MATHEMATICS

PART I

A Textbook for Grade 12

M12TB-I

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Foreword

Liberia, having gone through a period of utmost turmoil till 2003, due to the civil wars, is still reeling under its effect and the added trauma of Ebola in 2014 and effects of the COVID-19 outbreak in 2020. The Liberian government, in the past decade, has made valiant efforts to bring order to the lives of its people. In one such effort, the Ministry of Education (MoE) brought changes to the National Curriculum Framework which are relevant to the present generation, and which would prepare them to meet the challenges of the changing trends of the world. The National Curriculum Framework (NCF) 2018 recommends a change in basic assumptions in the teaching learning process from behaviorist to constructivist approach — moving from hardcore print material to the digital world. Keeping in consideration the sociocultural context and varied experiences of learners as laid down in the Framework, our Teaching Learning Materials are expected to be competent to use multiple methods and techniques like e-learning resources, energized textbooks, and readily available reference material to engage the learners.

As a first initiative, the MoE, through its World Bank-funded Improving Results in Secondary Education (IRISE) project, has adapted textbooks for Grades 10 to 12 in five subjects — English Language and Literature, Mathematics, Biology, Physics and Chemistry.

The National Curriculum Framework, 2018, recommends that children's learning at school is a reflection of their life outside the school and shows them the path to become a responsible citizen who makes knowledge-based choices.This principle marks a departure from the legacy of teacher centered learning to student centered learning. The syllabi and textbooks developed on the basis of the NCF indicate a serious attempt to implement the idea of Activity Base Learning (ABL). We hope these measures will take us ahead in the direction of building a system of education as outlined in the NCF.

Combined with the efforts by the school principals and teachers this will encourage children to reflect on their own learning and to pursue imaginative activities and questions. With this in mind, perhaps for the first time in our country, we are able to provide separate subject specific textbooks accompanied with guides for teachers for 10–12 grades. Not only have these been developed, adapted and modified to the Liberian context, each of the eight Minimum Learning Competencies (MLCs) have been included in each textbook. So as to reach every high school student, for the first time in the country's history we have included the digitized form of the textbook accessible by a Quick Response (QR) code given in each book. Not only does it have the digitized textbook, but it provides additional learning materials for use by students, teachers and interested persons. The links to these e-resources and digitized material is being made available on the MoE's website.

The Textbooks and Teacher Guides have reached the hands of the students after a rigorous quality evaluation by carefully handpicked subject specialists by the MoE, to whom the Ministry expresses gratitude. For the success of this project, I acknowledge the contributions of the IRISE Project Team in the World Bank, and in particular, the Task-Team Leaders; the Project Implementation Team in Liberia headed by its Coordinator Abraham A. Kiazolu II, supported by the Executive Director of the Center of Excellence for Curriculum Development and Textbooks Research, Mrs. Julia K. Sandiman-Gbeyai and her technical working group (TWG), and the International Textbook Consultant and Advisor, Dr Shveta Uppal engaged by the MoE. These notwithstanding would not have been possible without the guidance of the Senior Management Team (SMT) of the Ministry of Education, and in particular, the Deputy Ministers for Instructions, Administration, and Planning, Research and Development, respectively.

Professor Dao Ansu Sonii, Sr.
Minister of Education
Republic of Liberia

Monrovia, Republic of Liberia
January 24, 2023

Acknowledgments

The development of textbooks contributes to the quality of teaching and learning that go on in the classroom.

The Ministry of Education (MoE) has aligned its Curriculum for Grades 10–12 to the National Curriculum Framework (NCF) of 2018. To ensure the provision of Teaching Learning Materials (TLMs) that support the revised curriculum, the Ministry has sought, reviewed and adapted a new set of textbooks and teacher guides along with digitized contents and e-learning resources for the five core subjects taught at the Senior Secondary education level, namely English Language and Literature, Mathematics, Biology, Chemistry and Physics, through an internationally competitive bidding process from the market supported by the World Bank funded Improving Results in Secondary Education (IRISE) Project.

With profound gratitude and honor, we recognize the Senior Management Team of the Ministry, headed by the Coach, Professor D. Ansu Sonii, Sr., for the strategic decision to make teaching learning materials available and accessible to all in the Liberian Senior Secondary School System, and for providing directions through the process of securing these textbooks and other teaching learning materials for our students and teachers. Our special thanks and appreciation to the World Bank for the financial support towards this policy intervention, and its education task-team including Alonso Sanchez, Oni Lusk-Stover and Binta B. Massaquoi for all their technical inputs offered throughout the process to ensure the kind of quality TLMs the Liberian students deserve are made available for improved learning outcomes.

We would like to specifically recognize the invaluable contributions of the 15 subject experts selected by the MoE from across the various education systems and the West African Examinations Council (WAEC) to evaluate, review and sign off on these teaching learning materials. They didn't just deliver according to our expectations, but also ensured the contextual relevance of the materials

to the Liberian Secondary Education Curriculum and its minimum learning competencies (MLCs). These subject experts include Professor Isaac Saye-Lakpoh Zawolo – *Superintendent* of the Monrovia Consolidated School System (MCSS), Mr. Matthew V.Z. Darblo, Sr. – *Mathematics Instructor* at the University of Liberia (UL), Mr. Charles Tieh Bropleh – *Mathematics Specialist* (MoE), Mrs. Linda Y. Dean – *English Specialist*, Mr. Hassan M. Bangura – *English Language and Literature Expert*, Mr. J. Emmanuel Milton – *English Specialist* (MoE), Mr. Moses K.M. Togbah – *Physics Specialist*, Mr. Prince A. Dossen – *Physics Specialist*, Mr. Benjamin Koryah – *Physics Instructor* at the University of Liberia (UL), Mr. Dominic Dugbe Doe – *Chemistry Specialist*, Mr. Patrick A. Anderson, Sr. – *Director* of the Division of Technical and Vocational Education (MoE), Mr. Kandakai Massaquoi – *Chemistry Specialist*, Ms. Patricia N. Doe – *Head* of Biology Department, African Methodist Episcopal University (AMEU), Mr. Job Carpenter – *Biology Specialist* and Mr. Prince Philip K.A. Aderibigbe – *Biology Specialist*.

The MoE is sincerely grateful to Dr Shveta Uppal, the *International Textbook Consultant* engaged by the IRISE Project to provide technical guidance and quality assurance support to the revising of the Textbooks Management Guidelines (TMG) and the procurement process leading to the provision of textbooks, teacher guides, digital contents and e-learning resources for the Senior Secondary School System in Liberia in accordance with the revised TMG. Heartfelt thanks and appreciations also to the *Executive Director* for the Center of Excellence for Curriculum Development and Textbooks Research, Mrs. Julia K. Sandiman-Gbeyai, and members of her Technical Working Group (TWG) for taking up the responsibility to lead the process of making textbooks and other TLMs available to Liberian students and teachers.

Lastly, we acknowledge the IRISE Project Delivery Team led by Mr. Abraham A. Kiazolu, II – *Project Coordinator*, Mr. Fuseini A. Abu – *International Procurement Specialist* and Mr. Lawrence S. Taylor – *Project Control Specialist* who coordinated the entire process.

We remain grateful to you all!

Hon. Alexander N. Duopu, Sr.,
Deputy Minister for Instruction
Ministry of Education, Republic of Liberia
#The Teacher

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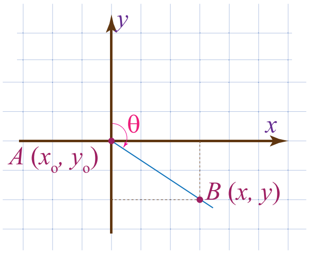

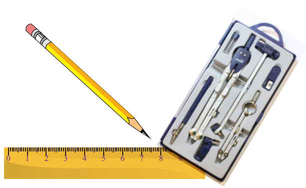

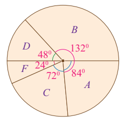

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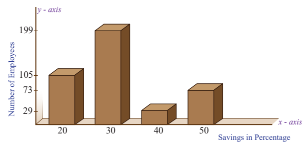

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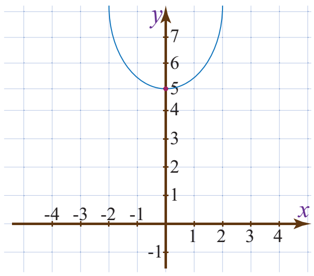

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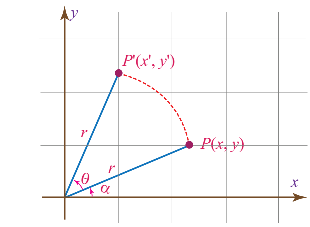

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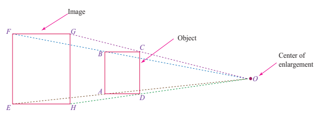

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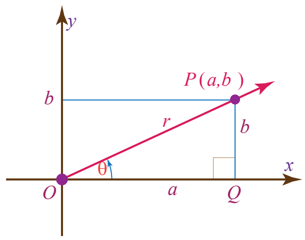

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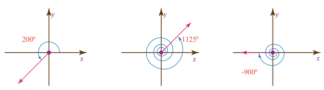

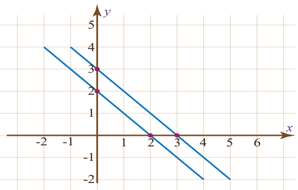
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MATHEMATICS PART II

CONTENTS

GRADE 12

M12CH01

CHAPTER 1

SEQUENCE AND SERIES

Chapter Content

1.1 Arithmetic Sequences
1.2 Terms of an Arithmetic Progression
1.3 Sum of Terms of an Arithmetic Progression
1.4 Sum of Terms of a Geometric Progression
- Key Terms
- Summary
- Exercises

Chapter Outcomes

Upon completion of this chapter, learners will:

- define and identify sequence;
- define and discuss arithmetic sequence or arithmetic progression;
- state the formula for arithmetic sequence or arithmetic progression and use it to solve problems;
- define and discuss geometric sequence or geometric progression;
- state the formula for geometric sequence or geometric progression and use it to solve problems;
- state the equation or formula for finding the sum of an arithmetic series and use the formula to solve problems;
- state the formula for finding the sum of a geometric series and use it to solve problems.

Introduction

In our day-to-day activities, we see different patterns, such as things that are placed in a particular order following a particular set of rules. For example, we can have 1, 3, 5, 7, … and 1, 2, 4, 8, … each representing a pattern. How do you find the next number in each pattern?

In the pattern 1, 3, 5, 7, … we find the next number by adding 2 to the preceding number and multiplying by 2 in the second pattern. We can also have a pattern of type 1, 4, 7, 10, …, or 1, 1/3, 1/9, 1/27, …; or other types possessing different rules.

A sequence is a group or sequential arrangement of numbers in a particular order or set of rules. A sequence can be finite or infinite. For example, 1, 2, 3, 4 is a finite sequence having four terms, whereas 1, 2, 3, 4, … is an infinite sequence.

Example 1

If we arrange numbers counting by 2 beginning with 1, we find 1, 3, 5, 7, 9, 11, …; this forms a sequence and the rule is adding 2. From this sequence it is possible to find the number at any position; e.g., the 50^{th} position. In a sequence, the number at the n^{th} place is mathematically known as the n^{th} term.

There are various types of sequences depending upon the set of rules that are used to form the sequence. The type of sequence as in Example 1 determined by adding a fixed number is known as an *arithmetic sequence*. The sequence determined by multiplying a term by a fixed number is called a *geometric sequence*.

Though there are different types of sequences, in this unit we will focus on arithmetic sequences obtained by adding a fixed number to the preceding term and geometric sequences obtained by multiplying the preceding term by a fixed number respectively.

Beyond identifying the type of a sequence, we will study how to find the n^{th} term and the sum of the first n-terms of a sequence.

Example 2

If we are asked to count by two beginning with 1, we will say: 1, 3, 5, 7, 9, … Here the numbers have a pattern, they form a sequence, and it is possible to find what the 50^{th} term of this sequence will be.

In the above sequence, you can see that to obtain the next term, we will add 2 to the preceding term.

In this unit, we will focus on sequences obtained by:

- Adding a fixed number on the preceding term.
- Multiplying the preceding term by a fixed number.

Such sequences which are called arithmetic sequences and geometric sequences will be discussed. We will also study how to find the n^{th} term and the sum of the first n-terms of such sequences.

A sequence is a list of numbers in some order. It has first term, second term, third term, and so on. So we can think of a sequence as a function whose domain is the set of positive integers {1, 2, 3, 4, ..}

Activity 1

Consider the following sequences of numbers.

(i) 5, 10, 15, 20, 25, …

(ii) 3, 6, 12, 24, 48, 96, …

1. In sequence (i), what is the next number after 25?
2. In the sequence (ii), what is the next number after 96?
3. In the above questions, how did you find the next number, was it by adding a fixed number or by multiplying by a fixed number?

Activity 2

1. For the following number patterns:
   - 1, 2, 3, 4, 5, … (set of positive integers).
   - 1, 3, 5, 7, 9, … (set of odd positive integers).
   - 2, 4, 6, 8, 10, … (set of even positive integers).
   - 5, 10, 15, 20, … (set of positive multiples of 5).
   - 2, 3, 5, 7, 11, … (set of positive prime numbers).

   (a) Tell how to generate the next term.

   (b) Is that possible to tell how to generate the next term for all patterns? Which one is possible and which one is impossible? Discuss the issues in detail in groups.

2. Consider the following patterns of numbers and discuss how to obtain the next term.

(a) 2, 3, 5, 8, 13, 21, …

(b) 4, 1, 5, 6, 11, 17, 28, …

(c) –8, –5, –2, 1, 4, 7, …

(d) 5, 10, 20, 40, …

(e) 1, –3, 9, –27, …

3. Give more examples of patterns that you know and discuss how to obtain the next term.

From the above activities you have seen different sequences, where some of the sequences were formed by adding a fixed number, some by multiplying by a fixed number, and some by other methods.

Definition

An arithmetic sequence or arithmetic progression is a sequence in which each term after the first term is obtained by adding a fixed number to the preceding term to find the next term.

- In arithmetic progression, the fixed number is called the common difference.

Notation

The common difference in an arithmetic progression (A.P) is usually denoted by (d).

In an arithmetic progression, we denote the n^{th} term of a sequence by A_n, where n is a natural number.

Definition

Let d be a real number. Then the sequence $\{A_n\}$ is an arithmetic sequence, if $A_{k+1} - A_k = d$, for all positive integers k (the difference between any two consecutive terms is always the same).

The number d is called the common difference.

Example 3

If the sequence 5, 8, 11, 14, 17, 20, … is denoted by $\{A_n\}$

The first term $A_1 = 5$, the second term $A_2 = 8$, $A_3 = 11$, $A_4 = 14$, …

To check whether the sequence 5, 8, 11, 14, 17, 20, … is an Arithmetic progression or not we need to check,

$A_{k+1} - A_k = d$, for any $k \in \mathbb{Z}^+$ (positive integer)

$A_5 - A_4 = 17 - 14 = 3$

$A_3 - A_2 = 11 - 8 = 3$

$A_2 - A_1 = 8 - 5 = 3.$

You can check for any $k \in \mathbb{Z}^+$, and from this you can conclude that to find the next term we will always add a fixed number to the preceding term.

The sequence 5, 8, 11, 14, 17, 20, … is an arithmetic progression, the fixed number or the common difference is $d = 3$. it is $A_2 - A_1 = 8 - 5 = 3$

or

$A_3 - A_2 = 14 - 11 = 3$, … etc.

Example 4

The sequence 12, 17, 22, 27, … is an arithmetic sequence, since the difference between each successive term is 5.

The sequence 1, 1, 2, 3, 5, 8, … is not an arithmetic sequence, since the difference between the first two terms is zero, but the difference between the second and the third term is 1. The difference between consecutive terms is not a fixed number.

Exercises

1. Observe each of the following sequences and identify those which are arithmetic progressions.
   (a) 5, 5, 5, 5, …
   (b) 8, 18, 28, 38, 48, …
   (c) 5, 10, 20, 40, 80, …
   (d) 100, 500, 250, 125, …
   (e) 10, 8, 6, 4, 2, 0, −2, …
   (f) −30, −20, −10, 9, …
2. From the above sequences, find the common differences and the first terms for those which are arithmetic progressions.

