(0, -1)$ .

So,  $(x, y) = (0, -1)$ .

Since  $\cos \theta = x$ ,  $\cos 270^\circ = 0$ .

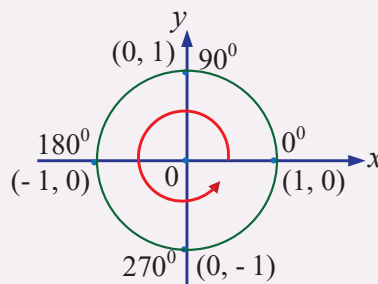


Figure 35.

If two angles are formed at point  $P$  and  $P'$  in a unit circle having the same  $x$  coordinates, then the cosine of both the angles are same.

**EXAMPLE 21**

Find two values of  $\theta$ , to the nearest degree, for which  $\cos \theta = -0.5$ .

**Solution**

Given  $\cos \theta = -0.5$

$$\theta = \cos^{-1}(-0.5)$$

$$\theta = 120^\circ$$

Also,  $\cos \theta = \cos (360^\circ - \theta)$

$$\cos \theta = \cos (360^\circ - 120^\circ)$$

$$\cos \theta = \cos 240^\circ$$

$$\theta = 240^\circ$$

$$\theta = 120^\circ \text{ and } 240^\circ$$

**Case 1:** When  $x$  is positive

Figure 36 represents two angles  $\theta_1$  and  $\theta_2$  having coordinates

$P(x, y_1)$  and  $P'(x, y_2)$  with the same value of *cosine*.

As shown in the figure 36, there is the same values of cosine in 1<sup>st</sup> and 4<sup>th</sup> quadrants. Here, we see that

$$\cos \theta_1 = \cos \theta_2$$

$$\text{also, } \theta_2 = 360^\circ - \theta_1$$

$$\cos \theta_2 = \cos (360^\circ - \theta_1)$$

$$\cos \theta_1 = \cos (360^\circ - \theta_1)$$

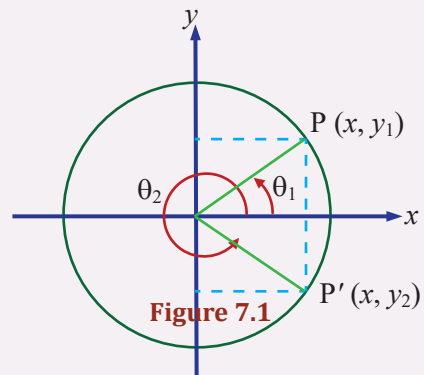


Figure 36.

**Case 2:** When  $x$  is negative

Figure 37 represents two angles  $\theta_1$  and  $\theta_2$  having coordinates  $P(-x, y_1)$  and  $P'(-x, y_2)$  with the same value of cosine:

As shown in the figure 37, there is the same values of cosine in 2<sup>nd</sup> and 3<sup>rd</sup> quadrants.

Here, we see that

$$\cos \theta_2 = \cos \theta_1$$

$$\cos \theta_2 = \cos (360^\circ - \theta_1)$$

$$\text{also, } \theta_2 = -\theta_1$$

$$\cos (-\theta_1) = \cos \theta_1$$

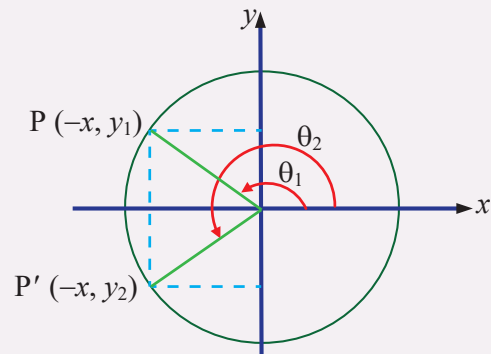


Figure 37.

**EXERCISES**

- The length of the hypotenuse of a right triangle is 19 cm. If the measure of one of the acute angles of the triangle is  $57^\circ$ , find the length of the side adjacent to this given angle.
- In  $\triangle ABC$ ,  $a = 32$  cm,  $b = 20$  cm and  $c = 40$  cm. Find the angle measure of B.
- In  $\triangle ABC$ ,  $m(\angle B) = 120^\circ$ ,  $a = 10$  cm and  $c = 15$  cm. Find  $b$ .
- With the help of a unit circle, find the value of:
 

(a) $\cos 45^\circ$	(c) $\cos 0^\circ$
(b) $\cos 270^\circ$	(d) $\cos 210^\circ$
- Find one value of  $\theta$ , to the nearest degree, for which:
 

(a) $\cos \theta = 0.50$	(c) $\cos \theta = 0.94$
(b) $\cos \theta = 0.30$	(d) $\cos \theta = -0.26$

The sine of an angle is defined using a right triangle. When we have a right triangle, the sine is equal to the length of the side opposite the angle divided by the length of the hypotenuse of the triangle. Sine is abbreviated by “*sin*”. In this-sub-topic, we will learn about sine of an angle and the sine rule.

**ACTIVITY 6**

- Discuss the sine of an angle in right angled-triangle.
- Using the sine ratio formula, express each given angle in terms of sides of the triangle.

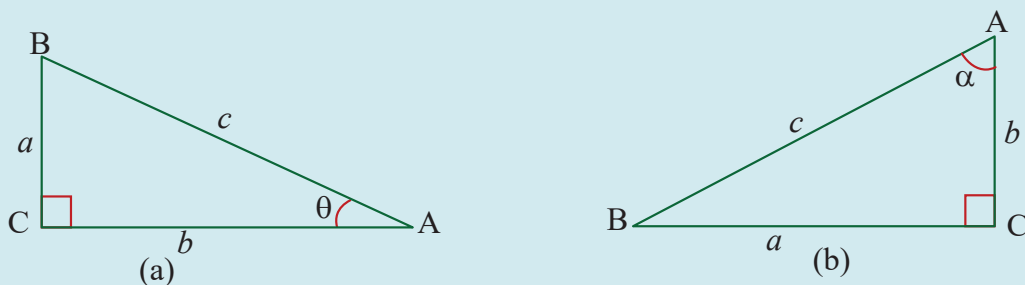


Figure 38.

Recall that solving a triangle involves finding the lengths of all sides and the measures of all angles in the triangle. In a right-angled triangle, the sine of an angle is the length of the side opposite the angle divided by the length of the hypotenuse. Sine of an angle is used to solve a right triangle.

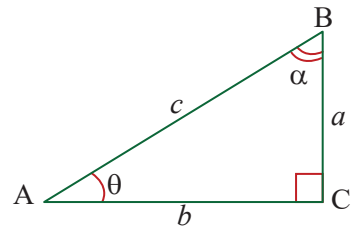


Figure 39.

In figure 39,  $\triangle ABC$  is right triangle. So, the side  $BC$  is the side opposite to angle  $\theta$  and the side  $AB$  is the hypotenuse of right  $\triangle ABC$ . In solving a right triangle, we use sine ratio formula:

$$\sin \theta = \frac{\text{side opposite to } \theta}{\text{hypotenuse}} = \frac{BC}{AB} = \frac{a}{c}. \text{ In short, } \sin \theta = \frac{a}{c}.$$

$$\text{Similarly, } \sin \alpha = \frac{\text{side opposite to } \alpha}{\text{hypotenuse}} = \frac{AC}{AB} = \frac{b}{c}. \text{ In short, } \sin \alpha = \frac{b}{c}.$$

### EXAMPLE 22

The length of the hypotenuse of a right triangle is 13 cm. If the measure of one of the acute angles of the triangle is  $30^\circ$ , find the length of the side opposite to this given angle.

#### Solution

Let the length of the side adjacent to the given angle be  $x$ . Then,

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 30^\circ = \frac{x}{13}$$

$$x = 13 \times \sin 30^\circ = 13 \times 0.5 = 6.5$$

Therefore, the length of the side adjacent to the given angle is 6.5 cm.

### EXAMPLE 23

In figure 40,  $\triangle ABC$  is right triangle with  $m(\angle B) = 53^\circ$  and  $BC = 8$  cm. Find the measure of  $\alpha$  and the values of  $b$  and  $c$ .

#### Solution

$$m(\angle A) + m(\angle B) + m(\angle C) = 180^\circ$$

$$m(\angle A) + 53^\circ + 90^\circ = 180^\circ$$

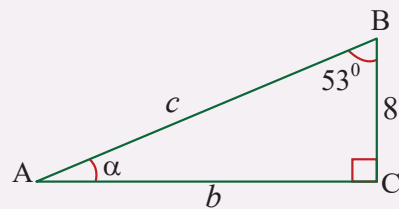


Figure 40.

$$m(\angle A) + 143^\circ = 180^\circ$$

$$m(\angle A) = \alpha = 37^\circ$$

$$\sin \alpha = \frac{a}{c}$$

$$\cos 37^\circ = \frac{8}{c}$$

$$c \times \cos 37^\circ = 8$$

$$c \times 0.6 = 8 \dots \dots \dots (\text{Use trigonometric table})$$

$$c = \frac{8}{0.6} = 13 \text{ cm to the nearest whole number.}$$

$$\sin(\angle B) = \frac{b}{c}$$

$$\sin 67^\circ = \frac{b}{13}$$

$$b = 13 \times \sin 67^\circ = 13 \times 0.92 = 12 \text{ cm to the nearest whole number.}$$

Therefore,  $\cos \frac{17}{12}\pi$  is negative.

The circle with center (0, 0) and radius 1 unit is called the **unit circle**.

Let the terminal side of  $\theta$  intersects the unit circle at point P(x, y). Since  $r = 1 = \sqrt{x^2 + y^2}$ , the sine, cosine and tangent function of  $\theta$  are given as follows:

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} = \frac{y}{1} \Rightarrow \sin \theta = y \text{ or } y = \sin \theta .$$

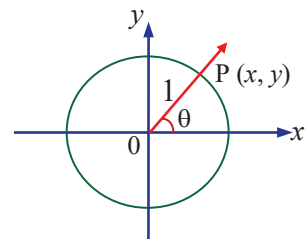


Figure 41.

**Note**

$\sin \theta$  is the y-coordinate of a point P(x, y) on a unit circle.

**EXAMPLE 24**

Using the unit circle, find the values of sine of:

- (a)  $0^\circ$                       (b)  $90^\circ$                       (c)  $180^\circ$                       (d)  $170^\circ$

**Solution**

To find the value of the sine of each angle, consider the figure 42.

(a) The terminal side of angle  $0^\circ$  intersects the unit circle at the point  $(1, 0)$ .

So,  $(x, y) = (1, 0)$ .

Since  $\sin \theta = y$ ,  $\sin 0^\circ = 0$ .

(b) The terminal side of angle  $90^\circ$  intersects the unit circle at  $(0, 1)$ .

So,  $(x, y) = (0, 1)$ .

Since  $\sin \theta = y$ ,  $\sin 90^\circ = 1$ .

(c) The terminal side of angle  $180^\circ$  intersects the unit circle at  $(-1, 0)$ .

So,  $(x, y) = (-1, 0)$ .

Since  $\sin \theta = y$ ,  $\sin 180^\circ = 0$ .

(d) The terminal side of angle  $270^\circ$  intersects the unit circle at  $(0, -1)$ .

So,  $(x, y) = (0, -1)$ .

Since  $\sin \theta = y$ ,  $\sin 270^\circ = -1$ .

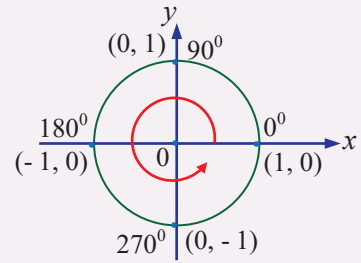


Figure 42.

If two angles are formed at point P and P' in a unit circle having the same y coordinates, then the sine of both the angles are the same.

### EXAMPLE 25

Find two values of  $\theta$  to the nearest degree for which  $\sin \theta = 0.39$ .

#### Solution

Given:  $\sin \theta = 0.39$

$$\theta = \sin^{-1}(0.39)$$

$$\theta \approx 23^\circ$$

Also,  $\sin \theta = \sin (180^\circ - \theta)$

$$\sin \theta = \sin (180^\circ - 23^\circ)$$

$$\sin \theta = \sin 157^\circ$$

$$\theta = 157^\circ.$$

**Case 1:** When  $y$  is positive

Figure 43 represents two angles,  $\theta_1$  and  $\theta_2$ , having coordinates  $P(x_1, y)$  and  $P'(x_2, y)$  with the same value of sine.

As shown in the figure 43, there is the same values of cosine in 1<sup>st</sup> and 2<sup>nd</sup> quadrants. Here, we see that

$$\sin \theta_2 = \sin \theta_1$$

$$\text{also, } \theta_2 = 180^\circ - \theta_1$$

$$\sin \theta_2 = \sin (180^\circ - \theta_1)$$

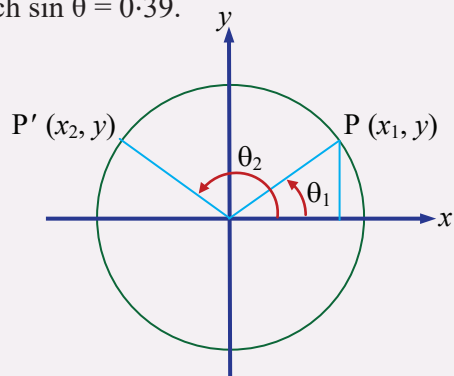


Figure 43.

**Case 2:** When  $y$  is negative

Figure 44 represents two angles,  $\theta_1$  and  $\theta_2$ , having coordinates  $P(x_1, -y)$  and  $P'(x_2, -y)$ . As shown in the figure 44, there is the same values of cosine in 3<sup>rd</sup> and 4<sup>th</sup> quadrants. Here, we see that

$$\sin \theta_2 = \sin \theta_1$$

$$\text{also, } \theta_2 = 360^\circ - \theta_1$$

$$\theta_1 = 180^\circ + \theta$$

$$\sin (180^\circ + \theta) = \sin (360^\circ - \theta)$$

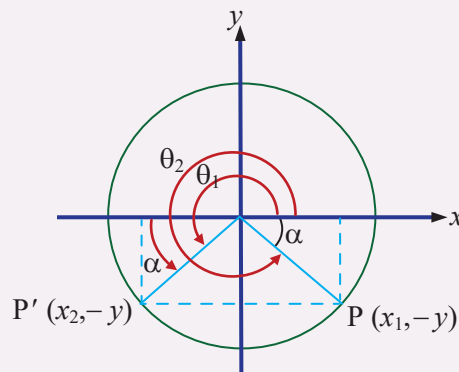


Figure 44.

**EXERCISES**

- The length of the hypotenuse of a right triangle is 17 cm. If the measure of one of the acute angles of a triangle is  $57^\circ$ , find the length of the side opposite to this given angle.
- In right angle triangle ABC,  $a = 30$  cm,  $b = 20$  cm and  $m(\angle A) = 300$ . Find the measure of  $\angle B, \angle C$  and the length of the third side.
- In right  $\triangle ABC$ ,  $m(\angle B) = 35^\circ$ ,  $a = 35$  cm and  $b = 21$  cm. Find measure of  $\angle B, \angle C$  and  $c$ .
- With the help of a unit circle, find the value of:
 

(a) $\sin 45^\circ$	(c) $\sin 90^\circ$
(b) $\sin 210^\circ$	(d) $\sin 150^\circ$
- Find one value of  $\theta$ , to the nearest degree, for which:
 

(a) $\cos \theta = 0.60$	(c) $\sin \theta = 0.50$
(b) $\cos \theta = 0.94$	(d) $\sin \theta = -0.72$
- Find the reference angle for:
 

(a) $165^\circ$	(c) $820^\circ$
(b) $345^\circ$	(d) $1250^\circ$

Recall that the trigonometry defines the relationship between sides and angles of a triangle. Trigonometry and its functions, sine, cosine and tangent, have an enormous number of uses in our daily life. For instance, it is used to measure height of a

building or a mountain, in surveying, in aviation and also in the satellite navigation system. In this sub-topic, we will consider some practical examples on real life problems to understand the use of trigonometry.

### ACTIVITY 7

1. Discuss why do we need trigonometry?
2. If the distance from where the building is observed is 120 meters from its base and the angle of elevation to the top of the building is  $45^\circ$ , then find the height of the building.

### EXAMPLE 26

A cat is lying on the level ground 20 meters away from the base of the tree. The cat can see a bird's nest directly above the base of the tree.

The angle of elevation from the cat to the bird's nest is  $45^\circ$  as shown in the figure 45. How far above the base of the tree is the bird's nest?