Activity 3

Consider the sequence given in the previous section: 5, 8, 11, 14, 17, … and complete the following.

The sequence is an A.P with first term $A_1 = 5$ and common difference $d = 3$.

$A_1 = 5$

$A_2 = 5 + 3 = 5 + (1 \times 3)$

$A_3 = 5 + 3 + 3 = 5 + (2 \times 3)$

$A_4 = 5 + 3 + 3 + 3 = 5 + (3 \times 3)$

$A_5 = 5 + 3 + 3 + 3 + 3 = 5 + (4 \times 3)$

.

.

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$A_{10} = \qquad\qquad = 5 + \square \times 3$

$A_{11} = \qquad\qquad = 5 + \square \times 3$

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.

.

$A_n = \qquad\qquad = 5 + \square \times 3$

In the above activity were you able to find the 10th term of the sequence?

It was $A_{10} = 9 \times 3$

(9 ↑ $(10 - 1)$; 3 ↓ d (common difference))

$A_n = 5 + (n - 1)\, 3$

(5 ↑ 1st term; 3 ↓ common difference)

Activity 4

Consider an arithmetic progression $\{A_n\}$ with first term A_1 and common difference (d)

First term	$A_1 = A_1$
Second term	$A_2 = A_1 + d$
Third term	$A_3 = A_2 + d = A_1 + d + d = A_1 + 2d$
Fourth term	$A_4 = A_3 + d = A_1 + 2d + d = A_1 + 3d$
Fifth term	$A_5 = A_4 + d = A_1 + 3d + d = A_1 + 4d$

.

.

.

10^{th} term	$= A_1 + \square\, d$
100^{th} term	$= A_1 + \square\, d$
n^{th} term	$= A_1 + \square\, d$

Note: Formula for terms of an Arithmetic sequence.

The n^{th} term or A_n of an arithmetic sequence $\{A_n\}$ with first term A_1 and common difference d is given by:

$$A_n = A_1 + (n - 1)\, d$$

Example 5

For the sequence 5, 8, 11, 14, …

(a) Find the 10^{th} term

(b) Find the 100^{th} term

(c) Find the n^{th} term, for $n \in \mathbb{Z}^+$

Solution

Here the sequence $\{A_n\}$ is arithmetic sequence (arithmetic progression)

$$A_2 - A_1 = A_3 - A_2 = d = 3$$

It is an arithmetic sequence with first term $A_1 = 5$ and common difference

$$d = A_2 - A_1 = 8 - 5 = 3$$

(a) $A_n = A_1 + (n-1)\,d$

$A_{10} = 5 + (10-1)3 \rightarrow$ (replacing $A_1 = 5$, $d = 3$, $n = 10$)

$= 5 + 9 \times 3$

$= 32$

(b) $A_{100} = 5 + (100-1)\,3 \rightarrow$ (replacing $A_1 = 5$, $n = 100$, $d = 3$)

$= 5 + 99 \times 3$

$= 302$

(c) $A_n = A_1 + (n-1)d \rightarrow$ (replacing $A_1 = 5$ and $d = 3$)

$A_n = 5 + (n-1)3$

$A_n = 5 + 3n - 3$

$A_n = 3n + 2 \rightarrow$ This is the explicit formula of the given sequence.

You can convince yourself by finding the 10^{th} term from the explicit formula found above

$A_n = 3n + 2$

$A_{10} = 3 \times 10 + 2 \rightarrow$ (replacing $n = 10$)

$A_{10} = 32 \rightarrow$ (the answer is the same as the one done on example 5(a) above)

Example 6

What is the 36^{th} term of an A.P. where the 3^{rd} term is 20 and the 4^{th} term is 25.

Solution

The common difference $d = A_4 - A_3 = 25 - 20 = 5$, difference of two consecutive terms.

To use the formula $A_n = A_1 + (n-1)d$, first, we should find A_1, by substituting n by 3 and $A_n = A_3$ by 20.

$A_n = A_1 + (n-1)d$

$A_3 = A_1 + (3-1)5 \rightarrow$ (replacing n by 3 and d by 5)

$20 = A_1 + 10$

$\therefore A_1 = 10$

The 36th term $A_{36} = A_1 + (n - 1)d$

$= 10 + (36 - 1)5$

$= 10 + (35 \times 5)$

$= 185.$

Example 7

What is the 356th odd natural number?

Solution

The odd natural numbers 1, 3, 5, 7, 9, … are in A.P. where the first term $A_1 = 1$ and the common difference $d = 2$ (the difference between consecutive terms is 2)

$A_n = A_1 + (n - 1)d$

$A_{356} = 1 + (356 - 1)2$

$= 711$ the 356th odd natural number.

Example 8

If the 3rd and 10th terms of an A.P are 10 and 59 respectively, then what is the 25th term?

Solution

First find A_1 and d

$A_n = A_1 + (n - 1)d$

$$\begin{cases} A_3 = A_1 + (3 - 1)d = 10 \\ A_{10} = A_1 + (10 - 1)d = 59 \end{cases}$$

$A_1 + 2d = 10$ ________ 1… (Equation −1)

$A_1 + 9d = 59$ ________ 2 … (Equation −2)

$-7d = -49$ (subtracting equation 2 from equation 1).

$d = 7$

Substituting $d = 7$ in one of the two equations (In this case Equation −1).

$A_1 + 2d = 10$ (equation −1)

$A_1 + 2 \times 7 = 10$ (replacing $d = 7$ in equation −1)

$A_1 = -4$

To find the 25[th] term or A_{25}

$A_n = A_1 + (n-1)d$ … (General formula for A_n)

$A_{25} = -4 + (25-1)7$ (replace $n = 25$)

$A_{25} = 164$.

Example 9

How many multiples of 3 are there between 124 and 789?

Solution

Multiples of 3 are 3, 6, 9, 12, 15, … and this sequence is in an A.P. where $A_1 = 3$ and $d = 3$.

then the n[th] terms is $A_n = A_1 + (n-1)d$

$= 3 + (n-1)3$

$A_n = 3n.$

To find the first multiple of 3 between 124 and 789

$A_n > 124$ for $n \in \mathbb{N}$ (natural number)

$3n > 124$

$n > 41\frac{1}{3}$ therefore $n = 42$ … $\left(\text{The first natural number greater than } 41\frac{1}{3}\right)$

The first multiple of 3 greater than 124 is 126 ($A_{42} = 3n = 3 \times 42 = 126$)

To find the last multiple of 3 between 124 and 789

$A_n < 789$

$3n < 789$

$n < 263$

$n = 262$ if $n = 262$ then $3n = 3 \times 262 = 786$.

The first multiple we got is the 42^{th} multiple of 3 and the last multiple we got (786) is the 262^{th} multiple of 3.

Note: There are 41 multiple of 3 not included here.

There are (262 − 41) = 221 multiples of 3 between 124 and 789

or the other way of solving this is to consider the sequence 126, 129, 132, … 786 (multiples of 3 between 124 and 789).

It forms an A.P where $A_1 = 126$ and $d = 3$

We can ask if 126 is the first term what term is 786?

substituting $A_1 = 126$, $d = 3$ in $A_n = A_1 + (n - 1)d$ gives

$$786 = 126 + (n - 1)3$$

$$126 + 3n - 3 = 786$$

$$3n = 663$$

$$n = 221$$

786 is the 221^{th} term of the sequence. This shows there are 221 multiples of 3 between 124 and 789.

Example 10

An office uses a big printer bought for LRD 60,000. If the value of this machine depreciates at the rate of LRD 3,000 per year, what is the value of the machine at the end of the 10th year?

Solution

The present values is LRD 60,000, at the end of the first year its value will be

LRD 60,000 − LRD 3,000 = LRD 57,000

in the same way at the end of the second year the value will be

LRD 57,000 − LRD 3,000 = LRD 54,000.

Thus the values at the end of consecutive years forms an A.P.

With first term $A_1 = 57{,}000$ and common difference $d = 3{,}000$.

The value at the end of the 10th year will be the 10th term of this sequence.

$$A_n = A_1 + (n - 1)\, d$$

$$A_{10} = 57{,}000 + (10 - 1) \times 3{,}000$$

$$= 57{,}000 - (9 \times 3{,}000)$$

$$= 57{,}000 - 27{,}000$$

$$= 30{,}000.$$

The value of the machine at the end of the 10th year will be LRD 30,000. $n < 263$

$n = 262$ if $n = 262$ then $3n = 3 \times 262 = 786$.

Activity 5

Form a group of two and discuss the following.

1. Write the first five odd natural numbers.
2. Write the first five even natural numbers.
3. Does the above sequence of numbers obtained in (1) and (2) above from an A.P.?
4. What is the nth odd natural number?
5. What is the nth even natural number?

Example 11

Show that the sequence $\{5n - 2\}$ is an arithmetic sequence and find the 10th term.

Solution

Let $A_n = 5n - 2$

This implies

$A_{n+1} = 5(n + 1) - 2$ (replacing n by the next number $(n + 1)$)

$$= 5n + 5 - 2$$

$$= 5n + 3$$

$$A_{n+1} - A_n = (5n + 3) - (5n - 2)$$

$$= 5n + 3 - 5n + 2$$

$$= 5,$$

which is a constant number, therefore $\{5n - 2\}$ is an A.P with first term

$$A_1 = 5 \times 1 - 2 = 3 \text{ and } d = 5$$

Here you can see that the 10th term

$$\begin{aligned} A_{10} &= A_1 + (n - 1)d \\ &= 3 + (10 - 1)5 \\ &= 3 + 45 \\ &= 48. \end{aligned}$$

The sequence is 3, 8, 13, 18, 23, … if you keep on writing the terms you can see that the 10th term is 48.

Exercises

1. True or False
   (a) The set of odd natural numbers forms an arithmetic progression.
   (b) The n^{th} odd natural number is $2n + 1$.
   (c) The n^{th} even natural number is n^2.
2. Work out
   (a) Identify the following sequences as an arithmetic progression or not arithmetic progression. If the sequence is an arithmetic progression find the common difference.
      (ii) 2, 7, 12, 17, …
      (iii) $f_n = 2n,\ n \in \mathbb{N}$.
      (iv) 3, 0, −3, −6, −9, …
      (v) $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, ...
      (vi) 3, 6, 12, 24, 48, …
   (b) What is the 10th term of each of the following sequences?
      (i) 1, 3, 5, 7, 9, …
      (ii) −250, −240, −230, …
      (iii) 5, 12, 19, 26, 33, …
      (iv) 7, 8, 9, 10, …
   (c) What is the 150th term of an arithmetic progression whose first term is 10 and common difference (−3)?
   (d) Find the 9th and 18th terms of the sequence 1, 5, 9, …

(e) If the 7th term of an arithmetic progression is 61 and the 14th term is 103 then find

(i) A_1 (iii) A_{101}

(ii) the common difference d

(f) In the sequence 4, 7, 10, 13, 16, which term is 304?

(g) If $A_1 = x$ and $d = -a$, then find the 10th term in terms of x and a.

(h) Find the first term of an A.P, in which the seventh term is 7 and the eleventh term is 55?

(i) Determine the common difference of the arithmetic progression with first term $21x$ and second term $-3x$.

(j) Find the first term of an A.P in which the third term is 7 and the 11th term is 55.

(k) Find the value of a so that -28, a, -3 will be in an A.P.

(l) How many multiples of 7 are there between 275 and 7,000?

Geometric Sequences

Activity 6

Observe the following sequences:

(i) 3, 6, 12, 24, 48, …

(ii) 3200, 1600, 800, 400, …

1. (a) For the first sequence what is the term after 48?

   (b) How are the consecutive terms obtained?

   (c) What is the ratio of consecutive terms, that is $\frac{6}{3}, \frac{12}{6}, \frac{24}{12}, \ldots$

2. For the second sequence

   (a) What is the term after 400?

   (b) How are the consecutive terms obtained?

   (c) What is the ratio of consecutive terms?

DEFINITION

A geometric sequence is a sequence where the ratio between consecutive terms is a non-zero constant.

The fixed number or the constant number in a geometric sequence is called the common ratio.

If $\{G_n\}$ is a geometric sequence $G_{n+1} = rG_n$; where r is a nonzero constant and G_{n+1} is the $(n + 1)^{th}$ term which is the common ratio times the n^{th} term

The common ratio $r = \frac{G_{n+1}}{G_n}$ for all $n \in \mathbb{N}$

Example 12

Is the sequence 3, 6, 12, 24, … a geometric sequence (Geometric progression).

Solution

To check whether the sequence is a geometric progression check the ratio of consecutive terms.

$$\frac{6}{3} = 2, \ \frac{12}{6} = 2, \ \frac{24}{12} = 2, \ldots$$

The ratio of consecutive terms is the same, hence the sequence is a G.P with first term $G_1 = 3$ and common ratio $r = 2$ (ratio of consecutive terms).

To continue the lesson we will see how to find the n^{th} term of such a sequence.

If $G_1, G_2, G_3, \ldots G_n, \ldots$ is a geometric sequence then

$$r = \frac{G_2}{G_1} = \frac{G_3}{G_2} = \frac{G_4}{G_3} = \ldots = \frac{G_{n+1}}{G_n}, \ n \in \mathbb{N}.$$

ACTIVITY 7

In a geometric sequence if G_1 is the first term and r is the common ratio, observe the pattern and find the n^{th} term G_n of the sequence.

$G_2 = rG_1$

$G_3 = rG_2 = r.rG_1 = r^2G_1$

$G_4 = rG_3 = r.r^2G_1 = r^3G_1$

$G_5 = rG_4 = r.r^3G_1 = r^4G_1$

.

.

.

$G_{10} = rG_9 =$ ________________ $= r\square\, G_1$

$G_{110} = rG_{109} =$ ______________ $= r\square\, G_1$

.

.

.

$G_n = r\, G_{n-1} =$ ______________ $= r\square\, G_1$.

From this pattern by finding what should be replaced in the empty box ($\square$) you can discover the formula for the nth term of a geometric sequence.

Note: If $\{G_n\}$ is a geometric progression with first term G_1 and common ratio r, then the n^{th} term of this sequence is given as

$$G_n = r^{n-1}G_1.$$

Example 13

If G_n is a geometric progression with fist term 3 and common ratio 2, then find the 5^{th}, 6^{th}, 10^{th} and n^{th} terms?

Solution

Given $G_1 = 3$, $r = 2$

Using $G_n = G_1r^{n-1}$ … (Formula for n^{th} term of a G.P).

$G_5 = 3 \times 2^{5-1} = 3 \times 2^4 = 3 \times 16 = 48$

$G_6 = 3 \times 2^{6-1} = 3 \times 2^5 = 3 \times 32 = 96$

$G_{10} = 3 \times 2^{10-1} = 3 \times 2^9 = 3 \times 512 = 1536.$

Example 14

If the 6^{th} and 8^{th} terms of a G.P. with positive common ratio are 256 and 1024 respectively, then find the first term, common ratio, second term and the n^{th} term?

If the 6th and 8th terms of a G.P. with positive common ratio are 256 and 1024 respectively, then find the first term, common ratio, second term and the n^{th} term?

Solution

Given $G_6 = 256$ and $G_8 = 1024$

$G_6.r = G_7$ and $G_7.r = G_8$

$G_6.r.r = G_6 r^2 = G_8$

$$r^2 = \frac{G_8}{G_6} = \frac{1024}{256} = 4$$

$r = 2$ (As r has to be positive, -2 is ignored)

to find the first term $\quad G_n = G_1\, r^{n-1}$

$$G_6 = G_1 r^5$$

$$256 = 32G_1$$

$$G_1 = \frac{256}{32} = 8$$

to find G_2, use $G_2 = G_1.r = 8 \times 2 = 16$

to find G_n, use $G_n = G_1\, r^{n-1} = 8 \times 2^{n-1} = 2^{n+2}$ … ($8 = 23$ and $2^{n-1} \times 23 = 2n + 2$)

Note: In solving such questions we can use relations such as

$G_3 = G_1\, r^2, \qquad G_6 = G_4 r^2,\ G_8 = G_5\, r^3$, etc

$G_{10} = G_8\, r^2, \qquad G_{12} = G_{10} r^2,\ G_{20} = G_1\, r^{19}$

Example 15

If G_n is a geometric progression with $G_3 = 6$ and $G_4 = 18$ find G_1, r and G_2.

Solution

Given $G_3 = 6$ and $G_4 = 18$

To find r use $G_4 = G_3.r$

$$18 = 6r$$

$$r = \frac{18}{6} = 3.$$

To find the first term use, $G_n = G_1 r^{n-1}$

$$G_3 = G_1\ 32 = 6$$

$$9G_1 = 6$$

$$G_1 = \frac{6}{9} = \frac{2}{3}.$$

To find G_2 use

$$G_2 = G_1.r;\ G_2 = \frac{2 \times 3}{3} = 2.$$

Exercises

1. True or False
   (a) The sequence $f_n = 2^n$ is a geometric sequence.
   (b) The set of even natural numbers forms a geometric sequence.
   (c) The sequence −2, 2, −2, 2, … is a geometric progression.
   (d) If the first term of a G.P. with common ratio (−1) is k, then $S_n = k$ for odd n.
2. Work out
   (a) Identify the following sequences as a geometric progression or not geometric progression.
      (i) 7, −7, 7, −7, …
      (ii) $f_n = (03)n$
      (iii) 2, 14, 9, 8, …
      (iv) 4, 2, 1, …
      (v) 2, 4, 6, 8, …
      (vi) −40, −20, −10, …
      (vii) $f_n = 3n + 5$
      (viii) $f_n = -3$
   (b) Find the 10th term of the sequence
      (i) 3, 6, 12, …
      (ii) 5, 5, −5, 5, …
      (iii) 5, 15, 45, …
      (iv) −60, −30, −15, …
      (v) $f_n = (-2)n$
      (vi) $g_n = \frac{1}{2^n}$

(c) Find the value of k so that $k + 1$, and $11k + 5$ form a geometric progression.

(d) If the first two terms of a G.P. are 2 and 1, respectively, which term of the sequence is equal to $\frac{1}{16}$?

(e) If the common ratio of a G.P. with first term $\frac{1}{64}$ is 2 then what is the 15^{th} term?

(f) Write the next three terms of the geometric progression

(i) 3, 6, 12, …

(g) If the 6^{th} term and the 12^{th} term of a G.P. are 2 and 64 respectively, find:

(i) the common ratio

(ii) the first term

(iii) the 20^{th} term

Historical Note

Carl Friedrich Gauss (1777 – 1855)

A teacher of Gauss, at his elementary school, asked him to add all the integers from 1 to 100. When Gauss returned with the correct answer after only a few moments, the teacher could only look at him in astounded silence. This is what Gauss did:

$$S_{100} = 1 + 2 + 3 + \ldots + 99 + 100$$
$$S_{100} = 100 + 99 + 98 + \ldots + 2 + 1$$
$$2S_{100} = 101 + \quad + \quad + \ldots + \quad + 101$$
$$S_{100} = \frac{100 \times 101}{2} = 5050.$$

If S_n is the sum of the first n-terms of an arithmetic progression with first term A_1 and common difference d:

$$S_n = A_1 + A_2 + A_3 + \ldots + A_{n-1} + A_n$$
$$S_n = A_1 + (A_1 + d) + (A_1 + 2d) + \ldots + (A_n - d) + A_n \ \ldots \text{(Equation −1)}$$

$S_n = A_n + (A_n - d) + (A_n - 2d) + \ldots + (A_1 + d) + A_1$ ...(Equation −2)
(reversing the order of terms in equation−1)

$2S_n = (A_1 + A_n) + (A_1 + A_n) + (A_1 + A_n) + \ldots + (A_1 + A_n) + (A_1 + A_n)$...
(Adding equation−1 and equation−2)

$$2\,S_n = n\,(A_1 + A_n)$$

$$S_n = \frac{n}{2}\left(A_1 + A_n\right)$$

The sum of the first n terms of an A.P. with first term A_1 and n^{th} term A_n is

$$S_n = \frac{n}{2}\left(A_1 + A_n\right)$$

In this formula, if we substitute $A_n = A_1 + (n - 1)d$

$$S_n = \frac{n}{2}(A_1 + (A_1 + (n-1)d)... \left(\text{since, } A_n = A_1 + (n-1)d\right)$$

$$= \frac{n}{2}\left(2A_1 + (n-1)d\right)$$

Note: The sum of the first n-terms of an arithmetic progression with first term A_1 and common difference d is

$$= \frac{n}{2}\left(2A_1 + (n-1)d\right)$$

Example 16

What is the sum of the first 100 natural numbers.

Solution

The set of natural numbers {1, 2, 3, …} forms an A.P. with 1^{st} term 1 and common difference 1.

$\therefore$ the first term $A_1 = 1$ and the last term $A_n = 100$.

The sum of the first 100 terms

$S_{100} = 1 + 2 + 3 + \ldots + 99 + 100$ 1

$S_{100} = 100 + 99 + 97 + \ldots + 2 + 1$ 2

$2\,S_{100} = (1 + 100) + (2 + 99) + (3 + 98) + \ldots + (99 + 2) + (100 + 1)$

$2\,S_{100} = 100 \times 101$

$$S_{100} = \frac{100\times101}{2} = 5050$$

or using the formula $S_n = \frac{n}{2}(A_1 + A_n)$ (n is number of terms)

$$= \frac{100}{2}(1+100) = 50\times101 = 5050$$

or we can use the second formula

$$S_n = \frac{n}{2}[2A_1 + (n+1)d]$$

$$S_{100} = \frac{100}{2}[2\times1 + (100-1)1] = 50\times(2+99) = 5050$$

by taking $A_1 = 1$, $d = 1$ and $n = 100$.

Example 17

What is the sum of the first n - odd natural numbers.

Solution

The set of odd natural numbers forms an A.P. with first term

$A_1 = 1$ and common difference $d = 2$

$$S_n = \frac{n}{2}[2A_1 + (n-1)d]$$

$$= \frac{n}{2}[2\times1 + (n-1)2]$$

$$= \frac{n}{2}[2n]$$

$S_n = n^2$.

Here we can easily see $S_2 = 1 + 3 = 4 = 2^2$

$S_3 = 1 + 3 + 5 = 9 = 3^2$

$S_4 = 1 + 3 + 5 + 7 = 16 = 4^2$

$S_5 = 1 + 3 + 5 + 7 + 9 = 25 = 5^2$ … sum of the first 5 odd natural numbers.

Example 18

In an A.P. if the first term is 3 and the common difference is 6, then find the sum of the first 11 terms.

Solution

$$S_n = \frac{n}{2}[2A_1 + (n-1)d]$$

$$= \frac{11}{2}[2\times 3 + (11-1)6]$$

$= 363.$

Example 19

If the 6th and 13th terms of an A.P. are 22 and 43, then what is the sum of the first 21 terms?

Solution

First find A_1 and d from the given terms A_6 and A_{13}

$A_6 = A_1 + (n - 1)d$ $\qquad$ $A_1 + 5d = 22$ ………….. 1

$A_{13} = A_1 + (n - 1)d$ $\qquad$ $A_1 + 12d = 43$ ……....... 2

Subtracting equation 2 from 1 we can see $-7d = -21$

$\therefore\ d = 3$ and $A_1 = 7$.

$$S_n = \frac{n}{2}[2A_1 + (n-1)d]$$

$$S_n = \frac{21}{2}[2\times 7 + (21-1)3]$$

$S_n = 777.$

Example 20

If the sum of the first n-terms of an A.P. is $S_n = n^2 + 4n$, find the n^{th} term and the common difference.

$$A_n = S_n - S_{(n-1)}$$
$$= n^2 + 4n - ((n-1)^2 + 4(n-1))$$
$$= n^2 + 4n - (n^2 - 2n + 1 + 4n - 4)$$
$$= n^2 + 4n - (n^2 + 2n - 3)$$
$$A_n = 2n + 3$$

The sequence is 5, 7, 9, 11, 13, …

$d = 2$ and $A_1 = 5$.

Exercises

1. True or False
   (a) The sum of the first n − odd natural numbers is n^2.
   (b) The sum of the first n − even natural numbers is $n^2 + n$.
   (c) The sequence 2, 2, 2, 2, … is an arithmetic progression.
2. Work out
   (a) What is the sum of the first 10 terms of the sequence
      (i) 1, 3, 5, 7, 9
      (ii) 5, 12, 15, 26, 33
      (iii) −250, −240, −230, …
      (iv) 7, 8, 9, 10, …
   (b) Find the sum of the terms 5, 9, 13, …, if the sequence consists of 20 terms.
   (c) An A.P. has its 10^{th} term 21 and 14^{th} term 37 find
      (i) the 1^{st} term
      (ii) d
      (iii) sum of the 1st 20 terms
      (iv) sum of the 1st n-terms
   (d) Find the first four terms of the sequence with the general term as given below
      (i) $f_n = (-1)^n \, 2^n$
      (ii) $f_n = (-1)^{n+1}$
      (iii) $f_n = \dfrac{n}{2n-1}$

(e) Given the favorable condition to reproduce, a culture of bacteria double itself every hour. If one bacteria is taken

How many bacteria's will there be after:

(i) 3hr?

(ii) 4hr

(iii) n − hours

(f) Find the sum of all even integers between −1000 and 1000.

(g) Find x and y so that x, y and $x - y - 1$ form an arithmetic progression with sum $13x$.

(h) What is the sum of multiples of 3 which are between 37 and 102?

If S_n is the sum of the first n-terms of a geometric progression with first term G_1 and common ratio r.

$$S_n = G_1 + G_2 + G_3 + \ldots + G^{n-1} + G_n$$

$$S_n = G_1 + G_1r + G_1r^2 + G_1r^3 + \ldots + G_1r^{n-2} + G_1r^n{-}1 \text{ (use } G_n = G_1(r^n{-}1)$$

$$rS_n = G_1r + G_1r^2 + G_1r^3 + G_1r^4 + \ldots + G_1r^n{-}1 + G_1r^n{-}1 + G_1rn \text{ (Multiplying (by } r)$$

$$rS_n - S_n = G_1r^n - G_1$$

$$S_n(r - 1) = G_1(r^n - 1)$$

$$S_n = \frac{G_1(r^n - 1)}{r - 1} \ldots \text{ if } r \neq 1.$$

But if $r = 1$, $S_n = G_1 + G_2 + \ldots + G_n$

$$S_n = G_1 + G_1 + \ldots + G_1$$

$$S_n = nG_1.$$

Note: The sum of the first n-terms of a geometric progression with first term G_1 and common ratio r is

$$S_n = \frac{G_1(r^n - 1)}{r - 1},\ r \neq 1$$

$$S_n = nG_1, \qquad \text{if } r = 1$$

Example 21

If G_n is a geometric progression with first term 5 and common ratio 2, then find the sum of the first ten terms.

$$S_n = \frac{G_1(r^n - 1)}{r - 1} \ldots,\ r \neq 1$$

$$S_{10} = 5(2^{10} - 1) = 5 \times 1023 = 5115.$$

Example 22

In a given G.P. if $G_3 = 6$ and $G_4 = 8$ find the sum of the first six terms.

Solution

From $G_3 = 6$ and $G_4 = 18$, $r = \frac{18}{6} = 3$ to find G_1 use $G_3 = G_1 r^2$

$$6 = G_1 3^2$$

$$G_1 = \frac{6}{9} = \frac{2}{3}$$

$$S_n = \frac{G_1(r^n - 1)}{r - 1}$$

$$S_n = \frac{\frac{2}{3}(3^6 - 1)}{3 - 1} = \frac{\frac{2}{3}(729 - 1)}{2} = \frac{728}{3}.$$

Example 23

Find the sum of the first 38 terms of a G.P., with third term 10 and common ratio 1.

Solution

Since $r = 1$ we will use the formula

$$S_n = nG_1$$

$$S_{38} = 10 \times 38 = 380 \ldots \text{note } G_1 = G_2 = G_3 = \ldots = G_n = 10.$$

Exercises

1. True or False
   - (a) The sequence $f_n = 2n$ is a geometric sequence.
   - (b) The sequence $-2, 2, -2, 2, \ldots$ is a geometric progression.
   - (c) If the first term of a G.P. with common ratio (-1) is k then $S_n = k$ for odd n.
2. Work - out
   - (d) Identify the following sequences as a geometric progression or not geometric progression.
     - (i) $7, -7, 7, -7, \ldots$
     - (ii) $f_n = (-3)n$
     - (iii) $2, 14, 9, 8, \ldots$
     - (iv) $4, 2, 1, \ldots$
     - (v) $2, 4, 6, 8$
     - (vi) $-40, -20, -10, \ldots$
     - (vii) $f_n = 3n + 5$
     - (viii) $f_n = -3$
   - (e) Find the 10th term of the sequence
     - (i) $3, 6, 12, \ldots$
     - (ii) $-5, 5, -5, 5, \ldots$
     - (iii) $5, 15, 45, \ldots$
     - (iv) $-60, -30, -15, \ldots$
     - (v) $f_n = (-2)n$
     - (vi) $g_n = \frac{1}{2^n}$
   - (f) Find the value of k so that $k + 1$, $5k - 1$ and $11k + 5$ form a geometric progression.
   - (g) If the first two terms of a G.P are 2 and 1, respectively, which term of the sequence is equal to $\frac{1}{16}$?
   - (h) If the common ratio of a G.P. with first term $\frac{1}{64}$ is 2, then what is the 15th term?
   - (i) Write the next three terms of the geometric progression
     - (i) $3, 6, 12, \ldots$
     - (ii) $x, 2, \frac{4}{x}, \ldots$
   - (j) If the 6th term and the 12th term of a G.P. are 2 and 64 respectively, then find:
     - (i) the common ratio
     - (ii) the first term
     - (iii) the 20th term

Key Terms

- Arithmetic sequence
- Common difference
- Common ratio
- General term
- Geometric sequence
- Sequence
- Sum of terms

Summary

- A sequence is a list of numbers.
- A sequence is a function whose domain is the set of natural numbers or sub set of consecutive natural numbers starting with 1.
- Arithmetic sequence or Arithmetic progression is a sequence in which the difference between any two consecutive terms a constant.
- Common difference in an A.P, is the difference between any two consecutive terms.
- The n^{th} term of an A.P. with first term A_1 and common difference d is
  $A_n = A_1 + (n-1)d.$
- sum of the first n terms of the above sequence is

$$S_n = \frac{n}{2}[A_1 + A_n]$$

$$S_n = \frac{n}{2}[2A_1 + (n-1)d].$$

- Geometric progression (G.P) or a Geometric sequence is a sequence in which the ratio of any two consecutive terms is a constant.
- This constant number in a G.P is called common ratio. It is usually denoted by r.
- The n^{th} term of a G.P with first term G_1 and common ratio r is $G_n = G_1 r^{n-1}$
- The sum of the first n-terms of a G.P. with first term G_1 and common ratio r is

$$S_n = \begin{cases} nG_1 \text{ if } r = 1 \\ \dfrac{G_1\left(r^{n-1}\right)}{r-1}, \text{ if } r \neq 1. \end{cases}$$

Exercises

I. True or False

1. The sequence 1, 2, 3, 4, … is an arithmetic progression.
2. The sequence of prime numbers is an arithmetic progression.
3. The sequence of multiples of 5 forms an arithmetic sequence.
4. The sequence $2^1, 2^2, 2^3, 2^4$, ...forms a geometric sequence
5. If an infinite geometric series is convergent then G_n approaches 0 for n - large.

II Matching

Column A	Column B
1. Sum of the first n-even natural numbers	(a) Geometric rogression
2. A sequence obtained by adding a fixed number	(b) $G_1 + (n-1)r$
3. A sequence obtained by multiplying by a fixed number	(c) nG_1
4. The sum of the first n-odd natural numbers.	(d) Common ratio
5. The nth term of a G.P. with common ratio r and first term G_1	(e) $G_1 r^{n-1}$
6. Sum of the first n - terms of a G.P. with first term G_1 and $r = 1$	(f) n^2
7. The fixed number in a G.P.	(g) series
8. The fixed number in an A.P.	(h) $n^2 + n$
	(i) Arithmetic progression
	(j) common difference

III Work Out

1. Find the first four terms of the sequence
   (a) $G_n = 2^{n-1}$
   (b) $f_{(n+1)} = f_n + 3n - 3$ where
   (c) $f_n = 2n^2 - 3n + 1$
   (d) $f_1 = -10$
2. Find the sum of the first ten terms of the sequence.
   (a) $G_n = \frac{1}{64} \times 2^{n-3}$
   (b) −600, −100 m, 400 m, 900, …
   (c) $A_n = 3n-51$
   (d) $G_n = (-25)(-1)^{n-1}$
3. Which term of the sequence 5, 14, 23, … is 239?