#### Solution

Let the bird's nest be  $h$  meters far above the base of the tree. Then,

$$\tan 45^\circ = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 45^\circ = \frac{h}{20}$$

$$h = 20 \times \tan 45^\circ = 20 \times 1 = 20 \text{ m.}$$

Therefore, the bird's nest is 20 meters far above the base of the tree.

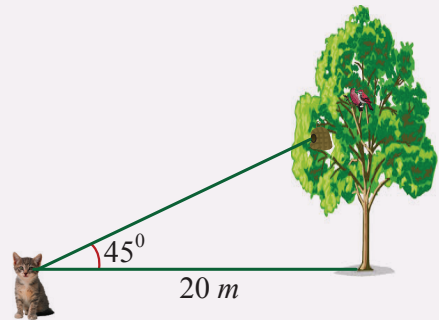


Figure 45.

### EXAMPLE 27

A surveyor stands 100 meters from the base of a building. The angle of elevation of the top of the building with respect to the location of the surveyor is  $75^\circ$ , as shown in figure 46. Find the height of the building.

#### Solution

From the figure, we see that the side adjacent to the angle is equal to 100 meters. The side opposite that angle is the height  $h$  of the building, which we need to find. Thus, using the definition of the tangent function, we have

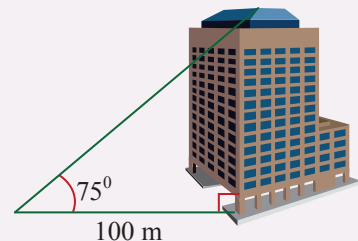


Figure 46.

$$\tan 78^\circ = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 78^\circ = \frac{h}{100}$$

$$h = 100 \times \tan 75^\circ$$

$$h = 100 \times 3.73 = 373 \text{ meters.}$$

### EXAMPLE 28

A traffic helicopter hovering 200 meters above ground observes a police car on the ground that has an angle of depression of  $30^\circ$ , as shown in figure 47. What is the straight line distance of the helicopter from the police car?

#### Solution

Using the definition of the sine function, we have

$$\sin 30^\circ = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 30^\circ = \frac{200}{d}$$

$$d \times \sin 30^\circ = 200$$

$$d = \frac{200}{\sin 30^\circ} = \frac{200}{0.5} = 400 \text{ meters.}$$

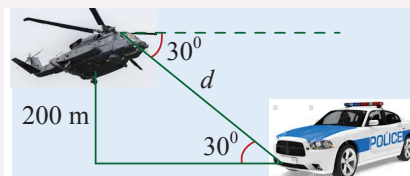


Figure 47.

### EXAMPLE 29

A ship with a heading of  $330^\circ$  first sighted a lighthouse at a bearing of  $N65^\circ E$ . After traveling 8.5 miles, the ship observed the lighthouse at a bearing of  $S50^\circ E$ . Find the distance from the ship to the lighthouse when the first sighting was made.

#### Solution

Use the given information to draw a diagram. See figure 48, which shows that the measure of

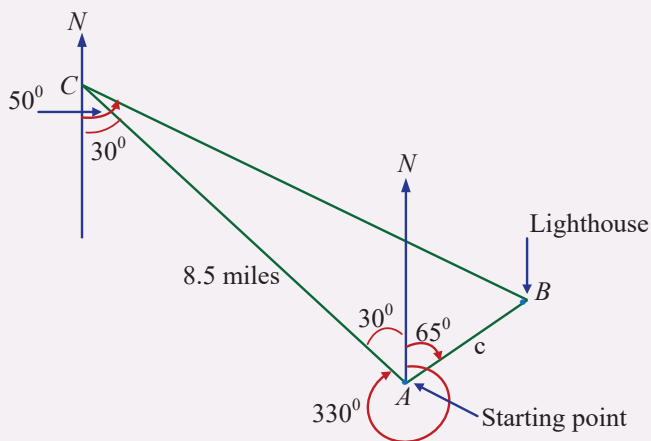


Figure 48.

$\angle CAB = 30^\circ + 65^\circ = 95^\circ$ , the measure of  $\angle ACB = 50^\circ - 30^\circ = 20^\circ$ , and  
 $\angle B = 180^\circ - 95^\circ - 20^\circ = 65^\circ$ . Use triangle ABC and the Law of Sines to find  $c$ .

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{8.5}{\sin 65^\circ} = \frac{c}{\sin 20^\circ}$$

$$c \times \sin 65^\circ = 8.5 \times \sin 20^\circ$$

$$c = \frac{8.5 \times \sin 20^\circ}{\sin 65^\circ} = \frac{8.5 \times 0.342}{0.906} \dots\dots\dots (\sin 20^\circ = 0.342, \text{ from trigonometric table})$$

$$= 3.2$$

The Law of Sines can be used to solve oblique triangles in which either two angles and a side or two sides and an angle opposite one of the sides are known.

### The Law of Sine

If A, B, and C are the measures of the angles of a triangle ABC and  $a$ ,  $b$ , and  $c$  are the lengths of the sides opposite these angles, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

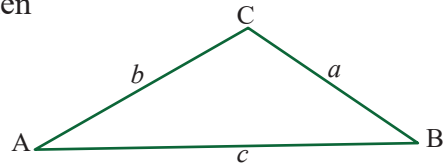


Figure 49.

### EXAMPLE 30

Using the sine law, solve  $\triangle ABC$  if  $\angle A = 43^\circ$ ,  $\angle B = 62^\circ$  and  $c = 18$  cm.

#### Solution

Find angle C by using the fact that the sum of the measures of the interior angles of a triangle is  $180^\circ$ .

$$\angle A + \angle B + \angle C = 180^\circ$$

$$42^\circ + 63^\circ + \angle C = 180^\circ$$

$$105^\circ + \angle C = 180^\circ$$

$$\angle C = 75^\circ$$

Use the Law of sines to find  $a$ .

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{a}{\sin 43^\circ} = \frac{18}{\sin 75^\circ}$$

$$a \times \sin 75^\circ = 18 \times \sin 43^\circ$$

$$a = \frac{18 \times \sin 43^\circ}{\sin 75^\circ} = \frac{18 \times 0.68}{0.97} = \frac{12.24}{0.97} = 12.6 \dots \dots \text{(using trigonometric table to find } \sin 43^\circ \text{ and } \sin 75^\circ \text{)}$$

Now, use the Law of sines to find  $b$ .

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{b}{\sin 62^\circ} = \frac{18}{\sin 75^\circ}$$

$$b \times \sin 75^\circ = 18 \times \sin 62^\circ$$

$$a = \frac{18 \times \sin 62^\circ}{\sin 75^\circ} = \frac{18 \times 0.89}{0.97} = \frac{10.02}{0.97} = 16.5 \dots \dots \text{(using trigonometric table)}$$

Therefore,  $\angle B = 75$ ,  $a \approx 13$  cm and  $b = 17$  cm.

The Law of Cosines can be used to solve triangles in which two sides and the included angle (SAS) are known or in which three sides (SSS) are known.

### The Law of Cosine

If  $A$ ,  $B$ , and  $C$  are the measures of the angles of a triangle  $ABC$  and  $a$ ,  $b$ , and  $c$  are the lengths of the sides opposite these angles, then

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

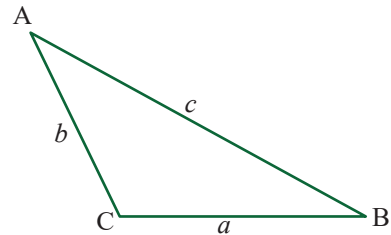


Figure 50.

### EXAMPLE 31

In figure 51,  $\triangle ABC$  has  $AB = 8$  cm,  $BC = 12$  cm and  $m(\angle B) = 120^\circ$ . Find the length of  $AC$ .

#### Solution

Using the law of cosine, we have

$$(AC)^2 = (AB)^2 + (BC)^2 - 2(AB)(BC) \cos(\angle B)$$

$$(AC)^2 = (8)^2 + (12)^2 - 2(8)(12) \cos 120^\circ$$

$$= 64 + 144 - 192(-0.5)$$

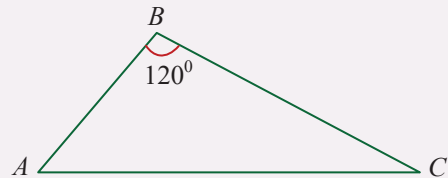


Figure 51.

$$= 208 + 96 = 304$$

$$AC = \sqrt{304} = 17.4 \text{ cm to the nearest one decimal place.}$$

Therefore, the length of side AC is 17.4 cm.

### EXERCISES

1. A boat sailed 3 kilometers at a heading of  $78^\circ$  and then turned to a heading of  $138^\circ$  and sailed another 4.3 kilometers. Find the distance and the bearing of the boat from the starting point.
2. A pole tilts toward the sun at  $15^\circ$  from the vertical and it casts a 22-meter shadow. The angle of elevation from the tip of the shadow to the top of the pole is  $45^\circ$ . How tall is the pole?
3. An electric pole 20 m high is supported by a wire fixing its one end on the ground at some distance from the pole. If the wire joining the top of the pole is inclined to the ground at an angle of  $30^\circ$ , find the length of the wire.
4. The circumference of a circular pond is 176 meters and a pillar is fixed at the center of the pond. If a person finds the angle of elevation of  $60^\circ$  of the top of the pillar from any point on the bank of the pond, find the height of the pillar above the water pillar.
5. A boy, 1.2 meter tall, is flying a kite. When the length of the string of the kite is 200 meters, it makes an angle of  $30^\circ$  with the horizontal line. At what height is the kite from the ground?
6. A man  $\sqrt{3}$  m tall is 72 m away from a tower  $25\sqrt{3}$  m high. Find the angle of elevation of the top of the tower from his eyes.
7. The top of a tree which is broken by the wind makes an angle of  $60^\circ$  with the ground at a distance of  $3\sqrt{3}$  m from the foot of the tree. Find the height of the tree before it was broken.
8. An observer measures an angle of elevation of  $30^\circ$  for the top of a lighthouse with respect to a certain point on the ground. She then walks 15 meters closer to the lighthouse and measures an angle of elevation of  $45^\circ$ . How tall is the lighthouse?
9. Using the *law of cosine* find the unknown parts of each of the following triangles.

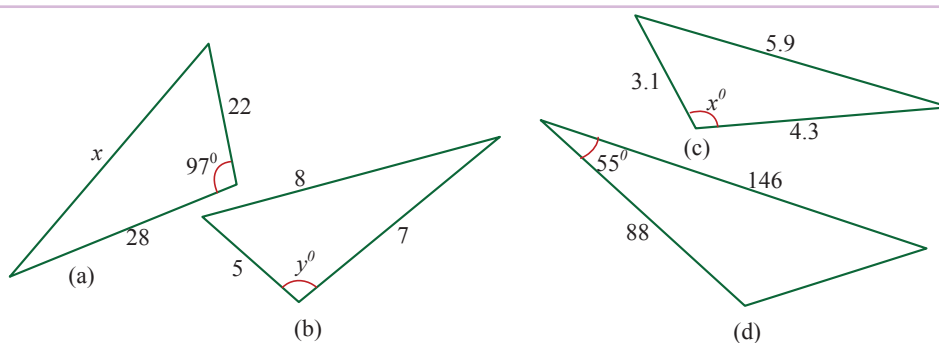


Figure 52.

10. Using the *law of sine* find the unknown parts of each of the following triangles.

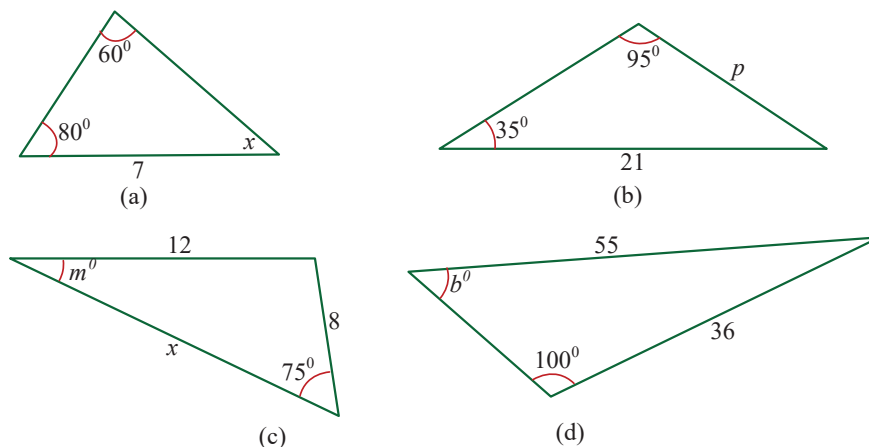


Figure 53.

Trigonometric ratios of some specific angle are defined as the ratio of the sides of a right-angled triangle with respect to any of its acute angles. Trigonometric ratios of some specific angle include  $0^\circ$ ,  $30^\circ$ ,  $60^\circ$  and  $90^\circ$ . In this sub-topic we will learn how to find the trigonometric ratios of  $30^\circ$ ,  $45^\circ$  and  $60^\circ$  in detail.

### ACTIVITY 8

- What are complementary angles?
- Find the complement of each of the following angles.
 

(a) $30^\circ$	(c) $60^\circ$	(e) $75^\circ$
(b) $45^\circ$	(d) $87^\circ$	(f) $3^\circ$

3. In figure 54,  $\triangle ABC$  is an equilateral triangle where  $BD \perp AC$  and  $AB = BC = AC = 2$  units.

Using the figure 54, determine the trigonometric ratios of  $30^\circ$  and  $60^\circ$ . That is,  $\sin 30^\circ$ ,  $\sin 60^\circ$ ,  $\cos 30^\circ$ ,  $\cos 60^\circ$ ,  $\tan 30^\circ$  and  $\tan 60^\circ$

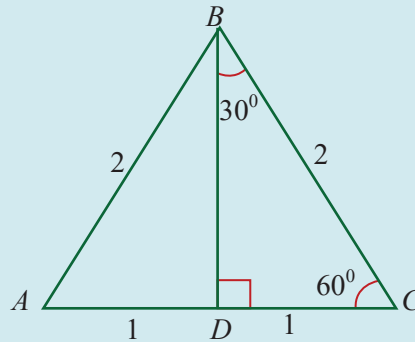


Figure 54.

The special angles with measures of  $30^\circ$ ,  $45^\circ$  and  $60^\circ$  have a way of showing up in various applications of trigonometry. Trigonometric ratios for these special angles can easily be deduced with the help of a right-angled triangle.

Using some elementary geometry, we can find the values of sine, cosine and tangent functions of the special angles. To begin with, consider an equilateral triangle  $ABC$  having sides of length 2 units, as shown in figure 55(a).

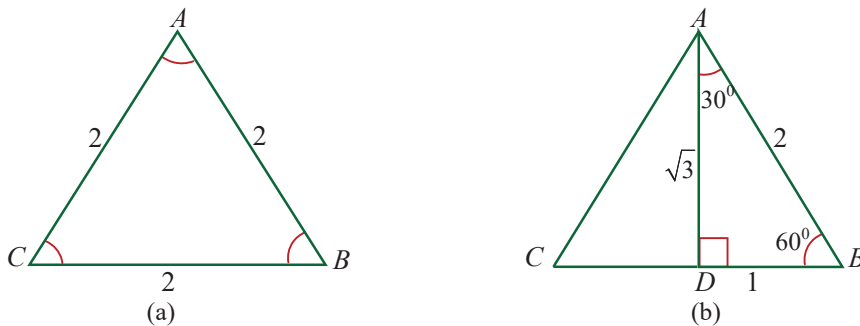


Figure 55.