4. How many consecutive integers beginning with 10 must be added for their sum to equal 2035?
5. How along will it take to pay of adept of L$ 880 if L$ 25 is paid the first month, L$ 27 the second month, L$ 29 the third month, etc.?
6. Find three numbers in an A.P. whose sum is 21 and whose product is 280?
7. How many terms of the G.P. 3, 6, 12, … must be added to yield the sum 3069?
8. What is the product $G_1 \times G_2 \times G_3 \times \ldots \times G_{12}$ of a G.P. where $G_n = 2^{n-3}$.
9. If the sum of the first n-natural numbers is 20300, then what is the value of n?
10. In a Geometric progression with common ratio 2 if the sum of the first ten terms is 10230 what is G_1?
11. What is the sum of multiples of 7 between 38 and 820?
12. What is the sum of the first n-terms of the sequence $f_n = 2^{n-5}$.
13. If the fifth term of a positive G.P. is $\frac{1}{64}$ and the 13th term is 4 find

    (a) G_1 (b) r (c) A_{10} (d) S_{10}
14. If the fifth term of an A.P. is −28 and the 11th term is −32, then find;

    (a) A_1 (b) d (c) A_{10} (d) S_{10}
15. If the 6th term of a G.P. is 8 and the 10th term is 32 find

    (a) G_1 (b) r (c) S_{10}
16. If the sum of the first 21 terms of an A.P. with first term −7 is 1953, then find

    (a) d (b) A_{21}
17. If the sum of the first n-terms of an A.P. is $S_n = \dfrac{3n^2 + 5n}{2}$, then what is A_n? (Hint use $A_n = S_n - S_{(n-1)}$).
18. In a G.P., if the sum of the first n-terms is $S_n = 5(2^n - 1)$ then what is the n^{th} term G_n?
19. How many terms are there in an A.P. where $A_1 = 3$, $d = 5$ and $S_n = 255$?

IV. Choose the Best Answer

20. If the third and the fifth terms of an arithmetic progression are respectively 0 and −5, then which of the following is not true about the progression?

    (a) The second term is $\frac{5}{2}$

    (b) The fourth term is negative of the first term.

    (c) The sum of the first five terms is zero.

    (d) None of the above

21. If the second and the eighth the terms of an arithmetic progression are respectively 1 and 10, then which of the following statements is true?

    (a) The first term is $\frac{1}{2}$

    (b) The sum of the first three terms is 4.

    (c) The sum of the first eleven terms is 77.

    (d) None of the above

22. The geometric series 0.7 + 0.07 + 0.007 + … is equal to

    (a) 0.777

    (b) $\frac{7}{9}$

    (c) $\frac{7}{10}$

    (d) None of the above

23. If the third and sixth terms of a geometric progression are respectively 1 and −27, then which of the following is true?

    (a) The fifth term is reciprocal of the first term.

    (b) The common ratio is 3.

    (c) The first term is $\frac{-1}{9}$

    (d) None of the above

24. A sequence of numbers is defined by
    $f(1) = f(2) = 1$ and $f(n) = f(n-1) - f(n-2)$, for all positive integers $n \geq 3$, then which of the following is not true?

    (a) The seventh term is 1

    (b) The sum of the first five terms is 0.

    (c) The sum of the third and the fourth term is 0.

    (d) None of the above.

25. If the first three terms of an arithmetic progression whose first term is 1 and the $(2n)^{th}$ term is twice the n^{th} term are:

    (a) 1, 0, −1, …

    (b) 1, 3, 5, …

    (c) 1, 2, 3, …

    (d) None of the above

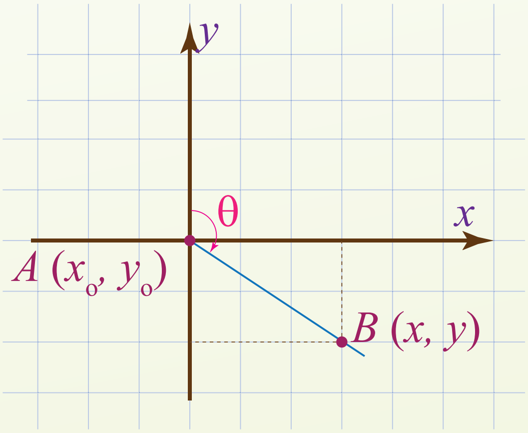
26. If the third and fifth terms of a geometric progression are respectively $\frac{1}{2}$ and $\frac{1}{8}$ and the second term is negative, then which of the following is not true about the progression?
   (a) The common ratio is $\frac{-1}{2}$
   (b) The fourth term is $\frac{-1}{4}$
   (c) The first term is 2
   (d) The sum of the first four term is (-1)
27. The 50th term of the arithmetic sequence $a - b, a + b, \ldots$ where $b \neq 0$ is
   (a) $a + 49b$
   (b) $a + 51b$
   (c) $a + 50b$
   (d) None of the above
28. If the third term of an arithmetic progression is 9 and the common difference is 2, then which of the following is true?
   (a) The first term is 3
   (b) The n^{th} term is $5 + 2n$
   (c) The 100th term is 203
   (d) None of the above
29. Let $a_1, a_2, a_3, \ldots$ be a sequence of numbers defined by $a_1 = \sqrt{2}$ and for every positive integer n,
   $a_{n+1} = a_n + 2$, Then $a_1 + a_2 + \ldots a_n$ is equal to
   (a) $n\left(n + \sqrt{2}\right)$
   (b) $n\left(n + \sqrt{2} - 1\right)$
   (c) $n\left(n + \sqrt{2} + 1\right)$
   (d) None of the above
30. If the fourth and eighth terms of an arithmetic progression are respectively 0 and -4, then which of the following is true about the progression?
   (a) The common difference is 1
   (b) The first term is 3
   (c) The eleventh term is -8
   (d) None of the above
31. A function f, defined on the set of positive integers, is given by $f(1) = 1$ and $f_{(n+1)} = (f_{(n)})^2 + 1$, for every positive integer n, then $f_{(5)}$ is
   (a) 26
   (b) 677
   (c) 676
   (d) None of the above

M12CH02

CHAPTER 2

BEARINGS

Chapter Contents



Chapter Outcomes

Upon completion of this chapter, learners will:

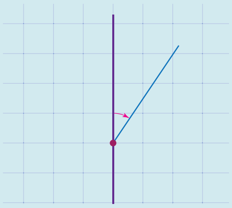
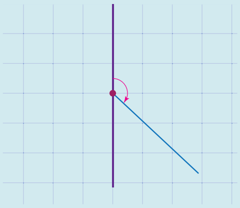
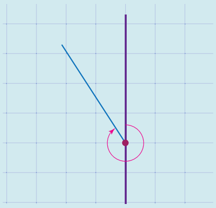
- interpret bearing as direction of one point from another;
- write bearing of one point from another as (r, q);
- find the bearing of a point A from another point, given the bearing of B.

Introduction

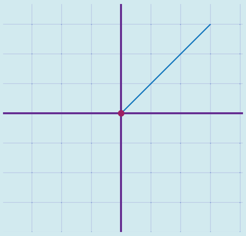
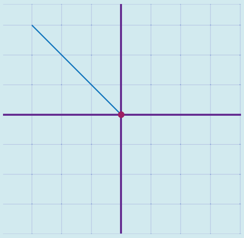
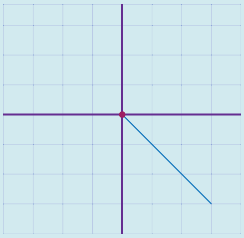
In this unit we are going to learn about bearings. Bearing measures the movement of an angle from some given point to some fixed line. So before defining the concept of bearings let us look at the following activities which revise measuring angles.

Activity 1

1. Use a protractor to measure the size of the following marked angles.

(a)  (b)  (c) 

2. Draw the XY-axis on square paper and label positive X-axis as East, negative Y-axis South, negative X-axis West and positive Y-axis North. Now measure the following angles clockwise from the North.

What we understand from the activities is that one can usually measure an angle between two intersecting lines using compass. So if a fixed line and a point on the same plane is given, one can measure the angle from that fixed line to the line drawn from the fixed line to the given point.

In this section the fixed line is specified to be the north direction. We always measure from the north direction to the line joining the north direction to the given point in the clockwise direction.

In navigation, bearings are used to express something about direction. The three characteristics of bearings are:

- The basis of a bearing is at the north direction.

- It always measures angle clockwise.
- It is written in three digit figures.

DEFINITION

A bearing is an angle, measured clockwise from the north direction to a given point.

Example 1

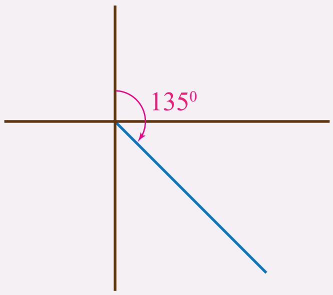
1. Imagine you are on an island in the middle of the ocean. You have a radio to ask for rescue. You received a message to turn 135 degrees. Where is 135 degrees?

Solution

(a) Draw a compass from the point and always face the north line.

(b) Moving clockwise, east will be 090°, south is 180°, and west is 270°. It will be between the Easterly and Southerly direction.

(c) Measuring the angle 135°, roughly, gives the following.



2. A ship leaves a port and travels 20 km on a bearing of 250°. At the same time, a second ship also leaves the same port and travels 14 km on a bearing of 190°. How far apart are the two ships?

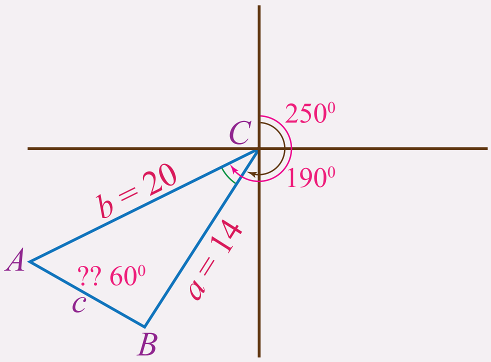
Solution

Let's make an illustration:

(a) Let's label the starting point or the port as point C.

(b) $\angle ACB = 250° - 190° = 60°$

(c) From the port (C) to the location of the first ship (A) is 20 km. The second ship (B) from the port (C) is 14 km.



(d) We will now use the cosine law:

$c^2 = a^2 + b^2 - 2ab \cos C$

$c^2 = 14^2 + 20^2 - 2(14)(20) \cos 60°$

$c = 17.78$ km

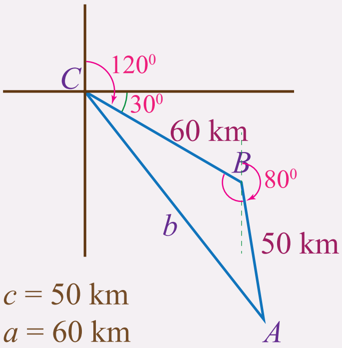
3. A ship from a port travels for 60 km on the bearing of 120°. It continuous to travel and changes direction for another 50 km on a bearing of 080°. How far is the ship from the starting point?

Solution

Let's make an illustration using a compass rose to trace where the ship travels.

We've formed a triangle. Now analyze the illustration:

(a) If the measure of angle A is 120°, then the measure of the remaining angle through the south is 60°.



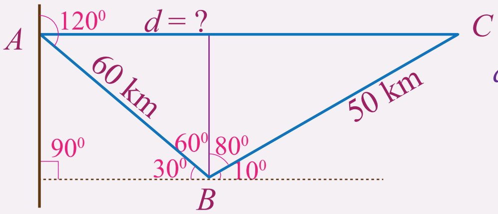
$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$b^2 = a^2 + c^2 - 2ac \cos 140°$$

$$= 60^2 + 50^2 - 2(60)(50) \cos (140°)$$

$$b = 103.42 \text{ km}$$

(b) Let's assume that from point A through the north the angle is 60°.



$$d^2 = 60^2 + 50^2 - 2(60)(50)\cos 140^o$$
$$= 3600 + 2500 - 6000\ (-??)$$
$$= 6100 - 6000\ (-??)$$

(c) Let's say the current location of the ship is point C.

$\angle ABC = 60° + 80° = 140°$.

(d) Remember, cosine law. It is used to find the measure of an angle or sides of a non right triangle.

Sometimes bearing coordinates of unknown point is measure from a survey station at known coordinates.

These technique apply to basic CAD program such as Auto CAD. Before discussing bearing coordinates let us consider the following rectangular and polar coordinates as an activity.

From the activities you observed that coordinates are used to locate a position.

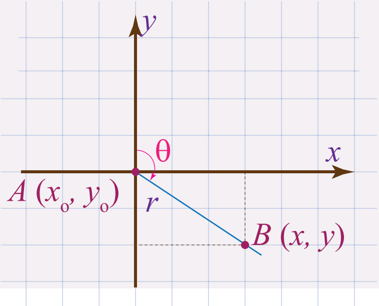
Activity 2

1. In rectangular and polar coordinate system which axes is taken as a reference line?
2. Describe both rectangular & polar coordinates by forming axes in the plane.
3. Which point is taken as fixed (or center) in both rectangular and polar coordinate?
4. Describe the relation between rectangular and polar coordinates.

Two such systems of coordinates are rectangular & polar. In similar fashion bearing coordinates are used in surveying to locate the position of a given unknown point given its bearing and its distance from a given fixed point (or enter). Here in this case the fixed point is not necessarily the origin like that of the rectangular and polar coordinates. Moreover, the reference line is not that of the positive x-axis rather the positive y-axis (North).

Example 2

For instance suppose known (or fixed) point A (x_o, y_o) with a measured bearing of a θ and a measured distance of r to an unknown point determine the Gradients of B.



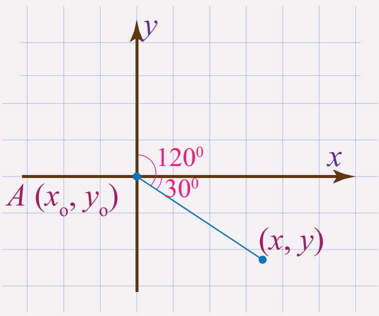
Solution

$$x = x_o + r \sin\theta$$
$$y = y_o + r \cos\theta$$

Example 3

Determine the coordinates of unknown point measured from a survey station at known coordinates (123 mE, 456 mN), given a measured bearing 120° and a measured distance of 36m.

Solution



$$\sin(120°) = \frac{1}{2}$$
$$\cos(120°) = -\sqrt{3}/2$$

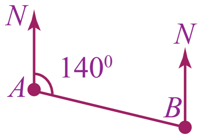
Easting: $x = 123 + 36\left(\frac{1}{2}\right) = 141\text{m}$

Northing: $y = 456 + 36\left(-\sqrt{3}/2\right) = 456 - 18\sqrt{3}.$

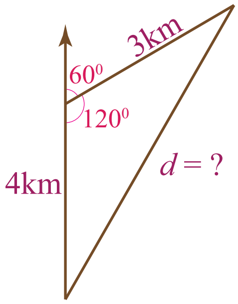
Therefore, the coordinates of the unknown point is (141 m E, $456 - 18\sqrt{3}$ m N)

Exercises

1. The diagram below shows the bearing of *B* from *A*. Find the bearing of *A* from *B*.



2. The location *C* is on a bearing of 140° from *A*. The bearing of *C* from *B* is 250°. Find the location of *C* and mark it on the diagram.
3. Two boats *A* and *B* are 5 km apart, and the bearing of *B* from *A* is 256. Using the scale 1 cm: 1 km, construct a diagram showing the relative positions of points *A* and *B*.
4. Find the components of vector *OA*, where *O* is the origin of the system of rectangular axes and *A* is a point 20 units away from the origin at a bearing of 030°.
5. A ship traveled a distance of 50 km from port *O* at a bearing of 120°, then another 100 km to port *B* at a bearing of 020°. What is the distance between ports *O* and *B*?
6. A ship is 4 km due north and then 3 km on a bearing of 060°. Find the distance from its original position.



Key Terms

- Bearing
- Clockwise
- North pole
- South pole

Summary

- Bearing are angles, measured clockwise from north.
- To measure a bearing, one has to know which direction is a clock wise direction
- All bearing need to be given in three figures; so if the angle measured is less than 100°, start the three figure bearing with zero

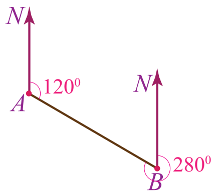
  Example:

  60° = 060°

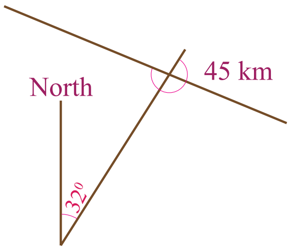
  99° = 099°

Exercises

1. The diagram below shows the bearing of A from B. Find the bearing of B from A.



2. A, B, and C are three ships. The bearing of A from B is 45. The bearing of C from A is 135. If $AB = 8$ km and $AC = 6$ km, what is the bearing of B from C?
3. The location of C is on a bearing of 140 from A. The bearing of C from B is 250. Find the location C and mark it using diagram.
4. Point A is 20 units away from the origin in the direction "N 30° W" and point B is 30 unit away from the origin in the direction "S 58° W". Find the components of vector AB.
5. A ship leaves a port and travels 21 km on a bearing of 032° and then 45 km on a bearing of 287°.
   (a) Calculate its distance from the port.
   (b) Calculate the bearing of the port from the ship.



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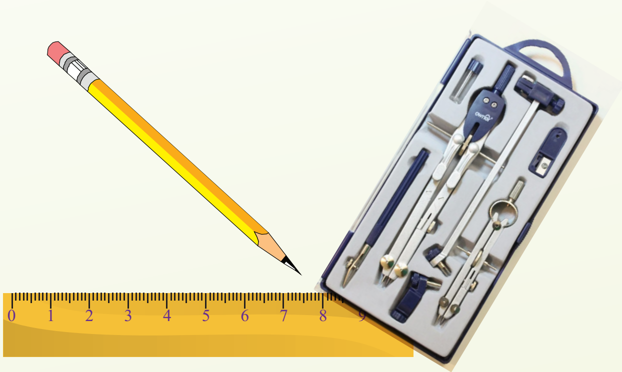
CHAPTER 3

CONSTRUCTIONS

Chapter Contents



Chapter Outcomes

Upon completion of this chapter, learners will:

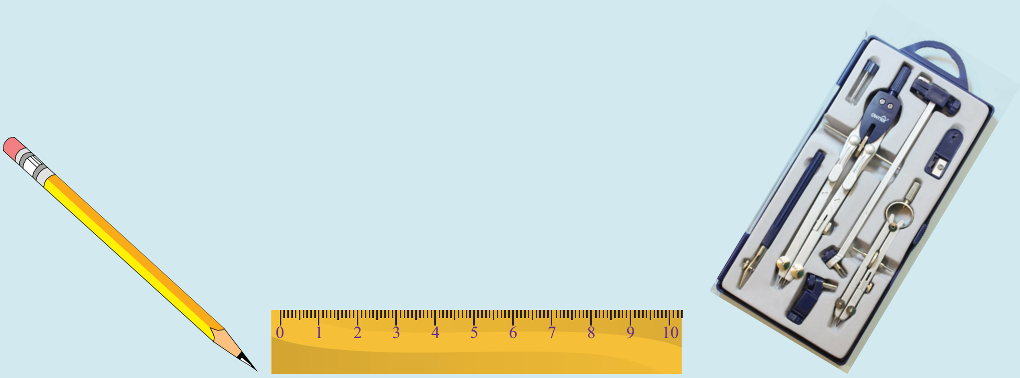
- copy line segment;
- copy angles;
- construct triangles and quadrilaterals;
- construct loci.

Introduction

Geometric construction refers to the process of drawing lines, angles and other geometric shapes and figures using only a compass and straightedge, without use of specific measurements of length, angle, etc.

Activity 1

1. From your previous construction, mention some of the instrument used to construct geometric figures.
2. Which instrument is used for some specific constructions; say line segment, angle, angle bisectors, parallel lines?



3.1 Construction of a Line Segment and its Copy Without Measurement

In geometry, construction of a line segment and its copy means drawing a line using pencils, rulers and compasses. This is pure geometric construction i.e. no measurements, no numbers. So let us see constructions of a line segment and its copy step by step in the following activities.

Activity 2

1. Draw any line segment using pencil on a paper mark its endpoints A and B.
2. Fix the compass pointer on A and the pencil end on B. The opening of the instrument now gives the length of AB.
3. Draw any line L and choose a point P on L. Without changing the compasses setting, place the pointer on P.
4. Draw an arc that cuts L at a point say Q. Hence PQ is the copy of AB.

From the activities one can observe that, a line segment can be drawn just using pencil and may be a ruler. But to draw the copy of a given line segment we must follow three steps. These three steps are the procedures from 2-3 given above.

Exercises

Construct a line segment of a given length using a compass (hint: follow the given steps).

1. Obtain the length of the line segment.
2. Mark a point A (say) on the plane of the paper and draw a line, say l, passing through it.
3. Place the metal point of the compass at zero marks on the ruler and open out it such that the pencil point on the arc indicates the length of the line segment on the ruler.
4. Transfer the compass as it is to the line l so that the metal point is on A.
5. With the pencil, point makes a minor stroke on l to cut it as B.
6. The segment AB so obtained is the required segment of the given length.

Bisecting a line segment

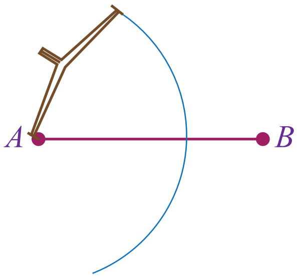
You can use a compass to bisect an angle or a line segment.

- For a line segment, open the compass to a bit more than half the line segment's length (visually, not by measuring).
- Place the needle on one endpoint and swing an arc above and below the line segment.
- Relocate the compass to the other end of the line segment and repeat the swing.
- The points where the two arcs intersect above and below the line segment are the two points to connect with your straightedge.
- That straight, perpendicular line will bisect the line segment!

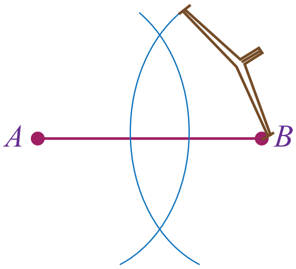
Midpoints and perpendicular bisectors

Construct a midpoint and a perpendicular bisector for line segment AB as given below:

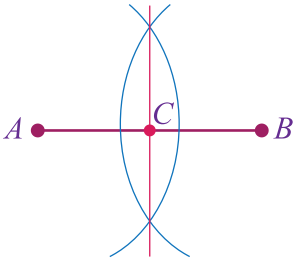
Step 1: Place the compass point on point A and open the compass such that its width is greater than half the length of the line segment. Maintain the width of the compass and draw an arc above and below the line segment. Keep in mind that the arc will need to intersect the arc drawn in the next step, so draw arc that are relatively long.



Step 2: Maintain the compass width, place the compass point on B, and draw arc above and below the line segment such that they intersect the first arc at two points.



Step 3: Draw a line through the intersections of the two arcs. Call the point of intersection of the line and line segment AB point C. Point C is the midpoint and the line is the perpendicular bisector of line segment AB.



Construction of angles is one of the key objectives in geometry. An angle is a shape formed by two rays called arms of angle that share a common point called vertex. Whether the arms are long or short, the angle size stays the same. There are **two angles at a vertex** so it is important to show which one we are talking about.

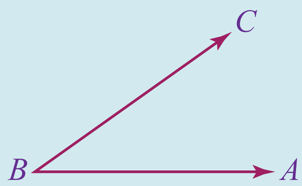
We can use protractor to construct various types of angles. Also there are methods by which we can construct some specific angles without using protractor. These angles can be constructed using a compass and ruler. Depending on the measure of the angle between the two arms, the six different types of angles are: acute, obtuse, right angle, straight, reflex, and full rotation.

Activity 3

Use your protractor to construct each of the six different types mentioned here.

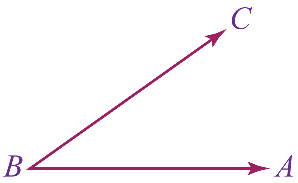
1. Define the six different types.
2. Draw a line segment *AB*.
3. Place the center of the protractor on point *A*, such that the line segment *AB* is aligned with line of the protractor.
4. Starting from 0 (in the protractor) mark the point *C* on the paper as per the required angle.
5. Join points *A* and *C*. $\angle BAC$ is the required angle.

In the activities, the procedures from 3–6 are the steps to construct any required angle.

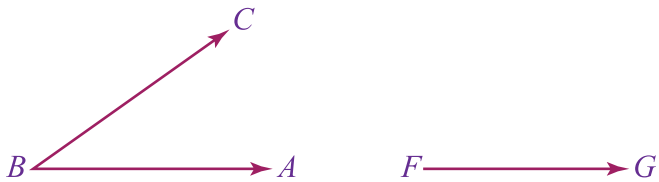


Construction of a copy of an angle

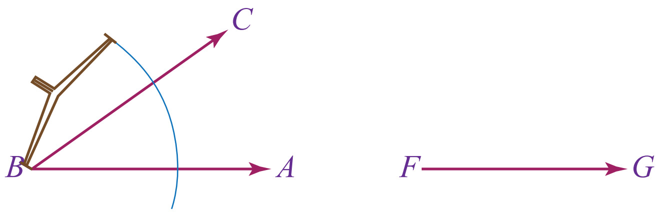
Given angle *ABC* as in the figure below, construct a copy of it using a compass and straightedge:



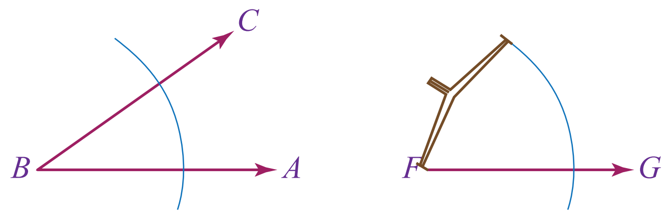
Step 1: Draw ray *FG*.



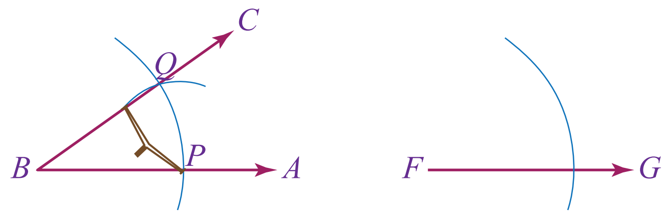
Step 2: Place the compass point on *B* and draw arc *DE* through rays *AB* and *BC*.



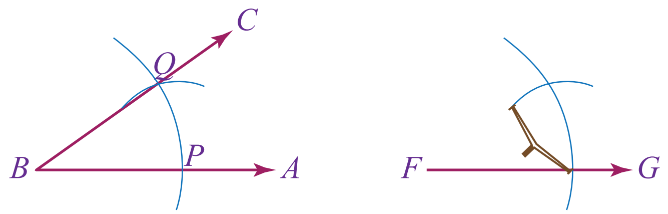
Step 3: Maintaining the compass width, place the compass point on F and draw an arc through ray *FG*.



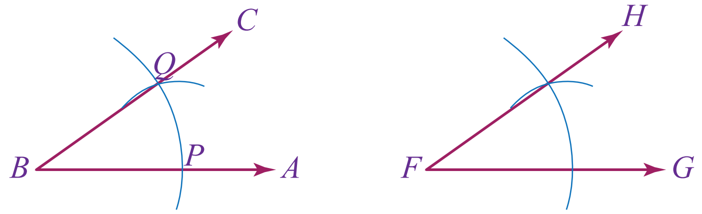
Step 4: Label the intersections of arc *DE* and ∠*ABC* (from step 2) *P* and *Q*. Place the compass point on *P* and draw an arc through *Q*. The radius of the arc is line segment *PQ*.



Step 5: Again, keeping the compass width the same, place the compass point on the intersection of the arc, drawn in step 3, and ray *FG*, then draw an arc above ray *FG* such that it intersects the other arc.



Step 6: Draw a ray from F through the intersection of the two arcs. Label the head of the ray *H*. The measure of ∠*GFH* should be equal to the measure of angle *ABC*.



Construction of the perpendicular bisector of a line segment and the bisector of a given angle.

Construct the perpendicular bisector of a given line segment.

Given: Line segment $\overline{AB}$

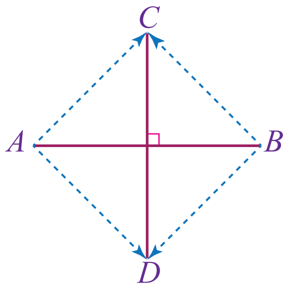
Required: To construct the perpendicular bisector of $\overline{AB}$

Step 1. Using point A as center, construct an arc above $\overline{AB}$ radius greater than half of the length of $\overline{AB}$, and construct another arc below $\overline{AB}$ with the same radius.

Step 2. Using point B as center and the same radius as in step 1, construct another arcs on opposite sides of $\overline{AB}$. Label the points at which the arcs intersect by the letters such as C and D.

Step 3. Construct $\overleftrightarrow{CD}$. $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$, intersecting $\overline{AB}$ at E.

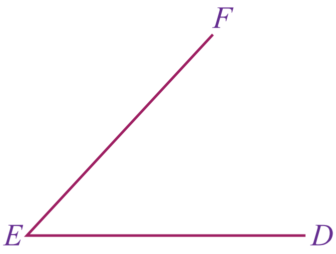
Therefore $\overline{AE} = \overline{BE}$.



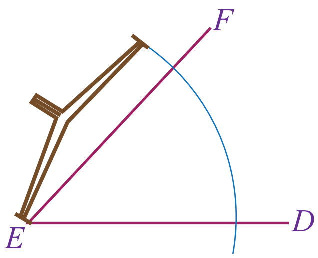
Procedures for Bisecting an Angle

- For an angle, hold the needle of the compass on the vertex. Open the compass to stretch most of the way along one ray.
- Swing an arc that crosses both rays.
- Without adjusting the compass, relocate the needle arm to the intersection of the arc and one ray.
- Swing an arc in the angle's interior area.
- Without adjusting the compass, relocate the compass to the other ray, where the first arc intersected.
- Swing an arc.
- Where the two arcs intersect, that point is an endpoint of a line segment that bisects the angle. Connect it with a straightedge to the vertex point.

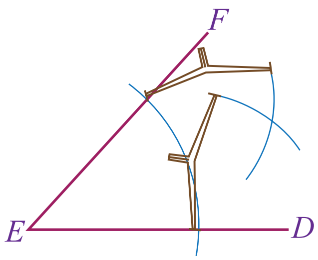
Construct an angle bisector for the following $\angle DEF$:



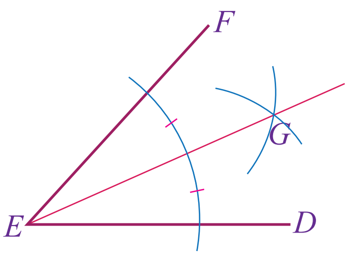
Step 1: Place the compass point on vertex *E* and draw an arc through both ray *ED* and *EF*.



Step 2: Place the compass point on the intersection of the arc (drawn in step 1) and ray *EF*, draw an arc in the interior of $\angle DEF$. Maintain the width of the compass and repeat this process with the compass point instead on the intersection of the arc and ray *ED*. Make sure the two arcs drawn in this step intersect.



Step 3: Label the intersection of the two interior arcs *G*. Draw a ray from *E* through *G*. Ray *EG* should bisect $\angle DEF$ forming congruent $\angle DEG$ and $\angle FEG$.



Construction of Lines Parallel or Perpendicular to a given line

Construction of a line perpendicular to a given line through a given point.