Recall from plane geometry that the vertex angle of an equilateral triangle all measures  $60^\circ$  and that the bisector  $\overline{AD}$  of vertex angle  $A$  is perpendicular bisector of the opposite side  $\overline{BC}$ , as shown in figure 55(b). Therefore,

$$BD = \frac{1}{2} \times BC = \frac{1}{2} \times 2 = 1 \text{ and } m(\angle DAB) = \frac{1}{2} m(\angle CAB) = \frac{1}{2} \times 60^\circ = 30^\circ.$$

Applying the Pythagorean theorem to right triangle ADB, we have

$$(AD)^2 + (DB)^2 = (AB)^2$$

$$(AD)^2 + 1^2 = 2^2$$

$$(AD)^2 + 1 = 4$$

$$(AD)^2 = 3$$

$$AD = \sqrt{3}.$$

Thus, from triangle ADB, we have

$$\sin 30^\circ = \frac{DB}{AB} = \frac{1}{2}. \text{ So, } \sin 30^\circ = \frac{1}{2}; \text{ and } \sin 60^\circ = \frac{AD}{AB} = \frac{\sqrt{3}}{2}. \text{ So, } \sin 60^\circ = \frac{\sqrt{3}}{2}.$$

$$\cos 30^\circ = \frac{AD}{AB} = \frac{\sqrt{3}}{2}. \text{ So, } \cos 30^\circ = \frac{\sqrt{3}}{2}; \text{ and } \cos 60^\circ = \frac{DB}{AB} = \frac{1}{2}. \text{ So, } \cos 60^\circ = \frac{1}{2}.$$

$$\sin 60^\circ = \frac{AD}{AB} = \frac{\sqrt{3}}{2}. \text{ So, } \sin 60^\circ = \frac{\sqrt{3}}{2}; \cos 60^\circ = \frac{DB}{AB} = \frac{1}{2}. \text{ So, } \cos 60^\circ = \frac{1}{2};$$

$$\tan 30^\circ = \frac{DB}{AD} = \frac{1}{\sqrt{3}}. \text{ So, } \tan 30^\circ = \frac{1}{\sqrt{3}} \text{ or } \tan 30^\circ = \frac{\sqrt{3}}{3}; \text{ and } \tan 60^\circ = \frac{AD}{DB} = \sqrt{3}$$

To find the values of the trigonometric functions of a  $45^\circ$ , we construct isosceles right triangle with two equal sides of length 1 unit, as shown in figure 56. We know from plane geometry that both of the acute angles in right triangle ABC measures  $45^\circ$ . By Pythagorean theorem, we have

$$(AC)^2 + (CB)^2 = (AB)^2$$

$$1^2 + 1^2 = (AB)^2$$

$$(AB)^2 = 1 + 1 = 2$$

$$AB = \sqrt{2}.$$

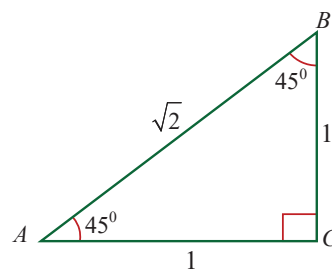


Figure 56.

Thus, from the isosceles right triangle ABC, we have

$$\sin 45^\circ = \frac{BC}{AB} = \frac{1}{\sqrt{2}}. \text{ So, } \sin 45^\circ = \frac{1}{\sqrt{2}} \text{ or } \sin 45^\circ = \frac{\sqrt{2}}{2};$$

$$\cos 45^\circ = \frac{AC}{AB} = \frac{1}{\sqrt{2}}. \text{ So, } \cos 45^\circ = \frac{1}{\sqrt{2}} \text{ or } \cos 45^\circ = \frac{\sqrt{2}}{2}; \text{ and}$$

$$\tan 45^\circ = \frac{BC}{AC} = \frac{1}{1}.$$

For memorizing, the three trigonometric ratios of the special angles  $30^\circ$ ,  $45^\circ$  and  $60^\circ$  are given in the following table.

	$30^\circ$	$45^\circ$	$60^\circ$
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

**EXAMPLE 32**

Evaluate the following without using a calculator.

(a)  $2 \sin 30^\circ + 3 \cos 60^\circ - 3 \tan 45^\circ$                       (b)  $3(\cos 30^\circ)^2 + 2(\sin 30^\circ)^2$

**Solution**

$$(a) \quad 2 \sin 30^\circ + 3 \cos 60^\circ - 3 \tan 45^\circ = 2 \times \frac{1}{2} + 3 \times \frac{1}{2} - 3 = 1 + \frac{3}{2} - 3 = -\frac{1}{2}$$

$$(b) \quad 3(\cos 30^\circ)^2 + 2(\sin 30^\circ)^2 = 3\left(\frac{\sqrt{3}}{2}\right)^2 + 2\left(\frac{1}{2}\right)^2 = 3 \times \frac{3}{4} + 2 \times \frac{1}{4} = \frac{9}{4} + \frac{2}{4} = \frac{11}{4}$$

**EXAMPLE 33**

In figure 57, find the value of  $x$  and  $y$ .

**Solution**

$$\cos 60^\circ = \frac{x}{16}$$

$$x = 16 \cos 60^\circ = 16 \times \frac{1}{2} = 8$$

$$\tan y = \frac{x}{6}$$

$$\tan y = \frac{8}{6} = \frac{4}{3}$$

$$y \approx 53^\circ.$$

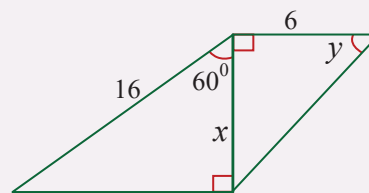


Figure 57.

**EXAMPLE 34**

An isosceles right triangle has two legs with a length of 2 units. If  $\theta$  is one of the acute angles, what are the sine, cosine, tangent, cosecant, secant and cotangent of  $\theta$ ?

**Solution**

Let  $a$  and  $b$  be the two legs of the given right triangle. Then,  $a = 2$  and  $b = 2$ . If the hypotenuse is denoted by  $c$ , then by Pythagorean theorem, we have

$$a^2 + b^2 = c^2 \Rightarrow 2^2 + 2^2 = c^2 \Rightarrow c^2 = 4 + 4 \Rightarrow c^2 = 8 \Rightarrow c = \sqrt{8} = 2\sqrt{2}.$$

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \text{and so} \quad \csc \theta = \frac{1}{\sin \theta} = \sqrt{2}.$$

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \text{and so} \quad \sec \theta = \frac{1}{\cos \theta} = \sqrt{2}.$$

**EXERCISES**

- Find the exact value of each of the following.
 

(i) $\cos 135^\circ$	(iii) $\sin 210^\circ$
(ii) $\tan (-405^\circ)$	(iv) $\tan 225^\circ$
- Find the exact value of each of the following.
 

(i) $\cos 300^\circ$	(iii) $\sin 1305^\circ$
(ii) $\sin (-420^\circ)$	(iv) $\tan 750^\circ$
- Evaluate the expression  $\sin 60^\circ \cos 30^\circ + \sin 30^\circ \cos 60^\circ$
- Evaluate the expression  $2(\tan 45^\circ)^2 + (\cos 30^\circ)^2 - (\sin 60^\circ)^2$
- Determine the exact value of each of the following.
 

(i) $\sin 30^\circ \tan 45^\circ + \tan 30^\circ \sin 60^\circ$
(ii) $\cos 30^\circ \sin 45^\circ + \sin 30^\circ \tan 30^\circ$

One type of angle that occurs in many applications is the angle of elevation of a given point with respect to a point that is at a lower elevation and at some positive horizontal distance from it.

Another important angle is the angle of depression of a given point with respect to a point that is at a higher elevation and at some positive horizontal distance from it.

In this sub-topic, we will see how trigonometry is used for finding the heights and distances of various objects without actually measuring them. For example, the

height of a tower, mountain, building or tree, distance of a ship from a light house, width of a river, etc. can be determined by using knowledge of trigonometry.

### ACTIVITY 9

1. What is angle of elevation?
2. What is angle of depression? Discuss!
3. Show that angle of elevation and angle of depression are congruent.

The process of finding heights and distances is the best example of applying trigonometry in real-life situations. We would explain these applications through some examples. Before studying methods to find heights and distances, we should understand some basic definitions.

### DEFINITION

The line which is drawn from the eyes of the observer to the point being viewed on the object is known as the *line of sight*.

In figure 58, the line PQ represents the line of sight, some times it can be called as *line vision*.

### DEFINITION

The *angle of elevation* of the point on the object (above the horizontal line) viewed by the observer is the angle which is formed by the line of sight with the horizontal line.

In Figure 58, the angle of elevation of Q with respect to P is  $\theta$  and the angle of depression of P with respect to Q is  $\alpha$ .

### DEFINITION

The *angle of depression* of the point on the object (below the horizontal line) viewed by the observer is the angle which is formed by the line of sight with the horizontal line.

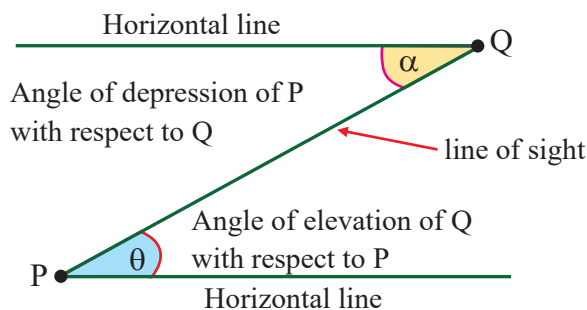


Figure 58.

### Note

The angle of elevation and the angle of depression are acute angles and are defined to have positive measure.

### GROUP WORK

A TV tower stands vertically on a bank of canal. The tower is watched from a point on the other bank directly opposite to it. The angle of elevation of the top of the tower is  $60^\circ$ . From another point 30 meter away from this point on the line joining this point to the foot of the tower, the angle of elevation of the top of the tower is  $30^\circ$ , as shown in figure 59. Find the height of the tower and the width of the canal.



Figure 59.

Using the concepts of angle of elevation and depression, let us solve some examples related to the problem of height and distance.

### EXAMPLE 35

In Figure 60, AB is a 7 m high pole and CD is a ladder inclined at an angle of  $60^\circ$  to the horizontal and reaches up to a point D of the pole. If  $AD = 3$  m, find the length of the ladder.

#### Solution

$$\begin{aligned} BD &= AB - AD \\ &= 7 \text{ m} - 3 \text{ m} = 4 \text{ m} \end{aligned}$$

$$\sin C = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{DB}{CD}$$

$$\sin 60^\circ = \frac{4}{CD}$$

$$CD \times \sin 60^\circ = 4$$

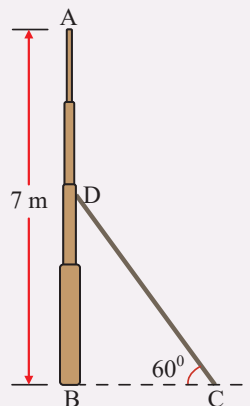


Figure 60.

$$CD = \frac{4}{\sin 60^\circ} = \frac{4}{0.866} = 4.6 \text{ m.}$$

Therefore, the length of the ladder is 4.6 meters.

### EXAMPLE 36

A vertical 6 m flag pole CD is held by two cables attached to the ground at A and B. Cable AD makes an angle of  $60^\circ$  with the ground. Cable BD makes an angle of  $30^\circ$  with the ground, as shown in figure 61. Find,

- the distance between A and B.
- the lengths of the cables.

#### Solution

$$\begin{aligned} \text{(a) } \tan 60^\circ &= \frac{6}{AC} \\ AC \times \tan 60^\circ &= 6 \\ AC &= \frac{6}{\tan 60^\circ} = \frac{6}{\sqrt{3}} = 2\sqrt{3} \text{ m.} \end{aligned}$$

$$\begin{aligned} \text{(b) } \tan 30^\circ &= \frac{6}{BC} \\ BC \times \tan 30^\circ &= 6 \\ BC &= \frac{6}{\tan 30^\circ} = \frac{6}{\frac{1}{\sqrt{3}}} = 6\sqrt{3} \text{ m} \end{aligned}$$

$$AB = AC + CB = 2\sqrt{3} + 6\sqrt{3} = 8\sqrt{3} \text{ m.}$$

Therefore, the distance between A and B is  $8\sqrt{3}$  m.

$$\begin{aligned} \text{(c) } \sin 60^\circ &= \frac{6}{AD} \\ AD \times \sin 60^\circ &= 6 \\ AD &= \frac{6}{\sin 60^\circ} = \frac{6}{\frac{\sqrt{3}}{2}} = 4\sqrt{3} \text{ m.} \end{aligned}$$

The length of cable AD is  $4\sqrt{3}$  m.

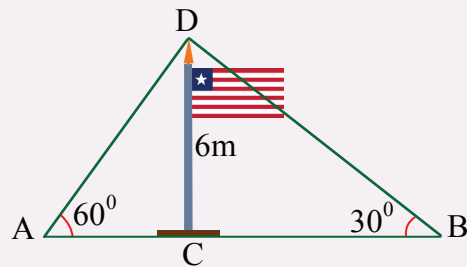


Figure 61.

$$(d) \sin 30^\circ = \frac{6}{BD}$$

$$BD \times \sin 30^\circ = 6$$

$$BD = \frac{6}{\sin 30^\circ} = \frac{6}{\frac{1}{2}} = 12 \text{ m.}$$

The length of cable BD is 12 m.

### EXAMPLE 37

A bird is at the top of a vertical pole posted on a level ground. At a point on the level of ground which is 15 meters away from the foot of the pole, the line of sight of the bird makes an angle of  $30^\circ$  with the ground as shown in figure 62. How far is the bird from point P?

#### Solution

Let the distance between the point P and the bird be  $d$  meters. Thus,

$$\cos 30^\circ = \frac{15}{d}$$

$$d \times \cos 30^\circ = 15$$

$$d = \frac{15}{\cos 30^\circ} = \frac{15}{\frac{\sqrt{3}}{2}} = 10\sqrt{3} \text{ m.}$$

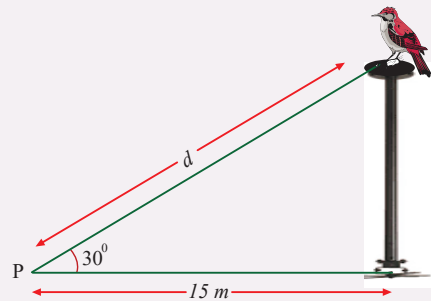


Figure 62.

### EXAMPLE 38

As shown in figure 63, the angle of elevation of the top of a cliff when seen from the foot and the top of the building are  $45^\circ$  and  $30^\circ$  respectively. If the height of the building is 20 m, then how high is the top of the cliff?

#### Solution

Let the height of the cliff be  $h$  meters and the distance between the foot of the building and the foot of the cliff be  $x$  meters. Thus,

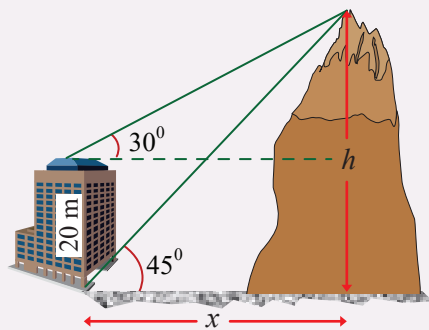


Figure 63.

$$\tan 45^\circ = \frac{h}{x}$$

$$x \tan 45^\circ = h$$

$$x = h \dots\dots\dots \text{(because } \tan 45^\circ = 1 \text{)}$$

$$\tan 30^\circ = \frac{h-20}{x}$$

$$x \tan 30^\circ = h - 20$$

$$\frac{x}{\sqrt{3}} = h - 20$$

$$x = h\sqrt{3} - 20\sqrt{3}$$

Now substituting  $x = h$  in equation  $x = h\sqrt{3} - 20\sqrt{3}$  gives

$$h = h\sqrt{3} - 20\sqrt{3}$$

$$h(\sqrt{3} - 1) = 20\sqrt{3}$$

$$h = \frac{20\sqrt{3}}{\sqrt{3} - 1} = 10(3 + \sqrt{3}) \text{ m.}$$

**EXAMPLE 39**

At a point P on a level ground, the angle of elevation of the top of a building is  $60^\circ$ . At point Q which is 10 meters away from P, the angle of elevation of the top of the building is found to be  $30^\circ$ , as shown in figure 64. What is the distance between point P and the foot of the building?

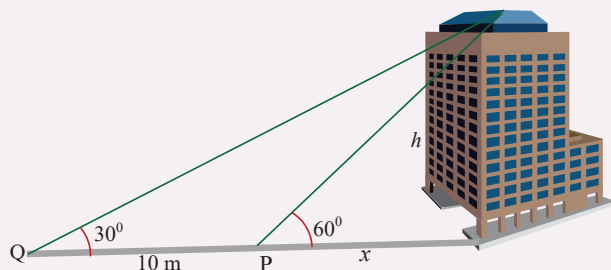


Figure 64.

**Solution**

Let the height of the building be  $h$  meters and the distance between the point P and the foot of the building be  $x$  meters. Thus,

$$\tan 30^\circ = \frac{h}{x+10}$$

$$\frac{1}{\sqrt{3}} = \frac{h}{x+10}$$

$$h = \frac{x+10}{\sqrt{3}} \dots\dots\dots(1)$$

$$\tan 60^\circ = \frac{h}{x}$$

$$\sqrt{3} = \frac{h}{x}$$

$$h = \sqrt{3}x \dots\dots\dots(2)$$

$$\sqrt{3}x = \frac{x+10}{\sqrt{3}}$$

$$3x = x + 10$$

$$2x = 10$$

$$x = 5.$$

Therefore, the distance between point P and the foot of the building is 5 meters.

**EXERCISES**

1. A tower stands vertically on the ground. From a point on the ground, which is 48 m away from the foot of the tower, the angle of elevation of the top of the tower is  $30^\circ$ . Find the height of the tower.
2. A kite is flying at a height of 87 m above the ground. The string attached to the kite is temporarily tied to a point on the ground. The inclination of the string with the ground is  $60^\circ$ . Find the length of the string, assuming that there is no slack in the string.
3. Two ships are sailing in the sea on either side of a lighthouse. The angle of elevation of the top of the lighthouse as observed from the ships are  $30^\circ$  and  $45^\circ$  respectively. If the lighthouse is 30 m high, find the distance between the two ships.
4. From a point on the ground, the angles of elevation of the bottom and top of a tower fixed at the top of a 40 m high building are  $45^\circ$  and  $60^\circ$  respectively. Find the height of the tower.
5. An observer on a level of ground is at a distance  $10\sqrt{3}$  m from a building. The angles of elevation to the bottom of the windows on the second and third floors are  $30^\circ$  and  $60^\circ$ , respectively. What is the distance between  $h$  the bottom of the windows?

6. If the distance from a helicopter to a boat is 300 feet and the angle of depression is  $60^\circ$ , find the distance on the ground from a point directly below the helicopter to the boat.
7. Two ships, one with angle of depression  $60^\circ$  due east and the other with  $30^\circ$  due west are observed from a plane 1,200 meters above a sea. If the two ships are on the same line, what is the distance between the two ships?
8. Two trees are standing on flat ground. The angle of elevation of the top of both the trees from a point P on the ground is  $40^\circ$ . If the horizontal distance between P and the smaller tree is 8 m and the distance between the top of the two trees is 20 m, calculate
  - (a) the distance between the point P and the top of the smaller tree.
  - (b) the horizontal distance between the two trees.

### KEY TERMS

- Slope
- Tangent
- Opposite side
- Gradient
- Cosine
- Adjacent side
- Inclination
- Sine
- Hypotenuse
- Height
- Trigonometry
- Line of sight
- Distance
- Rotation
- Trigonometric ratio
- Angle of elevation
- Angle of depression

### SUMMARY

- If the two points  $(x_1, y_1)$  and  $(x_2, y_2)$  are two distinct points on a line with  $x_2 - x_1 \neq 0$ , then the **gradient (slope)** of the line joining the two points is given by:  $m = \frac{y_2 - y_1}{x_2 - x_1}$ .
- The angle measured from positive  $x$ -axis to a line in counter clockwise direction is called the **inclination** of a line. The **tangent** of this angle is called the **slope** of the line. That is,

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = m$$

- Using the right triangle ABC which is a right angle at C and  $a$ ,  $b$ , and  $c$  are the lengths of the sides opposite these angles, then the tangent, cosine and sine of an angle in relation to  $\angle A$  can be defined as follows:

$$\sin \angle A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c};$$

$$\cos \angle A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}; \text{ and}$$

$$\tan \angle A = \frac{\text{adjacent}}{\text{opposite}} = \frac{a}{b}.$$

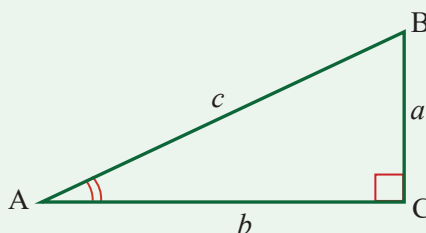


Figure 65.

- If we denote one of the two acute angles by  $\theta$ , we can express the relation to the angle  $\theta$  as follows:

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{a}{c}; \cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{b}{c}; \text{ and } \tan \theta = \frac{\text{adj}}{\text{opp}} = \frac{a}{b}.$$

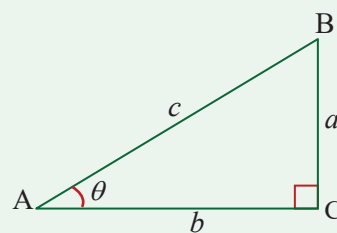


Figure 66.

- The point  $P(x, y)$  is the same as the point  $P(\cos \theta, \sin \theta)$  on the unit circle.
- The Law of Sine:** If  $A$ ,  $B$ , and  $C$  are the measures of the angles of a triangle  $ABC$  and  $a$ ,  $b$ , and  $c$  are the lengths of the sides opposite these angles, then
 
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
- The Law of Cosine:** If  $A$ ,  $B$ , and  $C$  are the measures of the angles of a triangle  $ABC$  and  $a$ ,  $b$ , and  $c$  are the lengths of the sides opposite these angles, then
 
$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$
- The line which is drawn from the eyes of the observer to the point being viewed on the object is known as the **line of sight**.
- The **angle of elevation** of the point on the object (above the horizontal line) viewed by the observer is the angle which is formed by the line of sight with the horizontal line.

- The *angle of depression* of the point on the object (below the horizontal line) viewed by the observer is the angle which is formed by the line of sight with the horizontal line.

**EXERCISES**

1. Find the slope of the line joining:
  - (a)  $(-1, -2)$  and  $(4, 8)$
  - (b)  $(-2, 3)$  and  $(3, -2)$
  - (c)  $(2, -3)$  and  $(-3, 7)$
  - (d)  $(9, -2)$  and  $(9, -9)$
2. Find the slope of a line if its inclination is
  - (a)  $150^\circ$
  - (b)  $-60^\circ$
  - (c)  $-30^\circ$
  - (d)  $120^\circ$
3. Find angle of a line if its slope is:
  - (a)  $-0.63$
  - (b)  $1.732$
  - (c)  $-0.58$
  - (d)  $-1.732$
4. Find the value of
  - (a)  $\cos 210^\circ$
  - (b)  $\sin 225^\circ$
  - (c)  $\tan 300^\circ$
  - (d)  $\cos 1250^\circ$
  - (e)  $\sin 680^\circ$
5. Find  $R_{90^\circ}$ ,  $R_{-90^\circ}$  and  $R_{180^\circ}$  and the center of rotation for the given equations of lines and represent them on the graph.
  - (a)  $x + 3y = 5$
  - (b)  $5x - 3y = 7$
  - (c)  $x - y = 7$
  - (d)  $4x - 3y = 12$

6. Triangle ABC has vertices  $A(-2, 3)$ ,  $B(3, 2)$ , and  $C(1, -5)$ . Draw triangle ABC and its image after a rotation of  $90^\circ$  counterclockwise about the origin.
7. Using the *law of cosine* find the unknown parts of each of the following triangles.

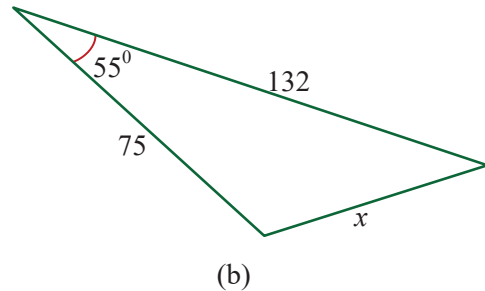
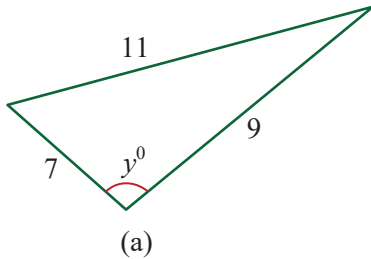


Figure 67.

8. Using the law of sine find the unknown parts of each of the following triangles.

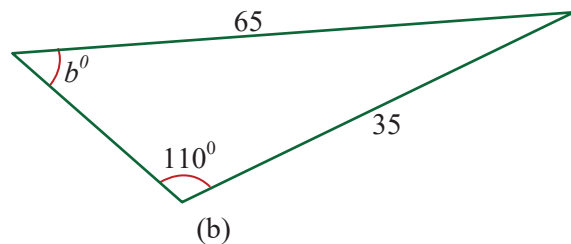
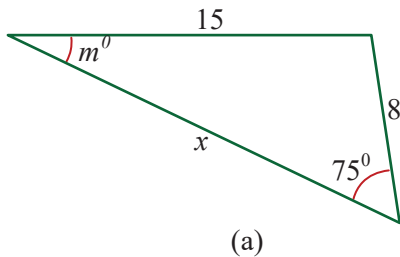


Figure 68.

9. In Figure 69, if  $\triangle ABC$  is right angled-triangle, then find  $\sin \alpha$ ,  $\sin \theta$ ,  $\cos \alpha$ ,  $\cos \theta$ ,  $\tan \alpha$  and  $\tan \theta$ .

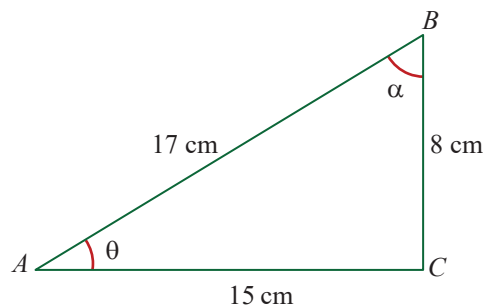


Figure 69.

10. From a stand which is 2 meters high, a 10 meters long ladder leans against a vertical wall and makes an angle of  $60^\circ$  with the wall. How high is the top of the ladder from the foot of the wall on the ground?
11. A boy notes that from a certain point on level ground, the angle of elevation to a point at the top of a tree is  $30^\circ$ . After walking 50 m closer to the tree, the backpacker notes that the angle of elevation is  $60^\circ$ . Find the height of the tree.
12. The angle of elevation and the angle of depression of the top of a building from the foot and the top of a tower are  $30^\circ$  and  $45^\circ$  respectively. If the building is 20-meter high, then what is the height of the tower?
13. The angle of depression of the top of a building from the top of a flag that is 100 m away from the flag pole is  $30^\circ$ . If the flag pole is 20 m tall, then what is the height of the building?
14. Two points on a level ground are 500 m apart. The angle of elevation from these points to the top of a tower are  $45^\circ$  and  $60^\circ$  respectively. If the two points and the base of the tower lie on a straight line, what is the height of the tower in meters?
15. If from the top of a hill 600 m above the sea level the angle of depressions of two boats in line with the observer were found to be  $30^\circ$  and  $60^\circ$ , then find the distance between the two boats.
16. Two men A and B, stand opposite sides of a 150 m tall tower. If A observes the top of the tower at angle of elevation of  $60^\circ$  and B observes the same point at an angle of elevation of  $30^\circ$ , then how far apart are the two men?
17. The angles of elevation and an airplane from two points A and B on level ground are  $45^\circ$  and  $60^\circ$  respectively. A and B are 8 km apart, and the airplane is to the east of A and B in the same vertical plane. Find the altitude of the airplane.
18. The angle of elevation and the angle of depression of the top of a building from the foot and the top of a tower are  $30^\circ$  and  $60^\circ$  respectively. If the building is 20 meter high, then what is the height of the tower?
19. A man walks from point A for 2 km in a direction N  $40^\circ$  W, to point B. He then turns and walks a further 2 km in a direction S  $80^\circ$  W to point C. He then walks back to A, his starting point.
  - (a) What distance does he walk from C to A?
  - (b) What is his bearing as he walks from C to A?

20. The angles of elevation to a jet from two missile stations A and B on the level ground are  $60^\circ$  and  $45^\circ$  respectively. The two stations A and B are 15km apart, and the jet is to the west of A and B in the same vertical plane.
- Find the altitude of the jet.
  - Find the distance between the foot of the altitude and the station B.

# CHAPTER



M11CH08

# 8

## PROBABILITY

### Chapter Contents

- 8.1 Probability
- 8.2 Relative Frequency
- 8.3 Compound Events
- 8.4 Union of Events
- 8.5 Intersection of Events
- 8.6 Review of the Basic Concept of Set, Venn, Tree Diagram and Contingency Tables
- 8.7 Sample Space and Event of an Experiment
- 8.8 The Probability of an Event
  - Key Terms
  - Summary
  - Exercises



## **Chapter Outcomes**

By the end of this chapter, you will be able to:

- define and discuss probability;
- determine the relative frequency;
- calculate compound events;
- compute union of events;
- compute intersection of events;
- compute independent events.

## Introduction

In our real life, we come across the statements involving words like “likely”, “probably”, “chances”, etc.

For example, consider the following statements:

- (a) It is likely to rain tomorrow.
- (b) Probably, the stock prices of agricultural products will go up next year.
- (c) Some parties will have 25% of chance of winning in an upcoming election.

All the above-mentioned statements have the element of uncertainty. These types of uncertainties can be measured numerically by using the concept of ‘probability’ in many cases.

**Probability** is a branch of mathematics that is widely used in the science, business, finance, and medicine. We use probability to determine the *chance* or *likelihood* of an event or events. In probability, events are the outcomes of an experiment. The probability of an event is the measure of the chance that the event will occur as an outcome of an experiment.

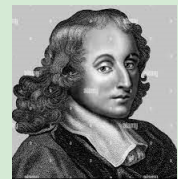
In this topic you learn how to apply your understanding of probability to solve a wide range of problems.

Probability refers to the likelihood that an event will occur. By analyzing the available data, we can predict the chances of a certain event in terms of a number. We can do simple experiments to become familiar with the different outcomes that are possible, and we can develop and apply some mathematical laws of probability to help us analyze such experiment.

We shall often use experiments like tossing coins, rolling dice, drawing cards from a pack and selecting colored balls from a box, because they are simple to visualize and will give us basic ideas of probability theory.

### HISTORICAL NOTE

A gambler’s dispute in 1654 led to the creation of a mathematical theory of probability by two famous French mathematicians, Blaise Pascal and Pierre de Fermat. Blaise Pascal (1623-62) suffered the most appalling ill-health throughout his short life. In correspondence with Fermat, he laid the foundation for the *theory of probability*. This correspondence consisted



of five letters and occurred in the summer of 1654. They considered the dice problem, already studied by Cardan, and the problem of points also considered by Cardan and around the same time, Pacioli and Tartaglia. The dice problem asks how many times one must throw a pair of dice before one expects a double six while the problem of points asks how to divide the stakes if a game of dice is incomplete. They solved the problem of points for a two player game but did not develop powerful enough mathematical methods to solve it for three or more players.

### ACTIVITY 1

1. Discuss the following basic concepts of probability.
  - (a) Outcomes
  - (b) Favorable outcome
2. Find the possible outcomes for rolling two dice
3. A bag contains balls made of leather. There are 4 white balls, 9 blue balls and 12 red balls in a bag. If a ball is drawn at random, then find the following:
  - (a) Probability that the ball drawn is made up of rubber
  - (b) Probability that the ball drawn is made up of leather.
  - (c) Probability that the ball drawn is a red colored ball
4. When we flip a coin in the air, what is the possibility of getting a head?

Probability denotes the possibility of the outcome of any extent to which any event is likely to happen. Probability of an event happening is a ratio of the number of favorable outcomes to the total number of outcomes.

Mathematically, probability of an event happening is a ratio of the number of favorable outcomes to the total number of outcomes. It can be formulated as follows:

$$\text{Probability} = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}$$

Range of probabilities is from 0 to 1, where 0 represents impossible event and 1 represents a probable event.

The outcome (sample point) is the result of an experiment. The outcomes which insure the happening of a particular result are said to be favorable outcomes to that particular result.

#### EXAMPLE 1

A fair die is thrown. How many favorable outcomes are there for getting an odd number? Find the probability of getting odd numbers.

**Solution**

In throwing a die there are six possible outcomes, they are  $\{1,2,3,4,5,6\}$

Since there are three odd numbers, there are 3 favorable outcomes. These are 1, 3 and 5.

$$\text{Probability} = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}} = \frac{3}{6} = \frac{1}{2}.$$

**EXAMPLE 2**

Two coins are tossed.

- Find the probability of at least one head appearing.
- Find the probability of no head appearing.

**Solution**

The possible outcomes in tossing two coins is  $S = \{HH, HT, TH, TT\}$ . There are 4 possible outcomes.

- Let  $E_1$  be an event consisting of the outcomes at least one head. Then,

$E_1 = \{HH, HT, TH\}$ . There are 3 outcomes of the event. Thus,

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{4}.$$

- Let  $E_2$  be an event consisting of the outcomes no head. Then,

$E_2 = \{TT\}$ . There is 1 outcome of the event. Thus,

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{4}.$$

**EXAMPLE 3**

A die is rolled twice. Find the probability of an event consisting of:

- all outcomes whose sum is equal to 5.
- two identical numbers.
- the two numbers are even numbers
- the two numbers are odd numbers.