Given: Point P is outside line $\overleftrightarrow{AB}$

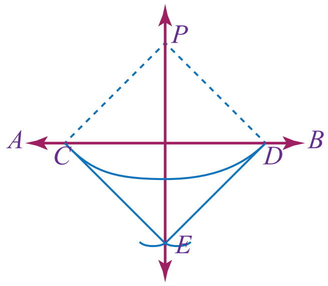
Required: To construct a line through P perpendicular to $\overleftrightarrow{AB}$

Procedures

Step 1: Construct an arc with center at P and a convenient radius that intersects $\overleftrightarrow{AB}$ at C and D.

Step 2: Construct arcs with centers at C and D and radius greater than half of the length of $\overline{CD}$ that intersect at E.

Step 3: Construct $\overleftrightarrow{EP}$. Therefore, $\overleftrightarrow{EP}$ is perpendicular to $\overleftrightarrow{AB}$.



Construction of a line parallel to a given line through a given point.

Given: Line $\overleftrightarrow{AB}$ and an external point P.

Procedures

Step 1: Through P, construct a transversal, intersecting $\overleftrightarrow{AB}$ at R.

Step 2: Construct $\angle SPD \cong \angle PRB$ that make a pair of corresponding angles.

Step 3: Construct $\overleftrightarrow{CD}$ through P.

Therefore, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

Construction of an Angle whose vertex is at a given point with measure.

(a) 60°

(b) 90°

(c) 30°

(d) 45°

Given: Point A

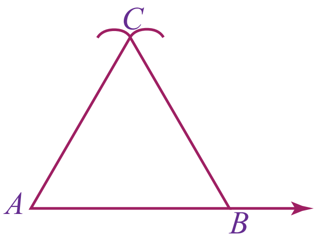
Required: To construct an angle with vertex at A that measures 60°.

Procedures:

Step 1: Construct any line segment $\overline{AB}$

Step 2: Construct arcs with centers at A and B and radius whose length is equal to AB on the same side of $\overline{AB}$ intersecting at C.

Step 3: Construct $\overline{AC}$ and $\overline{BC}$ so that ΔABC is an equilateral triangle.



Therefore, $m(\angle BAC) = 60°$.

Given: Point A

Required: To construct angle with vertex at A that measures 90°.

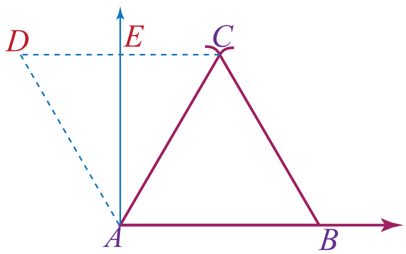
Procedures:

Step 1: Using the above procedure, construct any angle whose measure is 60°.

Step 2: Construct another angle at C with one side $\overrightarrow{CA}$ that measures 60° forming an equilateral triangle ACD different from ΔABC so that $m(\angle DAB) = 120°$.

Step 3: Construct $\overrightarrow{AE}$ which is the bisector of $\angle DAC$.

Therefore, $m(\angle EAB) = 90°$



Exercises

1. Construct an angle of a given measure using a pair of compasses and construct a copy of it.
2. Construct each of the following angles: 45°, 75°, 105°, 135°, 150°.
3. Construct and bisect each of the following angles: 30°, 60°, 105°, 120°, 135°, 150°.

Learning about geometric shapes such as triangle, rectangle, square, parallelogram, and in general any polygon is useful in whatever field you are.

Activity 4

1. What is a polygon?
2. What is regular polygon?
3. What is an irregular polygon?
4. Tell whether the following are regular or irregular polygons.

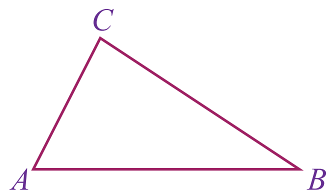
Equilateral triangle, isosceles triangle, rectangle, square

Note: You can call a polygon as the largest named group holding several geometric shapes. In this lesson, we would try to understand the importance and types of triangles and quadrilaterals which are important part of a polygon.

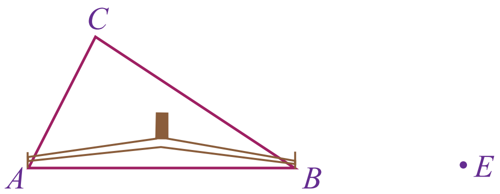
Construction of triangle

We can use some of the basic constructions shown in Section 3.1 and 3.2 to copy triangle.

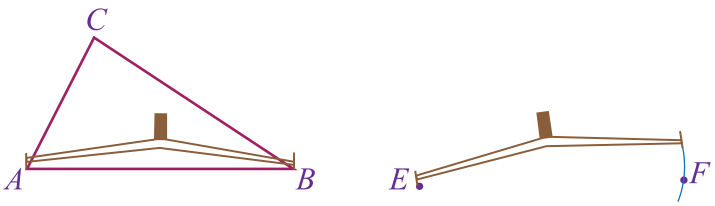
For example, to copy the following triangle *ABC* we follow the five steps given below:



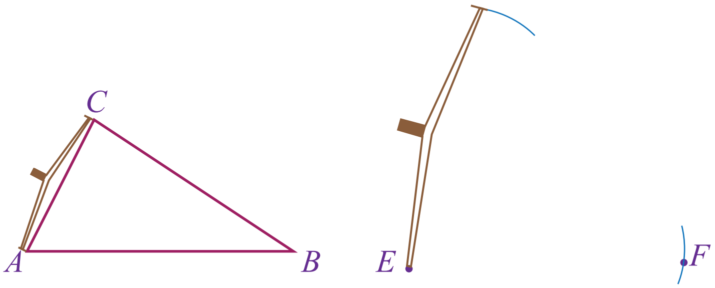
1. Measure the length of side *AB* by placing the compass point on *A* and the pencil end on point *B*. Draw a point, *E*.



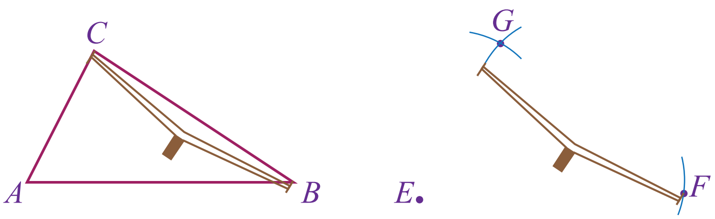
2. Maintain the width of the compass, place the compass point on *E*, and draw an arc to the right of *E*. Draw a point, *F*, somewhere on the arc. Line segment *EF* should be equal in length to *AB*.



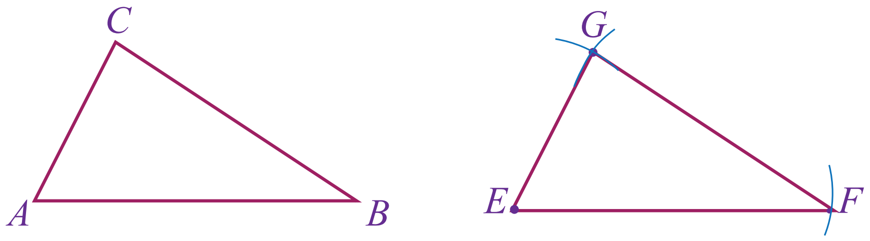
3. Measure the length of side *AC* by placing the compass point on *A* and the pencil end on point *C*. Maintain the width of the compass, place the compass point on *E*, and draw an arc.



4. Measure the length of side *BC* by placing the compass point on *B* and the pencil end on point *C*. Maintain the width of the compass, place the compass point on *F*, and draw an arc such that it intersects the arc drawn in step 3. Label the intersection of the two arcs *G*. Line segment *FG* should be equal to line segment *BC* and line segment *EG* should be equal to line segment *AC*.



5. With a straightedge draw line segments *EF*, *FG*, and *EG*. Triangle *EFG* should be congruent to triangle *ABC*.



Constructing triangles will include the construction of different triangles using a protractor, a compass and a ruler. *A* triangle is a three-sided polygon. It has three sides, three vertices and three angles. Construction of triangles is easy when the measurements are given to us based on different properties such as SSS, SAS and ASA.

How to construct triangles

To construct a triangle one should know these properties and rules:

- All three sides are given (SSS – Side-Side-Side)
- Two sides and included angle are given (SAS – Side-Angle-Side)
- Two angles and the included side is given (ASA – Angle-Side-Angle)
- The measure of the hypotenuse and a side is given in the right triangle (RHS – Right angle-Hypotenuse-Side)

For constructing triangles from given data, we generally make use of the given congruency conditions and construct the required triangle.

Constructing triangle with SSS property

When the length of three sides of the triangle is given, then follow the below steps to construct the required triangle.

- Draw a line segment *AB*, of length equal to the longest side of the triangle
- Now using a compass and ruler take the measure of the second side and draw an arc
- Again take the measure of the third side and cut the previous arc at a point *C*
- Now join the endpoints of the line segment to point *C* and get the required triangle *ABC*

Exercises

1. Construct triangle *ABC* such that *AB* = 5 cm, *BC* = 7 cm and *AC* = 6 cm.
2. Construct triangle *ABC* such that *AB* = 10 cm, *BC* = 12 cm and *AC* = 15 cm.
3. Is that possible to construct a triangle with the three sides given as 3 cm, 5 cm and 9 cm?

Constructing triangle with SAS property

When the length of two sides and the angle included between them are given, then use the following steps to construct the triangle.

- Draw a line segment *AB*, of length equal to the longest side of the triangle, using a ruler
- Put the center of protractor on one end of a line segment (say *A*) and measure the given angle. Join the points and construct a ray, such that the ray is nearer to the line segment *AB*
- Take the measure of another given side of the triangle using a compass and a ruler
- Put the compass at point *A* and cut the ray at another point, *C*
- Now join the other end of the line segment, i.e., *B* to the point *C*
- Hence, the triangle *ABC* is constructed.

Exercises

1. Construct triangle *ABC* such that *AB* = 5 cm, *BC* = 7 cm and $\angle ABC = 60°$.
2. Construct triangle *ABC* such that *AB* = 10 cm, *BC* = 12 cm and $\angle ABC = 75°$.

Constructing triangle with ASA property

When the measures of two angles and the side included between them are given of a triangle, then it is said to be ASA congruency. Follow the given steps to draw a triangle with ASA property.

- Draw a line segment *AB*, of length equal to the given side of the triangle, using a ruler
- At one endpoint of line segment (say *A*) measure one of the given angles and draw a ray *AR*
- At another endpoint of line segment (i.e.,*B*) measure the other angle using a protractor and draw the ray *BQ*, such that it cuts the previous ray at a point *P*
- Join the previous point *P*, with both the endpoints *A* and *B* of the line segment *AB*, to get the required triangle

Exercises

1. Construct triangle ABC such that $AB = 5$ cm, $\angle ABC = 60°$ and $\angle CAB = 45°$.
2. Construct triangle ABC such that $AB = 10$ cm, $\angle ABC = 70°$ and $\angle CAB = 30°$.

Construction triangle with RHS property

When the hypotenuse side and any one of the other two sides of right triangle are given, then it is RHS property. Follow the given steps to draw a triangle with RHS property.

- Draw the line segment AB, equal to the measure of other side
- At one endpoint, say A, of line-segment measure the angle equal to 90° and draw a ray, AR
- Measure the length of another given side i.e. hypotenuse and draw an arc to cut the ray AR at a point P
- Now join the point P and B to get the required right triangle.

Exercises

1. Construct triangle ABC such that $AB = 5$ cm, is the hypotenuse and $\angle ABC = 60°$.
2. Construct triangle ABC such that $AB = 10$ cm is the hypotenuse and $\angle CAB = 70°$.

Construction of quadrilateral

Activity 5

1. What is a quadrilateral?
2. Give examples of quadrilateral.

A unique quadrilateral can be drawn if five measurements of a quadrilateral are given. In a quadrilateral, there are four sides, four angles and two diagonals. A unique quadrilateral cannot be drawn when any four parts of a quadrilateral are known. But, if five parts of a quadrilateral are known, then a unique quadrilateral can be drawn. First, we draw a rough sketch of the quadrilateral and write the measurements of the five parts of the quadrilateral in the rough figure. Then, we draw the quadrilateral analyzing the given data (measurements).We divide the necessary quadrilateral into two triangles which can be easily constructed. These two triangles together will form a quadrilateral. Construction of unique quadrilateral is easy when the measurements are given to us based on different properties such as

1. If the lengths of its four sides and diagonal are given.
2. If its two diagonals and three sides are given.
3. If its two adjacent sides and three angles are given.
4. If its three sides and two included angles are given.
5. If four sides and one angle are given.

Exercises

Construct each of the following quadrilaterals.

1. A square *ABCD* of side 6 cm.
2. Rectangle *ABCD* of length $AB = 5$ cm and width $BC = 8$ cm.
3. Parallelogram *ABCD* given $AB = 6$ cm, $BC = 7$ cm and $\angle DAB = 60°$.

The definitions of a plane curve like circle, parabola, ellipse etc. are based on the concept of the locus. So, a locus is a set of points satisfying a certain condition. For example, the locus of points that are 1 cm from the origin is a circle of radius 1 cm centered on the origin, since all points on this circle are 1cm from the origin. Suppose *X* and *Y* are two fixed points in the two-dimensional coordinate plane. If a point *M* moves on this plane in such a manner that its distance from the points *X* and *Y* are always equal, then the point *M* will trace out a definite path on the plane. Again, if the point *M* moves path on the plane in such a manner that $MX^2 + MY^2 = 36$, then the moving point *M* will trace out another definite path on the plane. Thus, a moving point *M* trace out a definite path on the given plane if it satisfies some specified geometrical conditions. Such a path traced out by a moving point *M* on a plane is called its locus. For example, the locus of points in the plane equidistant from a given point is a circle, and the set of points in three-space equidistant from a given point is a sphere.

In Mathematics, a locus is a curve or other shape made by all the points satisfying a particular equation of the relation between the coordinates, or by a point, line, or moving surface. All the shapes such as circle, ellipse, parabola, hyperbola, etc. are defined by the locus as a set of points. In real-life you must have heard about the word ‘location’. The word location is derived from the word locus itself. Locus defines the position of something. When an object is situated somewhere, or something happened at a place, is described by locus. A locus is a set of all the points whose position is defined by certain conditions. For example, a range of the Southwest that has been the locus of a number of independent movements. Here, the locus is defining as the centre of any location. In Mathematics, a locus is the set of points represented by a particular rule or law, or equation.

ACTIVITY 6

1. Describe the following
   (a) Locus of a point.
   (b) Locus of a point equidistant from a fixed point.
   (c) Locus of a point equidistant from two fixed points.
   (d) Locus of a point equidistant from three fixed point.
2. Construct roughly the locus of points in (b), (c), and (d) under (1).

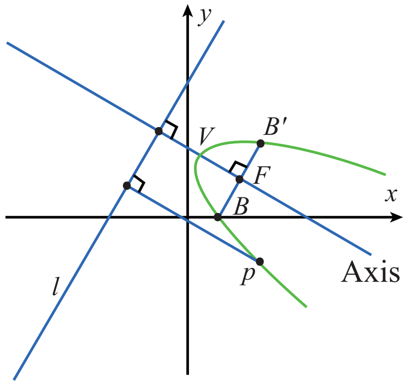
Some fundamental locus theorems on a plane

1. The locus of points equidistant from a fixed point is a circle.
2. The locus of points equidistant from two given intersecting lines is the bisector of the angles formed by the lines.
3. The locus of all points equidistant from two given points A and B is the perpendicular bisector of the line segment joining A and B.
4. The locus of points equidistant from two given parallel lines is the line parallel to the two given lines and located between these given lines.
5. The locus of points equidistant from the side of a given angle is the bisector of the angle.

Examples from plane geometry include:

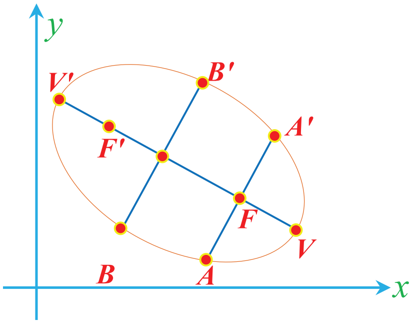
- The set of points equidistant from two points is a perpendicular bisector to the line segment connecting the two points.
- The set of points equidistant from two lines that cross is the angle bisector.
- All conic sections are loci:
  - Circle: the set of points for which the distance from a single point is constant (the radius).
  - Parabola: the set of points equidistant from a fixed point (the focus) and a line (the directrix.)
  - Hyperbola: the set of points for each of which the absolute value of the difference between the distances to two given foci is a constant.