**Solution**

The possible outcomes in rolling a die twice is:

$\{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$ .

There are 36 possible outcomes

- (a) Let  $E_1$  be an event consists of all outcomes whose sum is equal to 5. Then,  $E_1 = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$ . There are 4 outcomes of the event. Thus,
- $$P(E) = \frac{n(E)}{n(S)} = \frac{4}{36} = \frac{1}{9}.$$
- (b) Let  $E_2$  be an event consists of all outcomes that two identical numbers. Then,  $E_2 = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$ . There are 6 outcomes of the event. Thus,
- $$P(E) = \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}.$$
- (c) Let  $E_3$  be an event consists of all outcomes that two numbers are even numbers. Then,  $E_3 = \{(2, 2), (2, 4), (2, 6), (4, 2), (4, 4), (4, 6), (6, 2), (6, 4), (6, 6)\}$ . There are 9 outcomes of the event. Thus,  $P(E) = \frac{n(E)}{n(S)} = \frac{9}{36} = \frac{1}{4}$ .
- $$P(E) = \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}.$$
- (d) Let  $E_4$  be an event consists of all outcomes that two numbers are odd numbers. Then,  $E_4 = \{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5)\}$ . There are 9 outcomes of the event. Thus,  $P(E) = \frac{n(E)}{n(S)} = \frac{9}{36} = \frac{1}{4}$ .

### EXERCISES

- Alex takes out a ball from a box without looking at it. If box contains one white, one black and one green ball, then what is the probability that he takes out:
  - a white ball
  - a green ball
- Two dice are tossed simultaneously, what is the probability that the sum of the numbers is more than 10?
- Suppose three coins are tossed. Then, find the probability of:
  - exactly two heads appearing
  - at least three tails appearing
  - exactly two tails appearing.
  - at least two heads appearing.
- A bag contains 4 blue, 9 red, and 2 white marbles. One marble is drawn at random.
  - What is the probability that the marble is red?
  - What is the probability that the marble is not red?
  - What is the probability that the marble is red or not red?

The word “*relative*” is used to indicate that an event is being considered in relation or in proportion to something else. A relative frequency indicates how often a specific kind of event occurs within the total number of observations. It is a frequency that uses percentages, proportions and fractions. In this sub-topic, we will learn how to calculate relative frequency.

### ACTIVITY 2

1. What is relative frequency?
2. A coin is tossed 500 times with the frequencies of head 330 times and tail for remaining times. Find the probability for head and tail.
3. A dice is thrown with the following outcomes and frequencies in the following table:

Outcomes	1	2	3	4	5	6
Frequency	11	20	18	15	24	12

- (a) What is the probability that an even number appears?
- (b) What is the probability that a prime number appears?
- (c) What is the probability that a number 2 appears?
- (d) What is the probability that a number 5 appears?

Relative frequency of an event can be defined as the number of times an event occurs divided by the total number of events occurring in a given trials.

We can take the *relative frequency* of an event occurring as an estimate of the *probability* of that event occurring. The estimate improves as the number of trials is increased.

Suppose that we make  $n$  trials and that  $r$  of those trials are success, then, based on this evidence, an estimate for the probability of success or outcomes favorable to event  $E$  in the experiment is given by:

$$\text{Probability of the event } E = \frac{\text{Number of times E occurred}}{\text{Number of times the experiment was repeated}}$$

$$P(E) = \frac{r}{n}$$

### EXAMPLE 4

David randomly selects a card from a pack and notes whether it is a Heart, Spade, Diamond or Club. His results is shown in the table below.

Cards	H	S	D	C
Frequency	8	9	5	8

- (a) What was the relative frequency of selecting a Heart?  
 (b) What was the relative frequency of selecting a Diamond?

**Solution**

$$\begin{aligned} \text{(a) Relative frequency} &= \frac{\text{Number of times Heart occurred}}{\text{Total number of trials}} = \frac{8}{30} \\ &= 0.2667 = 26.67\%. \end{aligned}$$

$$\begin{aligned} \text{(b) Relative frequency} &= \frac{\text{Number of times Diamond occurred}}{\text{Total number of trials}} = \frac{5}{30} \\ &= 0.1667 = 16.67\%. \end{aligned}$$

Relative frequency calculations convert counts into percentages by taking the count of a specific type of event and dividing it by the total number of observations.

**EXAMPLE 5**

A die is tossed 50 times and lands 8 times on the number 5. What is the probability of observing the die land on the number 5?

**Solution**

The number times a die is tossed is 60 and the number of successful trials of getting number 5 is 10. Thus,

$$\begin{aligned} \text{Probability of getting 5} &= \frac{\text{Frequency of observing number 5 on the die}}{\text{Total number of trials}} \\ &= \frac{10}{60} = \frac{1}{6} = 0.1666 = 16.66\%. \end{aligned}$$

**EXAMPLE 6**

Anna has a packet containing 30 candies. The table below shows the frequency of each different candy color selected as she picked all 20 sweets one by one and finished them all.

Candy color	Yellow	Red	Green	Brown
Frequency	9	9	4	8

- (a) What is the probability of the picked candy being yellow or red?  
 (b) What is the probability for the brown candy?



4. The heights of 200 students of a school are shown in the following table:

height (in cm)	120-130	130-140	140-150	150-160	160-170	170-180	180-190
Number of students	30	28	32	48	43	15	4

Find the probability that a student's height lies in the interval:

- (a) 120-130 cm                      (b) 140-150 cm                      (c) 160-170 cm
5. 1500 families were randomly selected and a survey of their income was conducted. The result obtained is as follows:

Monthly income (in USD)	Less than 3500	3500-5000	5000-6500	6500-8000	More than 8000
Number of families	405	630	225	165	75

Suppose a family is chosen. Find the probability that the chosen family is as follows:

- (a) Earning L\$5,000 – L\$6,500 per month  
 (b) Earning L\$6,500 – L\$8,000 per month  
 (c) Earning L\$8,000 or more per month  
 (d) Earning less than L\$3,500 per month

A compound event is an event that has more than one possible outcome. In a compound event, an experiment gives more than one possible outcome. These outcomes may have different probabilities but they are all equally possible. In this sub-topic, we will see what we mean by compound events.

### ACTIVITY 3

- Define the following.
  - Simple Events
  - Impossible and sure events
  - Independent event and dependent events
- A number is selected at random from first 200 natural numbers. Find the probability that the number selected is a perfect square or a perfect cube.

An event is an occurrence that can be determined by a given level of certainty. A compound event combines two or more events, using the word “AND” or the word “OR”. Probability of a compound event can be written as an expression involving probabilities of simpler events. By doing this, calculation of compound probability becomes easier.

### Mutually exclusive events

Those events which cannot happen simultaneously are called *mutually exclusive events*. In other words, two events are mutually exclusive if they cannot occur at the same time. For two mutually exclusive events  $E_1$  and  $E_2$ ,  $E_1 \cap E_2 = \emptyset$ . For example, in a throw of die, we cannot get an even and odd outcome at the same time. Thus, getting an even number or odd number in a throw of a die are mutually exclusive events.

#### EXAMPLE 7

Determine which events are mutually exclusive and which are not, when a single die is rolled.

- Getting an odd number and getting an even number.
- Getting a 3 and getting an odd number.
- Getting an odd number and getting a number less than 4.
- Getting a number greater than 4 and getting a number less than 4.

#### Solution

- The events are mutually exclusive, since the first event can only be 1, 3, or 5 and the second event can only be 2, 4, or 6. Both events can't occur at the same time.
- The events are not mutually exclusive, since the first event is a 3 and the second can be 1, 3, or 5. Hence, 3 is contained in both events.
- The events are not mutually exclusive, since the first event can be 1, 3, or 5 and the second can be 1, 2, or 3. Hence, 1 and 3 are contained in both events.
- The events are mutually exclusive, since the first event can be 5 or 6 and the second event can be 1, 2, or 3.

The “OR” rule: For mutually exclusive events  $E_1$  and  $E_2$ .

$$P(E_1 \text{ or } E_2) = P(E_1) + P(E_2)$$

#### EXAMPLE 8

One ball is selected at random from a bag containing 5 red balls, 2 yellow balls and 4 white balls. Find the probability of selecting a red ball or a white ball.

**Solution**

The two events are mutually exclusive.

$$\begin{aligned} P(\text{red ball or yellow ball}) &= P(\text{Red ball}) + P(\text{Yellow ball}) \\ &= \frac{5}{11} + \frac{4}{11} = \frac{9}{11}. \end{aligned}$$

**Non-mutually exclusive events**

If there is at least one outcome which is common in the events, they are called **non-mutually exclusive events**. In other words, if two events have sample points in common, they are **not mutually exclusive**. For example, in drawing a card event of drawing a spade and an ace card are not mutually exclusive event because a spade can also be an ace card.

**EXAMPLE 9**

In rolling a die, we have a sample space:  $S = \{1, 2, 3, 4, 5, 6\}$ . Let  $E_1$  and  $E_2$  be two events such that  $E_1$  is getting even numbers and  $E_2$  is getting prime numbers. Then,  $E_1$  and  $E_2$  are not mutually exclusive events because they have common event or sample point  $\{2\}$ .

**EXAMPLE 10**

In a pack of 52 cards what is the probability of getting a queen or an ace?

**Solution**

These are mutually exclusive events, because it is impossible for a card to be a queen and an ace at the same time.

There are 4 queens in the pack. So,  $P(\text{queen}) = \frac{4}{52} = \frac{1}{13}$ .

There are 4 aces in the pack. So,  $P(\text{an ace}) = \frac{4}{52} = \frac{1}{13}$ .

Therefore,  $P(\text{queen or an ace}) = \frac{1}{13} + \frac{1}{13} = \frac{2}{13}$ .

**Independent events**

**Independent events** are events where the occurrence of one of the events does not affect the occurrence of the other event.

Consider an experiment of tossing a coin and throwing a die simultaneously. To determine the probability of getting a head on a coin and a "5" on a dice

simultaneously, we have to combine the outcomes of the coin and the dice. We know that the outcome for a coin is either head or tail. Outcomes for a die can be either 1, 2, 3, 4, 5 or 6. By combining these outcomes, we will have the following sample space:

$\{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$

Here,  $(H, 1)$  represents head on the coin and 1 on the dice.

One can observe that the total number of outcomes is 12, but only one is the favorable, that is,  $(H, 5)$ , so the required probability  $= \frac{1}{12}$ .

**The “AND” rule:** If  $E_1$  and  $E_2$  are independent events, then

$$P(E_1 \text{ and } E_2) = P(E_1) \times P(E_2).$$

### EXAMPLE 11

A fair coin is tossed and a fair die is rolled. Find the probability of obtaining a head and 6.

#### Solution

The two events are independent.

$$\begin{aligned} P(\text{Head and } 6) &= P(\text{Head}) \times P(6) \\ &= \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}. \end{aligned}$$

## Dependent events

*Dependent events* are events where the occurrence of one of the events does affect the occurrence of the other event. In other words, dependent events are events for which the occurrence of either event affects the next.

If  $E_1$  and  $E_2$  are dependent events, then  $P(E_1 \text{ then } E_2) = P(E_1) \times P(E_2)$  when  $E_1$  has occurred).

### EXAMPLE 12

A box contains 5 mathematics books and 7 physics books. Two books are randomly selected from the box without replacing them.

- Find the probability of both are mathematics books.
- Find the probability that the of first box is a mathematics book and the second is a physics book.

**Solution**

- (a) Probability that the first book drawn is a mathematics book  $\frac{5}{12}$ . After drawing one mathematics book, we are left with 4 mathematics books and 7 physics books. Probability that the second book drawn is a mathematics book  $= \frac{4}{11}$ .  
Probability that the both books drawn are mathematics books  $\frac{5}{12} \times \frac{4}{11} = \frac{5}{33}$ .
- (b) Probability that the second book drawn is a physics book  $= \frac{7}{11}$ .  
Probability that the first book drawn is a mathematics and the second is physics book is  $\frac{5}{12} \times \frac{7}{11} = \frac{35}{132}$ .

**EXERCISES**

- Define mutually exclusive events, and give an example of two events that are mutually exclusive and two events that are not mutually exclusive.
- Determine whether these events are mutually exclusive.
  - Roll a die: Get an even number, and get a number less than 3.
  - Roll a die: Get a prime number (2, 3, 5), and get an odd number.
  - Roll a die: Get a number greater than 3, and get a number less than 3.
- A standard die is rolled. What is the probability that a 2, 4, or 6 will be rolled?
- At a community swimming pool there are 2 managers, 8 lifeguards, 3 concession stand clerks, and 2 maintenance people. If a person is selected at random, find the probability that the person is either a lifeguard or a manager.
- A card is chosen at random from a standard deck of 52 playing cards. Without replacing it, a second card is chosen. What is the probability that the first card chosen is a queen and the second card chosen is a jack?
- Two dice are rolled. Find the probability of getting:
  - A sum of 5, 6, or 7.
  - Identical numbers or a sum of 6 or 8.
  - A sum greater than 8 or less than 3.

The union of two events  $E_1$  and  $E_2$ , denoted by  $E_1 \cup E_2$  is the event consisting of all outcomes that are either in  $E_1$  or in  $E_2$  or in both  $E_1$  and  $E_2$ .

## ACTIVITY 4

1. What is the union of two sets?
2. Let  $E_1 = \{0, 1, 2, 3, 4, 5, 6\}$  and  $E_2 = \{-1, 0, 2, 6, 7, 8\}$ . Then find  $E_1 \cup E_2$ .
3. A bag contains 7 green balls, 11 orange balls and 12 red balls. If a ball is drawn, find the probability that it is a green or an orange.
4. A card is drawn at random from a 52-deck of card. Find the probability of getting:
  - (a) A king or an ace.
  - (b) A number card or an ace.
  - (c) A black card or an a diamond.
  - (d) A red card or a club.
  - (e) A king or an odd number.

If two events  $E_1$  and  $E_2$  are associated with “OR”, then it means that either  $E_1$  or  $E_2$  or both  $E_1$  and  $E_2$ . The union symbol “ $\cup$ ” is used to represent “OR” in probability. Thus, the events  $E_1$  or  $E_2$  denotes

$$E_1 \cup E_2.$$

If two events  $E_1$  and  $E_2$  are not mutually exclusive, then they share outcomes in common. The overlap of the two sets is their intersection, that is,  $E_1 \cap E_2$ . The probability that  $E_1$  or  $E_2$  will happen is the sum of the probability of each event, minus the probability of the overlapping.

## DEFINITION

The union of two events is a new event that contains all of the outcomes that are in at least one of the two events.

The probability of the union of events  $E_1$  and  $E_2$ , denoted by  $P(E_1 \cup E_2)$ , is given by:

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2).$$

## EXAMPLE 13

William rolled a fair die and wished to find the probability of the number that turns up is even or a number greater than 3.

**Solution**

Sample space:  $S = \{1, 2, 3, 4, 5, 6\}$

Let the event that contains even numbers be  $E_1$ . Then,  $E_1 = \{2, 4, 6\}$  and  $P(E_1) = \frac{3}{6} = \frac{1}{2}$ .

Let the event that contains numbers greater than 3 be  $E_2$ . Then,  $E_2 = \{4, 5, 6\}$  and

$$P(E_2) = \frac{3}{6} = \frac{1}{2}.$$

$$E_1 \cap E_2 = \{4, 6\} \text{ and } P(E_1 \cap E_2) = \frac{2}{6} = \frac{1}{3}$$

$$\begin{aligned} P(E_1 \cup E_2) &= P(E_1) + P(E_2) - P(E_1 \cap E_2) \\ &= \frac{3}{6} + \frac{3}{6} - \frac{2}{6} = \frac{4}{6} = \frac{2}{3}. \end{aligned}$$

### Addition Rule of Probability

We can easily calculate probabilities of compound events by making use of the addition rule stated below.

If two events,  $E_1$  and  $E_2$ , are non-mutually exclusive, there is some overlapping between these events. The probability that  $E_1$  or  $E_2$  will happen is the sum of the probability of each event, minus the probability of the overlapping.

#### Addition Rule 1

If  $E_1$  and  $E_2$  are any two events, then  $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$ .

The subtraction of  $P(E_1 \cap E_2)$  is necessary because  $E_1$  and  $E_2$  may **overlap**.

#### EXAMPLE 14

What is the probability that when two die are rolled the sum will be less than 5 or the faces will show the same number?

##### Solution

Let event  $E_1$  be that the sum is less than 5. Thus,

$$E_1 = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)\} \text{ and } P(E_1) = \frac{6}{36} = \frac{1}{6}.$$

Let event  $E_2$  be that the sum is less than 5. Thus,

$$E_2 = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\} \text{ and } P(E_2) = \frac{6}{36} = \frac{1}{6}.$$

$$E_1 \cap E_2 = \{(1, 1), (2, 2)\} \text{ and } P(E_1 \cap E_2) = \frac{2}{36}.$$

Therefore, the probability of a sum less than 5 or probability of both faces the same is:

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = \frac{6}{36} + \frac{6}{36} - \frac{2}{36} = \frac{10}{36} = \frac{5}{18}.$$

Addition rule 1 of probability can be extended to three or more events. For three non-mutually exclusive events  $E_1$ ,  $E_2$ , and  $E_3$ ,

$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2 \cap E_3).$$

When two events  $E_1$  and  $E_2$  are mutually exclusive, it is impossible for them to happen together.

The probability of  $E_1$  and  $E_2$  together equals to 0 (impossible). That is,  $P(E_1 \text{ and } E_2) = 0$ .

### Addition Rule 2

If  $E_1$  and  $E_2$  are mutually exclusive, the probability of  $E_1$  or  $E_2$ , is the sum of the individual probabilities: That is,

$$P(E_1 \cup E_2) = P(E_1) + P(E_2).$$

### EXAMPLE 15

Benjamin rolled a fair die and wished to find the probability of the number that turns up is odd or even.

#### Solution

Sample space:  $S = \{1, 2, 3, 4, 5, 6\}$

Let the event that contains even numbers be  $E_1$ . Then,  $E_1 = \{2, 4, 6\}$  and  $P(E_1) = \frac{3}{6} = \frac{1}{2}$ .

Let the event that contains odd numbers be  $E_2$ . Then,  $E_2 = \{1, 3, 5\}$  and  $P(E_2) = \frac{3}{6} = \frac{1}{2}$ .

$$E_1 \cap E_2 = \emptyset \text{ and } P(\emptyset) = 0$$

$$\begin{aligned} P(E_1 \cup E_2) &= P(E_1) + P(E_2) - P(E_1 \cap E_2) \\ &= \frac{3}{6} + \frac{3}{6} - \frac{0}{6} = \frac{6}{6} = 1. \end{aligned}$$

### EXAMPLE 16

What is the probability of drawing a queen or a king from an ordinary deck of cards?

#### Solution

The two events are mutually exclusive.

$$P(\text{queen}) = \frac{4}{52} = \frac{1}{13} \text{ and } P(\text{king}) = \frac{4}{52} = \frac{1}{13}. \text{ Thus,}$$

$$\begin{aligned} P(\text{queen or king}) &= P(\text{queen}) + P(\text{king}) \\ &= \frac{1}{13} + \frac{1}{13} = \frac{2}{13}. \end{aligned}$$

**EXAMPLE 17**

A bag contains 7 white balls 11 orange balls and 12 red balls. If a ball is drawn, find the probability that it is a white or a red ball.

**Solution**

The three events are mutually exclusive.

$$P(\text{white ball}) = \frac{7}{30} \text{ and } P(\text{red ball}) = \frac{12}{30} = \frac{2}{5}. \text{ Thus,}$$

$$P(\text{white or red}) = P(\text{white}) + P(\text{red})$$

$$= \frac{7}{30} + \frac{12}{30} = \frac{19}{30}.$$

Addition rule 2 of probability can be extended to three or more events. For three mutually exclusive events  $E_1$ ,  $E_2$ , and  $E_3$

$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3).$$

**EXERCISES**

1. What is the probability of rolling a 3 or an even number after a single roll of a fair die?
2. A single card is drawn from a deck. Find the probability that it is a king or a club.
3. Oliver rolled a fair die and wished to find the probability of the number divisible by 5 turns up or an odd number turns up. Find the probability.
4. A box contains 3 glazed doughnuts, 4 jelly doughnuts, and 5 chocolate doughnuts. If a person selects a doughnut at random, find the probability that it is either a glazed doughnut or a chocolate doughnut.
5. What is the probability that a card drawn at random from a standard 52-card deck is either a face card or a spade?
6. A card is drawn randomly from a deck of ordinary playing cards. You win L\$10 if the card is a spade or an ace. What is the probability that you will win the game?
7. In a hospital unit there are 8 nurses and 5 physicians; 7 nurses and 3 physicians are females. If a staff person is selected, find the probability that the staff selected is a nurse or a male.
8. A bag contains 3 red balls, 6 green balls, 4 blue balls, 2 white balls and 5 yellow balls. One ball is selected at random. Find the probabilities of the following events.
  - (a) The ball is either red or green.
  - (b) The ball is not blue.
  - (c) The ball is either red or white or blue.

The intersection of two events  $E_1$  and  $E_2$ , denoted by  $E_1 \cap E_2$ , is the event consisting of all outcomes that are in both  $E_1$  and  $E_2$ . Sometimes  $E_1$  and  $E_2$  have no outcomes in common so that the intersection of  $A$  and  $B$  contains no outcomes. In this sub-topic, we will study the probability of events joined by the word “AND”, in other words, the *intersection* of events.

### ACTIVITY 5

1. What is intersection of two events?
2. Define independent events.
3. Let  $E_1 = \{1, 3, 5, 7, 9\}$  and  $E_2 = \{2, 3, 5, 7, 11\}$ . Then, find  $E_1 \cap E_2$ .
4. Suppose the die is fair. Find the probability that the number rolled is both odd and greater than two.

If two events  $E_1$  and  $E_2$  are associated with “AND”, then it means the intersection of elements which are common to both events. The intersection symbol “ $\cap$ ” is used to represent “AND” in probability.

Thus, the events  $E_1$  and  $E_2$  denotes  $E_1 \cap E_2$ .

### DEFINITION

The intersection of two events is a new event that contains all of the outcomes that are in both events. The probability of the intersection of events  $E_1$  and  $E_2$  is denoted by  $P(E_1 \cap E_2)$ .

### EXAMPLE 18

In the experiment of rolling a single die, find the intersection of “the number rolled is even” and “the number rolled is greater than two”.

#### Solution

Sample space:  $S = \{1, 2, 3, 4, 5, 6\}$

Let  $E_1$  be the number rolled is even. Then,  $E_1 = \{2, 4, 6\}$ .

Let  $E_2$  be the number rolled is greater than two. Then,  $E_2 = \{3, 4, 5, 6\}$ .

Thus,  $E_1 \cap E_2 = \{4, 6\}$ .

### EXAMPLE 19

Suppose the die is fair. Find the probability that the number rolled is both even and greater than two.

**Solution**

The sample space is  $S = \{1, 2, 3, 4, 5, 6\}$ . Since  $E_1 = \{2, 4, 6\}$  and  $E_2 = \{3, 4, 5, 6\}$ , we get  $E_1 \cap E_2 = \{4, 6\}$ . Since the die is fair, all outcomes are equally likely, so by counting we have

$$P(E_1 \cap E_2) = \frac{2}{6} = \frac{1}{3}.$$

The intersection of two events is the probability that the two events,  $E_1$  and  $E_2$ , will occur at the same time. If both events are mutually exclusive, then this probability will be 0 because both events cannot occur at the same time.

**Independent events** are events where the occurrence of one of the events does not affect the occurrence of the other event.

Consider an experiment of tossing a coin and throwing a die simultaneously. To determine the probability of getting a head on a coin and a “5” on a dice simultaneously, we have to combine the outcomes of the coin and the dice. We know that the outcome for a coin is either head or tail. Outcomes for a die can be either 1, 2, 3, 4, 5 or 6. By combining these outcomes, we will have the following sample space as follows:

$$(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}.$$

Here,  $(H, 1)$  represents head on the coin and 1 on the dice.

One can observe that the total number of outcomes is 12, but only one is the favorable, that is,  $(H, 5)$ , so the required probability  $= \frac{1}{12}$ .

**The “AND” rule:** If  $E_1$  and  $E_2$  are independent events, then

$$P(E_1 \text{ and } E_2) = P(E_1) \times P(E_2).$$

**EXAMPLE 20**

A fair coin is tossed and a fair die is rolled. Find the probability of obtaining a head and 6.

**Solution**

The two events are independent.

$$\begin{aligned} P(\text{Head and } 6) &= P(\text{Head}) \times P(6) \\ &= \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}. \end{aligned}$$

### *Multiplication rule of probability*

Multiplication rule of probability is based on the concepts of independence or dependence of events. To find the probability of two independent events that occur in sequence, one must find the probability of each event occurring separately and then multiply the answers. For example, if a coin is tossed twice, the probability of getting two heads is  $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ . This result can be verified by looking at the sample space  $\{HH, HT, TH, TT\}$ . Then,  $P(HH) = \frac{1}{4}$ .

If  $E_1$  and  $E_2$  are two independent events for a random experiment, then the probability of simultaneous occurrence of two independent events will be equal to the product of their probabilities.

If both events  $E_1$  and  $E_2$  are independent, then the probability of  $E_1 \cap E_2$  is given by:

$$P(E_1 \cap E_2) = P(E_1)P(E_2)$$

#### EXAMPLE 21

Suppose a card is selected at random from a pack of 52 playing cards. The card is replaced, and a second card is selected at random. The event  $H$  is the first card is a heart. The event  $D$  is the second card is a diamond. Find, the probability of getting:

- hearts
- diamond
- hearts and diamond.

#### **Solution**

There are 52 cards in the pack, of which 13 are hearts and 13 are diamonds. Thus,

(a) the probability of heart:  $P(H) = \frac{13}{52} = \frac{1}{4}$ .

(b) the probability of diamond:  $P(D) = \frac{13}{52} = \frac{1}{4}$ .

- (c) Since each card is replaced after being selected, the occurrence of each event is independent of the preceding event. Hence, the probability of heart and diamond:

$$P(H \text{ and } D) = P(H) \times P(D) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}.$$

Independent events are events for which the occurrence of either event does not affect the next.

**Note**

If  $E_1$ ,  $E_2$  and  $E_3$  are all independent events, then

$$P(E_1 \text{ and } E_2 \text{ and } E_3) = P(E_1) \times P(E_2) \times P(E_3).$$

Recall that dependent events are events for which the occurrence of either event affects the next.

If  $E_1$  and  $E_2$  are dependent events, then

$$P(E_1 \text{ then } E_2) = P(E_1) \times P(E_2 \text{ when } E_1 \text{ has occurred}).$$

**EXAMPLE 22**

A box contains 5 mathematics books and 7 physics books. Two books are randomly selected from the box without replacing them.

- Find the probability of both are mathematics books.
- Find the probability that the first book is a mathematics book and the second is a physics book.

**Solution**

- (a) Probability that the first book drawn is a mathematics book  $\frac{5}{12}$ . After drawing one mathematics book, we are left with 4 mathematics books and 7 physics books. Probability that the second book drawn is a mathematics book  $= \frac{4}{11}$ .

$$\text{Probability that the both books drawn are mathematics books } \frac{5}{12} \times \frac{4}{11} = \frac{5}{33}.$$

- (b) Probability that the second book drawn is a physics book  $= \frac{7}{11}$ .

$$\text{Probability that the first book drawn is a mathematics and the second is physics book is } \frac{5}{12} \times \frac{7}{11} = \frac{35}{132}.$$

**Multiplication rule of probability**

Multiplication rule of probability is based on the concepts of independence or dependence of events discussed earlier.

If  $E_1$  and  $E_2$  are any two events, the probability that both events occur, is denoted by  $P(E_1 \text{ and } E_2) = P(E_1 \cap E_2)$  is given by:

$$(i) \quad P(E_1 \cap E_2) = P(E_1) \times P(E_2/E_1), \text{ if } P(E_1) \neq 0.$$

$$(ii) \quad P(E_1 \cap E_2) = P(E_2) \times P(E_1/E_2), \text{ if } P(E_2) \neq 0.$$

**EXAMPLE 23**

A box contains six red marbles and four black marbles. Two marbles are drawn *without replacement* from the box. What is the probability that both marbles are black?

**Solution**

Let event  $E_1$  be the event that the first marble is black; and let event  $E_2$  be the event that the second marble is black.

In the beginning, there are 10 marbles in the box, 4 of which are black. Thus,  $P(E_1) = \frac{4}{10}$ . After the first selection, there are 9 marbles in the box, 3 of which are black. Thus,

$$P(E_2/E_1) = \frac{3}{9} = \frac{1}{3}.$$

Therefore, based on the rule of multiplication, we have

$$P(E_2 \cap E_1) = P(E_1) \times P(E_2/E_1) = \frac{4}{10} \times \frac{3}{9} = \frac{12}{90} = \frac{2}{15}.$$

**EXERCISES**

1. A coin is flipped and a die is rolled. Find the probability of getting a head on the coin and a 4 on the die.
2. A card is drawn from a deck and replaced; then a second card is drawn. Find the probability of getting a queen and then an ace.
3. A die is rolled two times. Find the probability that the outcome is 1 on the first and a prime number on the second roll
4. Henry rolls a red die and a green die. What is the probability that he gets 5 on the red die and an even number on the green die?
5. A box contains eight red and four yellow balls. Two balls are drawn from the box one after the other without replacement. What is the probability that both drawn balls are red?
6. A bag contains 3 red balls, 2 blue balls and 5 white balls. A ball is selected and its color noted. Then, it is replaced. A second ball is selected and its color noted. Find the probability of each of the following events.
  - (a) Selecting 2 blue balls
  - (b) Selecting 1 blue ball and then 1 white ball
  - (c) Selecting 1 red ball and then 1 blue ball

7. From a box having 10 white and 8 green balls two balls are drawn at random one by one with replacement. Find the probability that:
  - (a) both are green.
  - (b) both are white.
  - (c) The first is green and the second is white.
8. A person owns a collection of 30 CDs, of which 5 are country music. If 2 CDs are selected at random, find the probability that both are country music.

## 8.6 REVIEW OF THE BASIC CONCEPTS OF SET, VENN, TREE DIAGRAM AND CONTINGENCY TABLES

In mathematics, a collection of distinct well-defined objects is called a set. The objects in a set are called the *elements* or *members* of the set. If  $a$  is an element of set  $A$ , we write  $a \in A$ . If  $a$  is not an element of set  $A$ , we write  $a \notin A$ . A set that contains no elements is called a null set or an empty set. If every element in Set  $A$  is also in Set  $B$ , then Set  $A$  is a *subset* of Set  $B$ .

To specify the fact that a set  $A$  contains four elements 0, 1, 2, 3, we write  $A = \{0, 1, 2, 3\}$ . We have learned about Venn diagrams in the topic of sets. In this sub-topic, we will review the basic concept of set and we will use a Venn diagram and tree diagram to solve the problems based on probability.

### ACTIVITY 6

1. What types of events are mutually exclusive?
2. What is a set?
3. Discuss about Venn diagram and try to explain the use of Venn diagram.
4. Let  $A = \{0, 1, 2, 3, 4, 5, 7, 9\}$  and  $B = \{1, 3, 4, 5, 6, 8, 9\}$ . Then, find  $A \cup B$  and  $A \cap B$ .
5. Consider the Venn diagram below and answer the questions that follow. The Venn diagram below shows the favorite subject between Mathematics and Physics of 60 students in a certain High school.

- (a) How many students like Mathematics?
- (b) How many students like Physics?
- (c) How many students like Mathematics and Physics?
- (d) How many students like Mathematics or Physics?

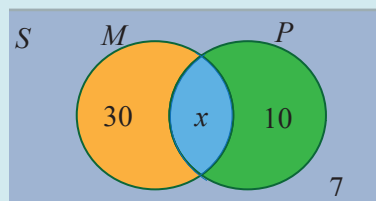


Figure 1.

## Venn diagrams

From the knowledge of sets, we can define the union and intersection of events. The union of two events  $E_1$  and  $E_2$  is denoted by  $E_1 \cup E_2$ .

$E_1 \cup E_2$  includes all outcomes that are in  $E_1$  or  $E_2$ . The intersection of two events  $E_1$  and  $E_2$  is denoted by  $E_1 \cap E_2$ .

$E_1 \cap E_2$  includes all outcomes that are in both  $E_1$  and  $E_2$ .

A useful graphical tool for studying the complement, union and intersection of events within a sample space  $S$  is known as a Venn diagram. In probability, a Venn diagram is a figure with one or more circles inside a rectangle that describes logical relations between events. The rectangle in a Venn diagram represents the sample space or the universal set, that is, the set of all possible outcomes. A circle inside the rectangle represents an event, that is, a subset of the sample space.