- Ellipse: the set of points for each of which the sum of the distances to two given foci is a constant



Here are some terminologies for parabolas.

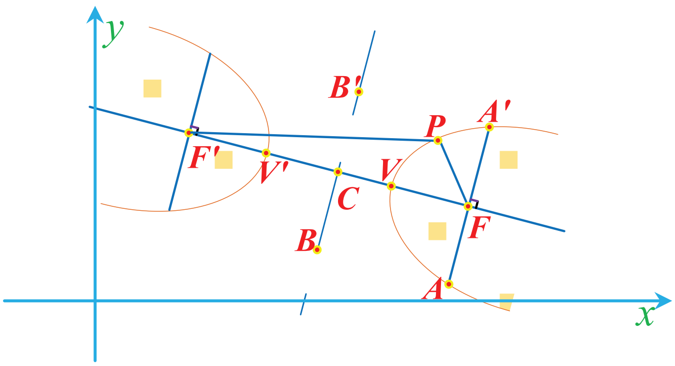
- F is the focus of the parabola.
- The line ℓ is the directrix of the parabola.
- The line which passes through the focus F and is perpendicular to the directrix ℓ is called the axis of the parabola.
- The point V on the parabola which lies on the axis of the parabola is called the vertex of the parabola.
- The chord $\overline{BB'}$ through the focus and perpendicular to the axis is called the latus rectum of the parabola.
- The distance $p = VF$ from the vertex to the focus is called the focal length of the parabola.



Here are some terminologies for ellipses.

- F and F' are foci.
- V, V', B and B' are called vertices of the ellipse.
- $\overline{V'V}$ is called the major axis and $\overline{B'B}$ is called the minor axis.

- C, which is the intersection point of the major and minor axes is called the centre of the ellipse.
- $\overline{CV}$ and $\overline{CV'}$ are called semi-major axes and $\overline{CB}$ and $\overline{CB'}$ are called semi-minor axes.
- Chord $\overline{AA'}$ which is perpendicular to the major axis at F is called the latus rectum of the ellipse.
- The distance from the centre to a focus is denoted by c.
- The length of the semi-major axis is denoted by a and the length of the semi-minor axis is denoted by b.
- The eccentricity of an ellipse, usually denoted by e, is the ratio of the distance between the two foci to the length of the major axis.



Here are some terminologies for hyperbolas.

- F and F' are the foci of the hyperbola.
- C is the centre of the hyperbola.
- The points V and V' on each branch of the hyperbola nearest to the centre are called vertices.
- $\overline{V'V}$ is called the transverse axis of the hyperbola and $CV = CV'$ is denoted by a and $CF = CF'$ is denoted by c
- Denote $c_2 - a_2$ by b_2, so that $b = \sqrt{c^2 - a^2}$.
- The segment of symmetry perpendicular to the transverse axis at the centre, which has length $2b$, is called the conjugate axis.
- The end points B and B' of the conjugate axis of the hyperbola are called co-vertices.
- The eccentricity of the hyperbola, usually denoted by e, is the ratio of the distance between the two foci to the length of the transverse axis.

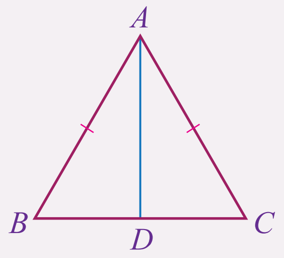
Here are some of the locus examples in two-dimensional geometry:

Perpendicular bisector: the locus of points that is equidistant from two fixed points. From the word perpendicular and bisector, let's make a description. Perpendicular refers to lines, rays or segments that intersect and form a 90° angle. The bisector is a line, segment or ray that cut into two equal parts.

Angle bisector: the locus of points that is equidistant from two lines. A line must cut an angle into two equal portions.

Example 1

In an isosceles triangle show that the bisector of the angle formed between two similar sides is also an altitude, a median, and a perpendicular bisector of the side of that triangle.



Solution

In ΔABD and ΔACD,

$AB = AC$

$\angle BAD = \angle CAD$ AD, is bisector of $\angle A$

Again, $AD = AD$

Therefore, $\Delta ABD \cong \Delta ACD$

Thus, $BD = CD$

AD is also a median

and $\angle ADB = \angle ADC = 90°$

Hence, AD is the perpendicular bisector of BC.

Example 2

In the figure given below, two isosceles triangles ΔPBC and ΔQBC lie on both sides of BC at common base BC. Prove that line joining P and Q bisects line BC at 90°.

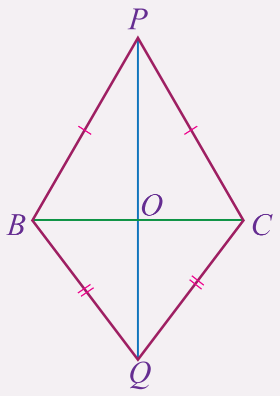
Solution

Given = ΔPBC and ΔQBC are the two isosceles triangles which lie on both sides of base BC.

Here, $PB = PC$ and $BQ = CQ$

To prove: $\angle POB = \angle POC = 90°$

or, $\angle QOB = \angle QOC = 90°$



In ΔPBC, $PB = PC$ (Given)

Therefore, $\angle PBO = \angle PCO$ (Equal sides)

$PO = PO$ Common by S.A.S. Congruence of

$\Delta POB \cong \Delta POC \Rightarrow \angle POB = \angle POC$ ….(i)

As we know that $\angle POB + \angle POC = 180°$

$\angle POB + \angle POB = 180°$ [From equation (i)]

$\angle POB = 180°$

$\angle POB = 180°/2 = 90°$

$\angle POB = \angle POC = 90°$

Similarly, $\angle QOB = \angle QOC = 90°$

Hence, PQ, bisects BC at 90°.

Key Terms

- Angle parallel line
- Angle bisector
- Compass
- Circle
- Hyperbola
- Locus
- Loci
- Line
- Parallelogram
- Parabola
- Quadrilateral
- Rectangle
- Ruler
- Set square
- Straightedge
- Triangle

Summary

Geometric construction refers to the process of drawing geometric shapes, angles or lines using a pair of compasses and a ruler.
A pair of compasses is a drawing instrument used to construct circles and arcs.

- Locus is a curve or shape created by all points satisfying a given equation of the connection between coordinates, or by a point, line, or moving surface. Circles, ellipses, parabolas, and hyperbolas, are defined by the locus as a set of points.
- A circle is described in terms of the locus of the points or loci as the set of all points equidistant from a fixed point, where the fixed point is the circle's center and the distance between the sets of points is the radius.
- A locus of collection of points that bisect an angle and are equally distant from two intersecting lines, which forms an angle is known as an angle bisector.
- Perpendicular bisector is the locus of points that are equidistant from two fixed points. From the word perpendicular and bisector, let's make a description. Perpendicular refers to lines, rays or segments that intersect and form a 90° angle. The bisector is a line, segment or ray that cut into two equal parts.

Exercises

1. Construct a line parallel to a given line at a given distance.
2. A farmer has tied a cow around a post on a rope 4 m long. What is the locus of the cow as it moves around the post?
3. Using a pair of compasses and ruler only, construct a parallelogram $ABCD$ with $AC = 10$ cm $BD = 8$ cm intersecting at point P and $\angle BPC = 60°$.
4. Define the following terms:
   (a) Median of a triangle
   (b) Altitude of a triangle

5. For the terms in Question number 4 construct a triangle and show its median and altitude.
6. Define and construct the following
   (a) Trapezium
   (b) Rhombus
   (c) Regular polygon with 5, 6, 7 and 8 sides
7. Consider a line segment *AB* of length 8 cm. Use construction to show that there are exactly two triangles: ΔABC and ΔABD with AC = AD = 5 cm and $BC = BD$ = 6 cm. Is ΔABC a right - angled triangle?

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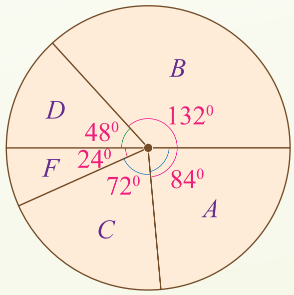
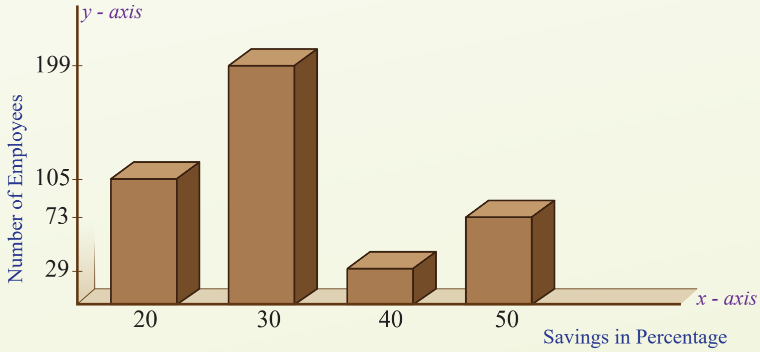
CHAPTER 4

STATISTICS I

Chapter Contents





Chapter Outcomes

Upon completion of this chapter, learners will:

- express data in graphical forms using bar chart pie chart;
- define grouped data and construct the frequency table;
- calculate the mean and mode of grouped data;
- calculate the median;
- construct the cumulative frequency curve;
- calculate quartiles and percentiles.

Introduction

Statistics is the subject that involves the study of how to collect, summarize and interpret numerical data. Many organizations and proffessions make decisions based on important conclusions which are drawn from data. For instance, data like unemployment figures or inflation figures used to make policy decisions by government officials.

To make investment decisions, finanical planners use recent trends in stock market prices. Also, physicians and hospitals use data on the effectiveness of drugs and surgical procedures to provide patients with the best possible treatments.

This topic is concerned with presenting data in graphical forms.

Activity 1

Raw data is a figure that has been collected but not organized in any way.

Example 1

Any list of numbers such as:

0, 0, 1, 2, 1, 5, 3, 4, 6, 2, 1, 6 is a raw data.

If we list this raw data in tabular form showing each number and the number of times it occurs, then the table giving this information is said to be a frequency table.

Look at the following frequency table

Value x	0	1	2	3	4	5	6
Frequencies, f	2	3	2	1	9	1	2

Qualitative and quantitative variables

Variables can be classified as qualitative or quantitative. Consider the people of a city.

(i) If the people in the city are classified by their gender (male or female) or by their mother tongue or by religion, a person takes on values that are not numbers. Such a variable, here the people of the city, is said to be **qualitative variable**.

(ii) If the people in the city are classified by age, height or weight a person takes on values that are numbers. Such a variable, in this case the people of the city, is said to be **quantitative variable**.

In general a variable is said to be qualitative if it takes on values that are names or labels and a variable is quantitative if it takes on values that are numbers.

Example 2

The following variables are qualitative.

(a) A persons gender

(b) Satisfaction of a person by his/her salary

(c) The advertising medium used to promote a product by radio, newspaper, and television. etc.

Example 3

Each of the following variables is quantitative.

(a) The number of grade 11 students in your school.

(b) The profit for a company.

(c) The total number of new graduates of a country in 2012.

Discrete variable and continuous variable

Quantitative variables are classified as discrete or continuous variable.

Let x be the number of exercise books you bought last year. The value might be 0, 1, 2, 3, …

Here x is a variable that can be counted.

If y is the number of students who are promoted to grade 10 out of 80 students then y might be 0, 1, 2, …, 80, which is a variable that can be counted. Such a variable is said to be a discrete variable.

In general, a variable is said to be discrete variable if its possible values can be counted or listed as 0, 1, 2,…,

Suppose you measured the height h of a boy and found it 170 cm rounded to the nearest centimeters.

Then h lies in the interval, $169.5 < h < 170.5$. This shows that height cannot be measured exactly. It lies between two specific values. Also, another student with height 170 cm lies in this range, but the two students do not have exactly the same height. Such a variable is said to be continuous variable.

In general, a variable is said to be continuous if it has values in one or more interval on the real number line.

Exercises

Decide whether each of the following variables is discrete or continuous

1. The number of mobile phones sold by a company
2. The weight of students in your class.
3. Your body temperature in degree Fahrenheit.
4. The number of new babies born in Liberia in 2019.
5. The number of students who joined university last year in Liberia.

In grade 10 you have learned how to present data using table and diagrams. In this section you continue on presenting data using specific charts called Bar chart and Pie chart.

Activity 2

1. Do you know what graphs or diagrams are?
2. Why diagrams? Merits and demerits

Note: A graph is a visual, concise means of presenting information. Graphs are visual representation of data. Data can be in the form of a table but graphical representation is easier to understand. With graphs we can easily compare the data.

Bar graph

A bar graph is a chart with rectangular bars with lengths proportional to the values that they represent and have equal width. The bars can be plotted vertically or horizontally.

Properties of bar graph

Some of the important properties of a bar graph are as follows:

- All the bars should have common base.
- Each column in the bar graph should have equal width.
- The height of the bar should correspond to the data value.
- The distance between each bar should be the same.

Bar graphs are used to match things between different groups or to trace changes over time. Yet, when trying to estimate change over time, bar graphs are most suitable when the changes are bigger.

Bar charts possess a discrete domain of divisions and are normally scaled so that all the data can fit on the graph. When there is no regular order of the divisions being matched, bars on the chart may be organized in any order.

Advantages and disadvantages of bar chart

Advantages

- Bar graph summarizes the large set of data in simple visual form.
- It displays each category of data in the frequency distribution.
- It clarifies the trend of data better than the table.
- It helps in estimating the key values at a glance.

Disadvantages

- Sometimes, the bar graph fails to reveal the patterns, cause, effects, etc.
- It can be easily manipulated to yield fake information.

Important notes

Some of the important notes related to the bar graph are as follows:

- In the bar graph, there should be an equal spacing between the bars.
- It is advisable to use the bar graph if the frequency of the data is very large.
- Understand the data that should be presented on the x - axis and y - axis and the relation between the two.

How to draw a bar graph

Let us consider an example, we have four different types of animals, such as sheep, goat, horse, and donkey and the corresponding numbers are 32, 25, 12 and 10 respectively.

In order to visually represent the data using the bar graph, we need to follow the steps given below.

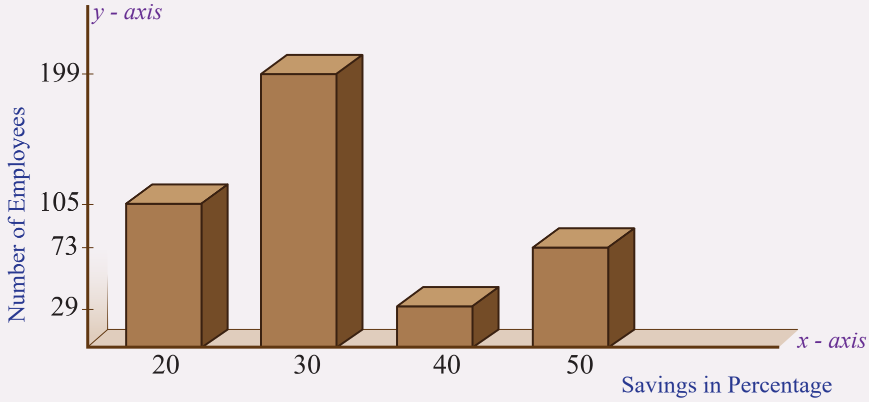
- **Step 1:** First, decide the title of the bar graph.
- **Step 2:** Draw the horizontal axis and vertical axis.
- **Step 3:** Now, label the horizontal axis.
- **Step 4:** Write the names on the horizontal axis, such as sheep, goat, horse, and donkey.
- **Step 5:** Now, label the vertical axis. (For example, Number of animals)
- **Step 6:** Finalize the scale range for the given data.
- **Step 7:** Finally, draw the bar graph that should represent each category of the animals with their respective numbers.