### EXAMPLE 24

A class contains 100 students. 70 of them like Mathematics, 60 like Physics, and 40 like both. If a student is chosen at random, using a Venn diagram, find the probability that a student

- |  |  |
|--|--|
| (a) likes only Physics                 | (d) likes Physics but not Mathematics.     |
| (b) likes both Mathematics and Physics | (e) likes Mathematics or Physics           |
| (c) does not like Mathematics          | (f) likes neither Mathematics nor Physics. |

#### Solution

We begin by drawing an empty Venn diagram to represent this problem as shown in figure 2.

We know that the overlapping region in a Venn diagram represents the intersection of the events. We are given that 40 students belong to this intersection, since they like both

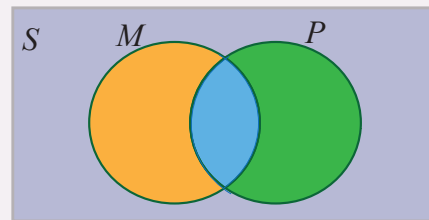


Figure 2.

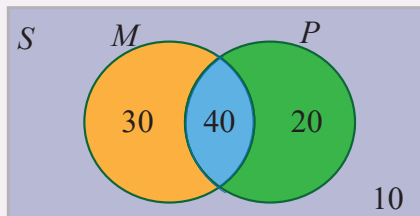


Figure 3.

Mathematics and Physics. 70 students like Mathematics, and 40 of them also like Physics. This tells us that the number of students who like Mathematics only is  $70 - 40 = 30$ . Similarly, we can deduce that  $60 - 40 = 20$  students like Physics only. This leads to the Venn diagram shown in figure 3.



In Venn diagram, the sets are shown in the Venn diagram shown in figure 4.

Number of employees who use both laptop and a desktop =  $57 + 42 - 87 = 12$ .

Number of employees who use only a laptop is  $n(L - D) = 42 - 12 = 30$ .

Number of employees who use only desktop is  $n(D - L) = 57 - 12 = 45$ .

Number of employees who neither use a laptop nor a desktop is

$$n(E) - n(L \cup M) = 150 - 87 = 63.$$

(a) Probability that the employee uses only a laptop =  $\frac{30}{150} = \frac{1}{5}$ .

(b) Probability that the employee uses only a desktop =  $\frac{45}{150} = \frac{3}{10}$ .

(c) Probability that the employee uses neither a laptop nor a desktop =  $\frac{63}{150} = \frac{21}{50}$ .

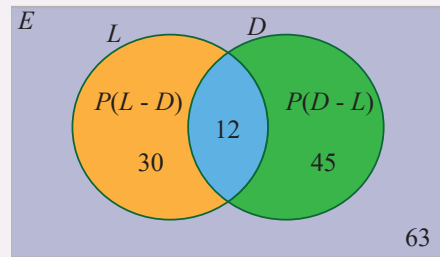


Figure 4.

## Tree diagrams

A device that can be helpful in identifying all possible outcomes of a random experiment, particularly one that can be viewed as proceeding in stages, is what is called a **tree diagram**. A tree diagram is used to count all the possible outcomes of a sequence of experiments or simple events where each event can occur in a finite number of ways.

Tree diagram in Figure 5 illustrates the possible outcomes when drawing two balls from a bag containing several balls of each of the colors white, blue and orange.

Let  $W$  be the event of getting a white color ball

$B$  be the event of getting a blue color ball

$O$  be the event of getting an orange color ball.

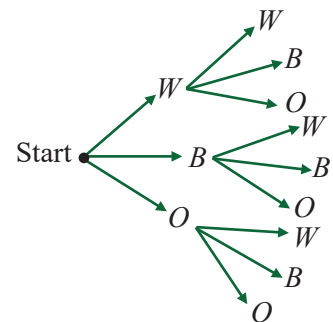


Figure 5.

### EXAMPLE 26

Draw a tree diagram showing the possible outcomes ( $H$  for head,  $T$  for tail) from flipping a coin three times.

(a) Use the tree to list the outcomes.

(b) Find the probability that exactly two outcomes will be heads.

(c) Find the probability that at least two outcomes will be heads.

(d) Find the probability that at least one outcome will be tails.

### Solution

The tree diagram showing the possible outcomes from flipping a coin three times can be drawn as shown in figure 6.

(a) From the tree diagram, we can see that the outcomes is:

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

(b) There are only three outcomes having exactly two heads. These are HHT, THT and THH. Thus,

$$P(RRB) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}. \text{ Therefore,}$$

$$P(\text{exactly 2 heads}) = 3 \times \frac{1}{8} = \frac{3}{8}.$$

(c) At least two means two or three. Thus,

$$P(HHH) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}. \text{ Thus,}$$

$$P(\text{at least 2 heads}) = P(2 \text{ heads}) + P(3 \text{ heads}) = \frac{3}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}.$$

(d) At least one means one, two or three. Thus,

$$P(\text{at least one tail}) = 1 - P(HHH) = 1 - \frac{1}{8} = \frac{7}{8}.$$

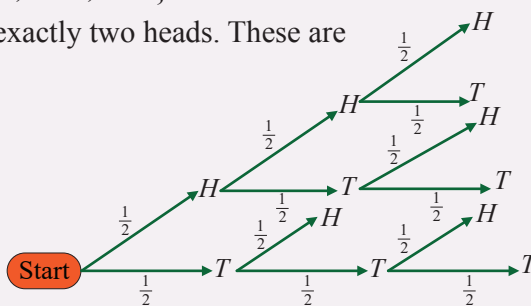


Figure 6.

### EXAMPLE 27

A drawer contains 4 red and three blue socks. Draw three socks randomly and replace each one before drawing another (draw with replacement).

(a) Find the probability of drawing exactly two red socks.

(b) Find the probability of drawing at least two red socks.

### Solution

First, let us draw a tree diagram as shown in figure 7.

(a) There are only three outcomes having exactly two red socks. These are

$RRB$ ,  $RBR$  and  $BRR$ . Thus,

$$P(RRB) = \frac{4}{7} \times \frac{4}{7} \times \frac{3}{7} = \frac{48}{343}. \text{ Therefore, } P(\text{exactly 2 red}) = 3 \times \frac{48}{343} = \frac{144}{343}.$$

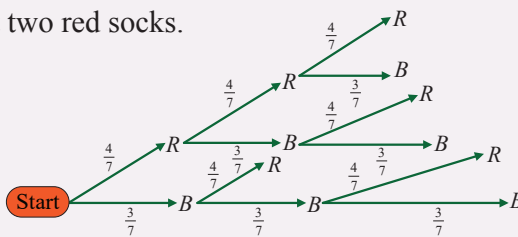


Figure 7.

(b) At least two means two or three.

$$P(RRR) = \frac{4}{7} \times \frac{4}{7} \times \frac{4}{7} = \frac{64}{343}. \text{ Thus,}$$

$$\begin{aligned} P(\text{at least 2 red}) &= P(2 \text{ red}) + P(3 \text{ red}) \\ &= \frac{144}{343} + \frac{64}{343} = \frac{208}{343}. \end{aligned}$$

### EXERCISES

- In a class of 65 students, 24 play cricket, 43 play chess, and 5 play neither of these sports. A student is randomly chosen from the class. Determine the probability that the student:
  - plays chess.
  - does not play chess.
  - plays at least one of the games.
- In a class of 38 students, 20 like burger, 25 like pastry, and 10 like both of these items. Represent this information on a Venn diagram, and hence determine the probability that a randomly selected student likes:
  - both burger and pastry.
  - at least one of these items.
  - burger, but not pastry.
- In a class, there are 6 girls and 7 boys. Two students are selected at random.
  - Represent the information on a tree diagram.
  - Find the probability that both the students are girls.
- Peter has a probability of  $\frac{3}{4}$  for passing Biology and probability of  $\frac{4}{5}$  for passing Chemistry.
  - Display this information on a tree diagram.
  - What is the probability that Peter passed in one of the subjects?
- A drawer contains 4 red and 3 blue socks. If we draw three socks randomly without replacement.
  - Find the probability of drawing exactly two red socks.
  - Find the probability of drawing at least two red socks.

Rolling an ordinary six-sided die is a familiar example of a *random experiment*, an action for which all possible outcomes can be listed, but for which the actual outcome on any given trial of the experiment cannot be predicted with certainty. In such a situation we wish to assign to each outcome, such as rolling a two, a number, called the *probability* of the outcome, that indicates how likely it is that the outcome will occur.

Similarly, we would like to assign a probability to any *event*, or collection of outcomes, such as rolling an even number, which indicates how likely it is that the event will occur if the experiment is performed. In this sub-topic, we will discuss about sample space and event of an experiment.

### ACTIVITY 7

1. Discuss the term sample space
2. Construct a sample space for the experiment that consists of rolling a single die. Find the events that correspond to the phrases “an even number is rolled” and “a number greater than two is rolled.”
3. Write the sample space for the following cases:
  - (a) When two coins are tossed.
  - (b) When three coins are tossed.
  - (c) When two dice are tossed.

One of the most basic concepts in probability (and statistics) is that of a *random experiment*. A random experiment is an activity that is carried out repeated numerous times under a certain set of conditions.

An activity which can produce some well-defined outcomes is called an *experiment*. An experiment in which all possible outcomes are known, and the exact outcome cannot be predicted in advance is called a *random experiment*.

### DEFINITION

A random experiment is a mechanism that produces a definite outcome that cannot be predicted with certainty. The sample space associated with a random experiment is the set of all possible outcomes.

### EXAMPLE 28

Tossing a coin, throwing a dice, picking a card from a pack, etc., are some common random experiments.

In any random experiment, there is always uncertainty if whether a particular event will occur or will not occur. For example, when we toss a dice, the result of the experiment will be a number from the one of the numbers in the set  $\{1, 2, 3, 4, 5, 6\}$ . A random experiment can be repeated any number of times under the same set of conditions: and upon the completion of the activity, certain results are observed.

### DEFINITION

- (a) A collection or a set of all possible outcomes of a random experiment is called a **sample space** (or a **possibility set**). The sample space is represented using the symbol " $S$ "
- (b) The subset of possible outcomes of an experiment is called **events**, and it is denoted by  $E$ .

An event  $E$  occurs (or happens) if the outcome of the random experiment (that is, the sample point  $s$ ) belongs to  $E$ ,  $s \in E$ .  $E$  does not occur (or does not happen) if  $s \notin E$ . The event  $S$  always occurs and is called the **sure** or **certain event**.

### EXAMPLE 29

The following are some examples of random experiments and their sample spaces.

- (a) Tossing a coin. Then, with  $H$  and  $T$  standing for "heads" and "tails," respectively, a sample space is:  $S = \{H, T\}$ .
- (b) Rolling a die. Then, a sample space is:  $S = \{1, 2, 3, 4, 5, 6\}$ .

### DEFINITION

Outcomes of a random experiment are said to be **equally likely** when there is no reason to expect any one of the outcomes in preference to another. That is, each element has equal chance of being chosen.

### EXAMPLE 30

If a fair die is thrown, any one of the outcomes 1, 2, 3, 4, 5, 6 has an equal chance of appearing at the top. Therefore, they are considered as equally likely.

### EXAMPLE 31

A fair die is thrown. How many favorable outcomes are there for getting a prime number?

#### Solution

Since there are three prime numbers, there are 3 favorable outcomes. These are 2, 3 and 5.

**EXAMPLE 32**

Find the sample space for rolling two dice.

**Solution**

Since each die can land in six different ways, and two dice are rolled, the sample space can be presented by a rectangular array, as shown in the table below

Die 1	Die 2					
	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

The sample space is the list of pairs of numbers in the chart. Thus, the sample space is:

{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}.

**EXAMPLE 33**

Find the sample space for the gender of the children if a family has three children. Use *B* for boy and *G* for girl.

**Solution**

There are two genders, male and female, and each child could be either gender. We can use tree diagram to identify the required sample space.

As shown in the tree diagram, there are eight possibilities.

That is,  $2 \times 2 \times 2 = 8$ .

Therefore, the sample space (possibility set) is:

{*BBB*, *BBG*, *BGB*, *BGG*, *GBB*, *GBG*, *GGB*, *GGG*}

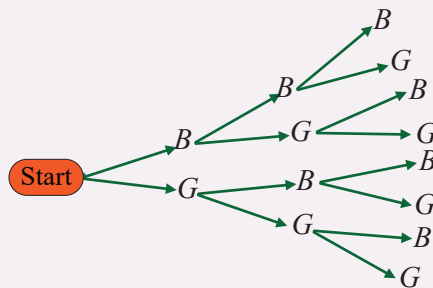


Figure 8

**EXERCISES**

1. A fair coin is tossed, and a fair die is thrown. Write down sample spaces for:
  - (a) the toss of the coin
  - (b) the throw of the die
  - (c) the combination of these experiments.
2. If an experiment consists of tossing a coin three times and recording the results in order. Then, find
  - (a) the sample space for this experiment
  - (b) the event  $E_1$  that consists of all outcomes with two heads.
  - (c) the event  $E_2$  that consists of all outcomes showing at least two heads.
  - (d) the event  $E_3$  that consists of all outcomes showing no heads.
3. A die is rolled twice. Determine the event that consists of all outcomes whose sum is:
 

(a) equal to 1	(c) less than 7
(b) equal to 4	(d) greater than or equal to 9.
4. A die is rolled twice. Determine the event that:
 

(a) consists of all outcomes whose sum is equal to 6.	(c) two even numbers
(b) two identical numbers.	(d) two odd numbers.
5. List all the possible outcomes of flipping a coin and rolling a die. How many possible outcomes is obtained from this experiment?

The *probability of an event* occurring is intuitively understood to be the likelihood or chance of it occurring. In the very simplest cases, the probability of a particular event  $A$  occurring from an experiment is obtained from the number of ways that  $A$  can occur divided by the total number of possible outcomes.

**ACTIVITY 8**

1. What does it mean by “odds in probability of a particular event”?
2. A die is rolled, find the probability that an even number is obtained.
3. Two dice are rolled, find the probability that the sum is;
 

(a) equal to 2	(b) equal to 8	(c) less than 9	(d) greater than 7
----------------	----------------	-----------------	--------------------

In general, the probability of an event can be defined as follows:

### DEFINITION

In a random experiment, the outcomes which insure the happening of a particular result are said to be *favorable outcomes* to that particular result.

Let  $S$  be the sample space of an experiment in which all outcomes are equally likely, and let  $E$  be an event. The probability of  $E$ , written  $P(E)$ , is

$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{number of elements in } E}{\text{number of elements in } S}$$

### Note

$0 \leq n(E) \leq n(S)$ , so the probability  $P(E)$  of an event is a number between 0 and 1. That is,

$$0 \leq P(E) \leq 1.$$

## The probability of an event

There are three different approaches to measure probabilities. These are *theoretical* probability, the *experimental* probability and the *axiomatic* probability.

### I. Theoretical probability

Theoretical probability is based on the possible chances of something to happen. The theoretical probability is mainly based on the reasoning behind probability. For example, if a coin is tossed, the theoretical probability of getting a head will be  $\frac{1}{2}$ .

In the theoretical probability, each outcome is assumed to have an equal probability of occurrence. In other words, the classical method of assigning probabilities assumes that each outcome is equally likely to occur.

### EXAMPLE 34

In rolling a fair die, it is equally likely to get a 1, 2, 3, 4, 5, or 6.

If an experiment has  $n$  possible outcomes and if  $n$  of these outcomes is favorable to an event  $E$ , then the probability of occurrence of an event  $E$  (called its *success*) is denoted and given by the formula:

$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{Number of outcomes favouring } E}{\text{Total number of possible outcomes}}$$

**EXAMPLE 35**

A fair die is rolled.

- What is the probability that an even number appears?
- What is the probability that a composite number appears?
- What is the probability that a number less than or equal to 4 appears?
- What is the probability that a number 7 appears?

**Solution**

- (a) Let an even number shows up be event  $E$ . Then,  $E = \{2, 4, 6\}$ .

$$\text{Thus, } P(E) = \frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2}.$$

- (b) Let a composite number shows up be event  $E$ . Then,  $E = \{4, 6\}$ .

$$\text{Thus, } P(E) = \frac{n(E)}{n(S)} = \frac{2}{6} = \frac{1}{3}.$$

- (c) Let a number less than or equal to 4 shows up be event  $E$ . Then,  $E = \{1, 2, 3, 4\}$ .

$$\text{Thus, } P(E) = \frac{n(E)}{n(S)} = \frac{4}{6} = \frac{2}{3}.$$

- (d) Let number 7 shows up be event  $E$ . Then,  $E = \{ \}$ .

$$\text{Thus, } P(E) = \frac{n(E)}{n(S)} = \frac{0}{6} = 0.$$

**Note**

- An event which occurs at each performance of an experiment is called a **certain event**. Thus,  $P(\text{certain event}) = 1$ .
- An event which can never occur at each performance of an experiment is called an **impossible event**. Thus,  $P(\text{impossible event}) = 0$ .

**II. Experimental Probability**

Experimental probability is based on the basis of the observation of an experiment. A random experiment is done and is repeated many times to determine their likelihood and each repetition is known as a **trial**.

The experimental probability can be calculated based on the number of possible outcomes by the total number of trials. For example, if a coin is tossed 50 times and record whether a “head” or a “tail” is obtained, then the experimental probability of

obtaining a “head” is calculated as a fraction of the number of heads obtained and the total number of tosses. That is,

$$P(\text{head}) = \frac{\text{Number of heads recorded}}{50 \text{ tosses}}.$$

### EXAMPLE 36

The number of pancakes prepared by Amelia per day this week is in order of 5, 7, 9, 8, 5, 6, 5. Relying on this data, what is the probability that Amelia makes less than 7 pancakes the next day?

#### Solution

The number of pancakes prepared per day less than 7 are 5, 5, 6, 5 which are 4 possibilities

$$\text{Mathematically, we get: } \frac{4}{7} = 0.57 = 57\%.$$

Therefore, the percentage probability of Amelia preparing less than 7 pancakes the next day is 57%.

### III. Axiomatic Probability

**Axiomatic probability** is just another way of describing the probability of an event. As, the word itself says, in this approach, some *axioms* are predefined before assigning probabilities. In axiomatic probability, a set of rules or axioms are set which applies to all types.

Let  $S$  be the sample space of a random experiment. With each event  $E$ , we associate a real number called the probability of  $E$ , denoted by  $P(E)$ , that satisfies the following properties called **axioms** (or **postulates**) of probability.

#### Note

- (i) The probability of an event is always between 0 and 1. That is,  $0 \leq P(E) \leq 1$ .
- (ii) If  $E = \emptyset$  (the impossible event), then  $P(\emptyset) = 0$  and if  $E = S$  (the certain event), then  $P(S) = 1$ .
- (iii) If  $E \cup E' = S$ , then  $P(E_1 \cup E_2) = P(S) = 1$  and  $P(E') = 1 - P(E)$ ,  $E'$  (not  $E$ ) =  $S/E$

### EXAMPLE 37

In a pack of 52 cards, what is the probability of getting a queen or an ace?

#### Solution

These are mutually exclusive events because it is impossible for a card to be a queen and an ace at the same time.

There are 4 queens in the pack. So,  $P(\text{Queen}) = \frac{4}{52} = \frac{1}{13}$ .

There are 4 aces in the pack. So,  $P(\text{Ace}) = \frac{4}{52} = \frac{1}{13}$ .

Therefore, the probability of a Queen or an Ace is:

$$\begin{aligned} P(\text{Queen or Ace}) &= P(\text{Queen}) \times P(\text{Ace}) \\ &= P(\text{Queen}) + P(\text{Ace}) = \frac{1}{13} + \frac{1}{13} = \frac{2}{13}. \end{aligned}$$

Another important concept in probability theory is that of complementary events.

When a die is rolled, for instance, the sample space consists of the outcomes 1, 2, 3, 4, 5, and 6. The event  $E$  of getting odd numbers consists of the outcomes 1, 3, and 5. The event of not getting an odd number is called the complement of event  $E$ , and it consists of the outcomes 2, 4, and 6.

### DEFINITION

The **complement** of an event  $E$  is the set of outcomes in the sample space that are not included in the outcomes of event  $E$ . The complement of  $E$  is denoted by  $E^1$ .

### EXAMPLE 38

Find the complement of each event.

- Rolling a die and getting a 4
- Selecting a letter of the alphabet and getting a vowel
- Selecting a month and getting a month that begins with a J
- Selecting a day of the week and getting a weekday

#### Solution

- Getting a 1, 2, 3, 5, or 6.
- Getting a consonant.
- Getting February, March, April, May, August, September, October, November, or December.
- Getting Saturday or Sunday.

### Note

$$P(E) = 1 - P(E^1) \text{ or } P(E^1) = 1 - P(E) \text{ or } P(E) + P(E^1) = 1.$$

## Odds in favor and odds against an event

Odds in probability of a particular event means the ratio between the number of favorable outcomes to the number of unfavorable outcomes. To find “*odds*”, we must first know how to determine the probability of success and the probability of failure.

*Odds* are divided into two types. These are odds *in favor* and odds *against*.

Odds *in favor* of a particular event  $E$  are the ratio of the probability that the event  $E$  will occur (success) to the probability that event  $E$  will not occur (failure). If we know the probability of an event  $E$ , we can determine the odds *in favor* of event by the formula:

$$\text{Odds in favor of } E = \frac{P(\text{Success})}{P(\text{Failure})} = \frac{P(E)}{P(\text{not } E)}.$$

### EXAMPLE 39

Find the odds in favor of rolling a die to get a 3.

#### Solution

Let  $E$  represent the event of rolling a 3. Find the probability of  $E$  occurring and the probability of  $E$  not occurring.

$$S = \{1, 2, 3, 4, 5, 6\} \text{ and } E = \{3\}. \text{ Thus, } P(E) = \frac{1}{6} \text{ and } P(\text{not } E) = \frac{5}{6}.$$

$$\text{Odds in favor of } E = \frac{P(E)}{P(\text{not } E)} = \frac{\frac{1}{6}}{\frac{5}{6}} = \frac{1}{6} \times \frac{6}{5} = \frac{1}{5}.$$

Therefore, odds in favor of throwing a die to get a 3 is 1:5 or  $\frac{1}{5}$ .

The odds *against* an event  $E$  are the ratio of the probability that the event  $E$  will fail to occur (failure) to the probability that the event  $E$  will occur (success). If we know the probability of an event  $E$ , we can determine the odds *against* event by the formula:

$$\text{Odds against } E = \frac{P(\text{Failure})}{P(\text{Success})} = \frac{P(\text{not } E)}{P(E)}$$

**EXAMPLE 40**

Find the odds against rolling a die to get a 2.

**Solution**

Let  $E$  represent the event of rolling a 2. Find the probability of  $E$  occurring and the probability of  $E$  not occurring.

$$S = \{1, 2, 3, 4, 5, 6\} \text{ and } E = \{2\}. \text{ Thus, } P(\text{not } E) = \frac{5}{6} \text{ and } P(E) = \frac{1}{6}.$$

$$\text{Odds in favor of } E = \frac{P(\text{not } E)}{P(E)} = \frac{\frac{5}{6}}{\frac{1}{6}} = \frac{5}{6} \times \frac{6}{1} = \frac{5}{1}.$$

Therefore, odds in favor of throwing a die to get a 2 is 5:1 or  $\frac{5}{1}$ .

**Note**

If  $m$  and  $n$  are probabilities of the occurrence and nonoccurrence of an event respectively, then the ratio  $m:n$  is called the odds *in favor* of the event.

The ratio  $n:m$  is called the odds *against* the event.

**Odds to probability**

If the odds in favor of event  $E$  are  $m$  to  $n$ , then the probability of the event is given by:

$$P(E) = \frac{a}{a+b}.$$

**EXAMPLE 41**

The odds in favor of an event are 5:7. Find the probability of its nonoccurrence.

**Solution**

Let  $E$  be the event. Then, we are given that number ( $E$ ) = 5 and number (not  $E$ ) = 7. Then,  $n(S) = n(E) + n(\text{not } E) = 5 + 7 = 12$ .

$$\text{Therefore, } P(E) = \frac{5}{12}.$$

**EXERCISES**

1. A die is rolled and a coin is tossed, find the probability that the die shows an odd number and the coin shows a head.

2. A card is drawn at random from a deck of cards. Find the probability of getting the 3 of diamond.
3. A jar contains 3 red marbles, 7 green marbles and 10 white marbles. If a marble is drawn from the jar at random, what is the probability that this marble is white?
4. The blood groups of 200 people is distributed as follows: 50 have type **A** blood, 65 have **B** blood type, 70 have **O** blood type and 15 have type **AB** blood. If a person from this group is selected at random, what is the probability that this person has **O** blood type?
5. Tickets numbered 1 to 20 are mixed up and then a ticket is drawn at random. What is the probability that the ticket drawn has a number which is a multiple of 3 or 5?

## KEY TERMS

- Complement of event
- Compound events
- Dependent event
- Experiment
- Favorable outcome
- Independent event
- Intersection of events
- Mutually exclusive events
- Odds against event
- Odds in favor event
- Outcome event
- Probability
- Possibility set
- Relative frequency
- Random experiment
- Simple event
- Sample space
- Tree diagram
- Union of events
- Venn diagram

## SUMMARY

- Relative frequency of an event can be defined as the number of times an event occurs divided by the total number of events occurring in a given trials.

$$\text{Probability of the event } E = \frac{\text{Number of times } E \text{ occurred}}{\text{Number of times the experiment was repeated}}$$

$$P(E) = \frac{r}{n}$$

- A compound event combines two or more events, using the word “**AND**” or the word “**OR**”.

- Those events which cannot happen simultaneously are called **mutually exclusive events**.
- If there is at least one outcome common in events, they are called **non-mutually exclusive events**.
- Independent events are events where the occurrence of one of the events does not affect the occurrence of the other event.
- **The “OR” rule:** For mutually exclusive events  $E_1$  and  $E_2$ ,
 
$$P(E_1 \text{ or } E_2) = P(E_1) + P(E_2)$$
- **The “AND” rule:** If  $E_1$  and  $E_2$  are independent events, then
 
$$P(E_1 \text{ and } E_2) = P(E_1) \times P(E_2).$$
- The union of two events is a new event that contains all of the outcomes that are in at least one of the two events. The probability of the union of events  $E_1$  and  $E_2$ , denoted by  $P(E_1 \cup E_2)$ , is given by:
 
$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2).$$
- **Addition Rule 1:** If  $E_1$  and  $E_2$  are any two events, then,
 
$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2).$$
- **Addition Rule 2:** If  $E_1$  and  $E_2$  are mutually exclusive, the probability of  $E_1$  or  $E_2$ , is the sum of the individual probabilities: That is,
 
$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$
- The intersection of two events is a new event that contains all of the outcomes that are in both events. The probability of the intersection of events  $E_1$  and  $E_2$  denoted by  $P(E_1 \cap E_2)$ .
- If  $E_1$  and  $E_2$  are independent events, then
 
$$P(E_1 \text{ and } E_2) = P(E_1) \times P(E_2).$$
- If both events are independent, then the probability of  $E_1 \cap E_2$  is given by:
 
$$P(E_1 \cap E_2) = P(E_1) P(E_2)$$
- A useful graphical tool for studying the complement, union and intersection of events within a sample space  $S$  is known as a Venn diagram.
- A device that can be helpful in identifying all possible outcomes of a random experiment, particularly one that can be viewed as proceeding in stages, is what is called a **tree diagram**.
- A random experiment is a mechanism that produces a definite outcome that cannot be predicted with certainty. The sample space associated with a random experiment is the set of all possible outcomes.

- A collection or a set of all possible outcomes of a random experiment is called a **sample space** (or a **possibility set**). The sample space is represented using the symbol " $S$ "
- The subset of possible outcomes of an experiment is called **events**, and it is denoted by  $E$ .
- Outcomes of a random experiment are said to be **equally likely** when there is no reason to expect any one of the outcomes in preference to another. That is, each element has equal chance of being chosen.
- In a random experiment, the outcomes which insure the happening of a particular result are said to be **favorable outcomes** to that particular result.
- Let  $S$  be the sample space of an experiment in which all outcomes are equally likely, and let  $E$  be an event. The probability of  $E$ , written  $P(E)$ , is

$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{number of elements in } E}{\text{number of elements in } S}$$

- (i) An event which occurs at each performance of an experiment is called a **certain event**. Thus,  $P(\text{certain event}) = 1$ .
- (ii) An event which can never occur at each performance of an experiment is called an **impossible event**. Thus,  $P(\text{impossible event}) = 0$ .
- (iii) The probability of an event is always between 0 and 1. That is,  $0 \leq P(E) \leq 1$ .
- (iv) If  $E = \emptyset$  (the impossible event), then  $P(\emptyset) = 0$  and if  $E = S$  (the certain event), then  $P(S) = 1$ .
- (v) If  $E \cup E' = S$ , then  $P(E_1 \cup E_2) = P(S) = 1$  and  $P(E') = 1 - P(E)$ ,  $E'$  (not  $E$ ) =  $S/E$
- The **complement** of an event  $E$  is the set of outcomes in the sample space that are not included in the outcomes of event  $E$ . The complement of  $E$  is denoted by  $E'$ .  
 $P(E) = 1 - P(E')$  or  $P(E') = 1 - P(E)$  or  $P(E) + P(E') = 1$ .
- If  $m$  and  $n$  are probabilities of the occurrence and nonoccurrence of an event respectively, then the ratio  $m : n$  is called the odds **in favor** of the event.  
 The ratio  $n : m$  is called the odds **against** the event.

## EXERCISES

- When a fair dice is thrown, what is the probability of getting:
  - the number 5
  - a number that is a multiple of 3
  - a number that is greater than 6
  - a number that is less than 7.

2. There are 4 blue marbles, 5 red marbles, 1 green marble and 2 black marbles in a bag. Suppose we you select one marble at random. Find the probability of:
- (a)  $P(\text{black})$  (d)  $P(\text{not green})$   
 (b)  $P(\text{blue})$  (e)  $P(\text{not purple})$   
 (c)  $P(\text{blue or black})$
3. A bag contains 6 red marbles and 4 black marbles are drawn without replacement from the bag. What is the probability that both of the marbles are black?
4. Determine which events are mutually exclusive and which are are not when a single card is drawn from a deck.
- (a) Getting a 7 and getting a jack.  
 (b) getting a club and getting a king.  
 (c) getting a face card and getting an ace.  
 (d) Getting a face card and getting a spade.
5. A sales representative who visits customers at supermarket finds she sells 0, 1, 2, 3 or 4 items according to the following frequency distribution.

Items sold	0	1	2	3	4
Frequency	8	10	3	2	1

Find the probability that she sells the following.

- (a) Exactly 1 item (c) At least 1 item  
 (b) More than 2 items (d) At most 3 items
6. The following frequency distribution table represents the length of time a patient spends in a hospital.

Days	0-3	4-7	8-11	12-15	16 and more
Frequency	2	15	8	6	9

If a patient is selected, find the probability of:

- (a) The patient spends 3 days or fewer in the hospital.  
 (b) The patient spends fewer than 8 days in the hospital.  
 (c) The patient spends 16 or more days in the hospital.  
 (d) The patient spends a maximum of 11 days in the hospital.
7. A recent study of 300 patients found that of 100 alcoholic patients, 87 had elevated cholesterol levels and of 200 nonalcoholic patients, 43 had elevated

- noncholesterol levels. If a patient is selected at random, find the probability that the patient is:
- (a) an alcoholic with elevated cholesterol level.
  - (b) a nonalcoholic.
  - (c) A nonalcoholic with non-elevated cholesterol level.
8. What is the probability of choosing an odd number from the first 50 natural numbers?
9. Two dice are thrown. What is the probability that the product of the score is more than 20?
10. Two dice are thrown at the same time. Find the probability that the sum of the two numbers appearing on the top of the dice is:
- (a) 6
  - (b) 9
  - (c) 11
  - (d) greater than 8
  - (e) 0

# WHAT IS BULLYING?

Any unwanted written, verbal, graphic, or physical act by an individual or group toward another person(s) that causes harm or distress.

## Types of Bullying

- Physical
- Verbal
- Social
- Emotional
- Cyber

## STOP BULLYING



## Signs of Bullying

- Headaches
- Depression
- Loss of friends
- School absenteeism
- Academic problems

## What You Can Do

### PREVENT

- Be a role model for positive communication, healthy relationships, and self-care.
- Reinforce acts of kindness, respect, and inclusion.
- Set policies and rules about bullying.

### RECOGNIZE

- Know the definition of bullying and its many forms.
- Talk with and actively listen to the youth who confide in you.
- Watch for warning signs of bullying.

### INTERVENE

- If you witness bullying behavior
- Respond quickly and consistently to send the message that it is not acceptable.
- Separate the students involved.
- Meet any immediate medical or mental health needs.
- Stay calm and model respectful behavior.



Source: Teacher's Diary on *Cyber-Crime Awareness* by UNODC, Cybercrime and MoE, Republic of Liberia