To understand the above types of bar graphs, consider the following examples:

Example 4

In a firm of 400 employees, the percentage of monthly salary saved by each employee is given in the following table. Represent it through a bar graph.

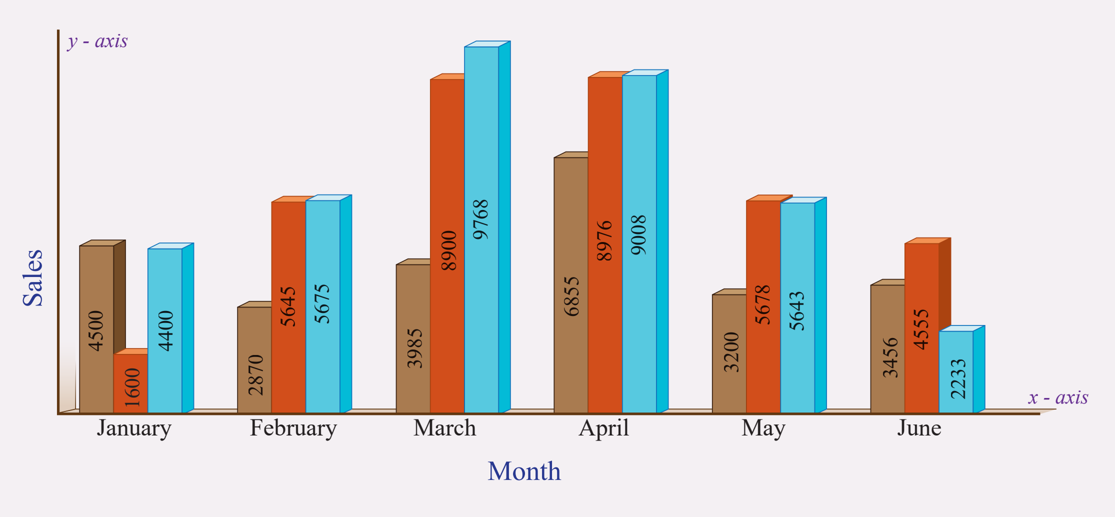
Saving (in percentage)	Number of Employees (Frequency)
20	105
30	199
40	29
50	73
Total	400



Example 5

A cosmetic company manufactures 3 different shades of lipstick. The sale for 6 months is shown in the table. Represent it using bar charts.

Month	Sales (in units)		
	Shade 1	Shade 2	Shade 3
January	4500	1600	4400
February	2870	5645	5675
March	3985	8900	9768
April	6855	8976	9008
May	3200	5678	5643
June	3456	4555	2233



Practice problems

1. A school conducted a survey to know the favorite sports of the students. The table below shows the results of this survey.

Name of the Sport	Total Number of Students
Cricket	45
Football	53
Basketball	99
Volleyball	44
Chess	66
Table Tennis	22
Badminton	37

From this data

(a) Draw a graph representing the sports and the total number of students.

(b) Calculate the range of the graph.

(c) Which sport is the most preferred one?

(d) Which two sports are almost equally preferred?

(e) List the sports in ascending order.

Pie chart

Pie chart is a circular diagram. It may be used in place of bar chart. It is a circle graph which is divided into a number of sectors. The sectors are proportional to the frequencies.

Construction of pie chart

- Various observations of the data are represented by the sectors of the circle.
- The total angle formed at the center is 360°.
- The whole circle represents the sum of the values of all the components.
- The angle at the center corresponding to the particular observation component is given by

$$\frac{\text{Value of the component}}{\text{Total value}} \times 360^\circ$$

If the values of observations/components are expressed in percentage, then the center angle corresponding to particular observation/Component is given by

$$\frac{\text{Percentage values of component}}{100} \times 360^\circ$$

Therefore, the following steps are helpful to construct a pie chart.

Step 1: Find the central angle for each component using the formula given above.

Step 2: Draw a circle of any radius.

Step 3: Draw a horizontal radius.

Step 4: Starting with the horizontal radius, draw radii, making central angles corresponding to the values of respective components.

Step 5: Repeat the process for all the components of the given data.

The graphs pieces are equal to the percentage of the total in each group. In other words, the size of each slice of the pie is proportional to the size of the group as a whole. The entire "pie" represents 100% of a total, while the "slices" represent parts of the whole.

At a glance, pie charts will help you understand the scale of your portions. They are commonly used in business presentations and education to display proportions among a broad range of categories, such a expenditures, population groups, and survey responses.

Example 6

1. The pie graph given below shows the breakup of the cost of construction of a house.

   Assuming that the total cost of construction is L$60,000, answer the question given below.

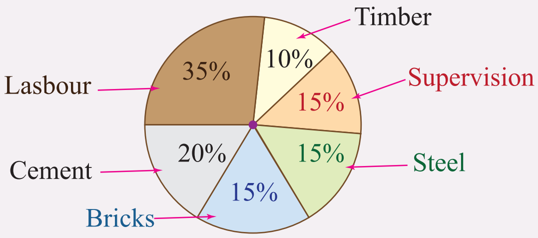
   Cost of construction of House

   Find

   (a) The sum spent one cement

   (b) The sum spent on Labour

   (c) The sum spent on steel.



2. There are 800 students in a Zwedru Multilateral School. The percentages of types of transportation used by the students as follow 10% car, 30% pehn-pehn, 45% keh-keh and 15% walking. Draw the pie chart.

Example 7

The table shows the grade achieved by 30 students in their end - of - year exam:

Grade	A	B	C	D	F
Frequency	7	11	6	4	2

To present this information on a pie chart, use the following steps:

(a) Find the total number of students:

(b) 7 + 11 + 6 + 4 + 2 = 30

(c) Calculate the angle of each segment

Grade	A	B	C	D	F
Frequency	7	11	6	4	2
Angle of each %	$\frac{7}{30}.360^{o} = 84^{o}$	$\frac{11}{30}.360^{o} = 132^{o}$	72^{o}	48^{o}	24^{o}

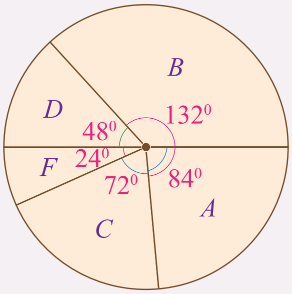
$$\frac{7}{30} \times 360^{o} = 84^{o}$$

$$\frac{11}{30} \times 360^{\circ} = 132^{\circ}$$

$$\frac{6}{30} \times 360^{\circ} = 72^{\circ}$$

$$\frac{4}{30} \times 360^{\circ} = 48^{\circ}$$

$$\frac{2}{30} \times 360^{\circ} = 24^{\circ}$$



Exercises

1. Project work: Go to any nearby health center and collect data on malaria, typhoid and measles for a particular month and construct bar charts and pie charts.

   (Directions: some group of students may work on malaria, some others on typhoid, and so on.)

2. A survey recorded the number of people living in each of 40 houses. The numbers were as follows:

   3 4 2 4 3 2 2 5 4 3 4 1 2 6 3 5 5 2 4 1 4 3 4 2 4 4 6 2 4
   3 2 5 4 5 6 4 2 3 2 4

   (a) Make a frequency table.

   (b) Draw a bar graph to illustrate your results.

   (c) What is the total number of people living in these 40 houses?

3. The table below shows how an income of L$ 400 was spent. Show these data on a bar graph and a pie chart.

Item	Food	Rent	Clothing	Transport	Savings
Amount	120	80	40	110	

4. A number of students were asked to name their favorite sport. $1/4$ of the students said kick-ball, $1/8$ said basketball, $1/3$ said football and the rest said volleyball.

   (a) What fraction said volleyball?

   (b) Calculating the value of x, if x is the angle of the sector representing basketball in the pie chart.

   (c) If 32 students chose football, how many said kick-ball?

In statistics, data can be classified as ungrouped data and grouped data. Data that is gathered at the first time but not organized in any way is ungrouped data. Raw data is ungrouped data.

Consider the following raw data that is consisting of 40 figures.

0	1	2	3	4	5	6	7
16	17	19	12	6	7	16	7
4	0	12	21	5	13	18	9
21	24	16	12	15	17	19	13
10	13	12	23	12	1	0	24

If you prepare a frequency table starting from the smallest number 0 to the largest number 24, then it may make the work lengthy.

Hence we need to classify the raw data into catagories called classes as follows. Suppose we divide it into 8 – equal classes, then the first class will be 0 to 4, the second class will be 5 to 9, …, the 8th class will be 20 to 24. Use tally marks to determine the frequency of each class i.e. the number of the figures in the raw data that fall in the class.

Class	Interval of values	Tally marks	Frequents
Class I	0 - 4	///	8
Class II	5 - 9	~~////~~ ///	8
Class III	10 - 14	~~////~~ /// ~~////~~ ///	10
Class IV	15 - 19	~~////~~ ////	9
Class V	20 - 24	~~////~~	5

This frequency table is said to be grouped frequency distribution. In general, data that has been organized into catagories is said to be grouped data.

Classes of a grouped data

The following are important steps in the construction of classes

1. Number of classes

In the construction of classes, the number of classes must be determined. There are different ways to determine the number of classes. In this book the number of classes will be given. Another point to be considered is classes should not overlap; for each number in the raw data there must be a class that includes the number. As much as possible classes should have equal lengths.

2. Class width (or class size)

The class width is given by the formula

$$I = \frac{\text{largest value} - \text{smallest value}}{\text{Number of classes}}$$

Class width is always a whole number.

Example 8

In the class 10 to 14 the third class, the class length $L = 5$, because the values in the interval are 5 number which are: 10, 11, 12, 13 and 14.
It should be rounded to a whole number if the quotient is not a whole number.

3. Class limits:

Class limits are the numbers that one can see in the frequency distribution table. Class limits are classified as upper and lower class limits.

Upper class limit: The upper class limit of a class is the largest number that can belong to the class.

Lower class limit: The lower class limit of a class is the smallest number that can belong to the class.

Example 9

In the class 10 to 14, the class limits are 10 and 14. In particular, the lower class limit is 10 and the upper class limit is 14.

4. Frequency:

The frequency for a class is the number of values that fall into the class.

5. Class mark (or Midpoint)

The class mark M of a class is the average of the upper and lower class limits of a class.

Example 10

In the class 10 to 14, the class mark is $\frac{10+14}{2} = 12$

6. Class boundary

Upper class boundary: An upper class boundary of a class is the value halfway between the upper class limit of a class and the lower class limit of the next class.

Lower class boundary: The lower class boundary of a class is halfway between the lower class limit of a class and the upper class boundary of the preceding class.

If 10 – 14 and 15 – 19 are the third and fourth classes then $\frac{15-14}{2}=0.5$,

the third class boundary is 10 – 0.5 to 14 + 0.5 which is 9.5 – 14.5, the lower class and the upper class boundaries of the third class are 9.5 and 14.5 respectively.

Also the 4th class boundary is 14.5 – 19.5 with lower and upper class boundaries 14.5 and 19.5 respectively.

Note that 0.5 is said to be correction factor.

EXAMPLE 11

If the intervals 10.3 – 14.3 and 14.4 – 18.4 are the 2nd and 3rd classes in a grouped frequency distribution, then $\frac{14.4-14.3}{2}=0.05$. Here the correction factor is 0.05.

Hence the lower class boundary of the 2nd class is 10.3 – 0.05 = 10.25 and

- the upper class boundary of the 2^{nd} class is 14.35.
- the lower class boundary of the 3^{rd} class is 14.4 – 0.5 = 14.35
- the upper class boundary is 18.4 + 0.5 = 18.45.
- If the classes have equal width, the 4^{th} class baundaries are 18.45 – 22.55, if there is a 4^{th} class.

EXAMPLE 12

The ages in years for 70 persons is given by the following grouped frequency distribution. The group consists of 8 classes. The largest value is 44 and the smallest value is 20.

Hence, the class width is $I=\frac{44-20}{8}=3$

The correction factor is $\frac{23-22}{2}=0.5$.

See the following table.

Class	Frequency	Lower class limit	Upper class limit	Class mark M	Class boundary	Lower class boundary	Upper class boundary
20–22	4	20	22	$\frac{20+22}{2}=21$	20 – 0.5 = 19.5 22 + 0.5 = 22.5 19.5 – 22.5	19.5	22.5
23–25	10	23	25	$\frac{23+25}{2}=24$	22.5–25.5	22.5	25.5
26–28	23	26	28	27	25.5–28.5	25.5	28.5
29–31	12	29	31	30	28.5–31.5	28.5	31.5
32–34	10	32	34	33	31.5–34.5	31.5	34.5
35–37	5	35	37	36	34.5–37.5	34.5	37.5
38–40	4	38	40	39	37.5–40.5	37.5	40.5
41–43	2	41	43	42	40.5–43.5	40.5	43.5

EXERCISES

Group the following data into six class of equal width

13	17	28	24	23	18	14	20	21	19
26	22	15	14	20	27	25	18	19	17
18	23	24	28	27	26	19	20	18	16
20	25	27	24	26	13	15	14	16	19

The mean

There are different types of means namely, the arithmetic mean (simply, the mean), the geometric mean and the harmonic mean.

In this topic we have only the arithmetic mean.

Consider the following grouped frequency distribution.

Value x	5 – 8	9 – 12	13 – 16	17 – 20	21 – 24
Frequency f	5	7	10	6	2

In the grouped frequency distribution, the individual values and their frequencies are not known. For instance in the first interval 5 – 8, you may have individual values like 6 and 7 with a given frequency.

Here, we consider the mid points M_i of each class. To represent all the values of the corresponding class.

M_i is the average of the lower class limit and the upper class limit of the i^{th} class.

Then, the mean, $\bar{x} = \dfrac{\sum f_i M_i}{\sum f_i}$

Also, M_i is the average of the i^{th} lower class boundary and upper class boundary.

Example 13

Find the mean from the grouped frequency distribution in the following table

Value x	**Class mark midpoint** (M_i)	**Frequency** f	$f_i M_i$
5 – 8	$\frac{5+8}{2} = 6.5$	5	$5 \times 6.5 = 32.5$
9 – 12	$\frac{9+12}{2} = 10.5$	7	$7 \times 10.5 = 73.5$
13 – 16	$\frac{13+16}{2} = 14.5$	10	$10 \times 14.5 = 145$
17 – 20	$\frac{17+24}{2} = 18.5$	6	$6 \times 18.5 = 111$
21 – 24	$\frac{21+24}{2} = 22.5$	2	$2 \times 22.5 = 45$
Total		**30**	**407**

Mean , $\bar{x} = \dfrac{407}{30} \approx 13.57$

Mode

Mode is another measure of central tendancy. For ungruped data, the mode is simply the value of a variable which occurs with the maximum frequency. The mode is the most common value. Unlike the mean and the median the mode may not be a single value. Data may have no mode or may have 1, 2, or 3, ... modes.

The mode from a grouped frequency distribution is computed as follows.

Mode, $M_o = L + \left(\dfrac{f - f_1}{2f - f_1 - f_2}\right) \times I$

Where,

$L =$ lower boundary of the modal class.

The modal class is the class with the maximum frequency.

f = the maximum frequency (frequency of the modal class).

f_1 = the frequency of the class just preceding the modal class

f_2 = the frequency of the class just following the modal class

I = width of the modal class

You know that median is a measure of central tendancy. For ungrouped data, the median is calculated as follows:

First arrange the series of the values of the variables in either ascending or descending order of magnitude.

If there are odd number of observations, then there is one, middle value that is the median. If n is odd, the median is the number in position $\dfrac{n+1}{2}$.

If there are even number of observations then there are two middle values. Hence the median is the average (or mean) of the middle values.

If n is even the median is the average of the numbers in position, $\dfrac{n}{2}$ and $\dfrac{n}{2}+1$.

Unlike the mean the median is not affected by the existance of extremly large or small value.

The median from a grouped frequency distribution is calculated as follows.

Median $M_d = L + \dfrac{\frac{n}{2} - f}{f_m} \times I$

Where,

L = lower boundary of the median class.

n = total frequency

f = Cumulative frequency preceding the median class.

f_m = frequency of the median class.

I = the width of the median class.

The median class is the class corresponding to the smallest cumulative frequency greater than or equal to $\dfrac{n}{2}$.

Exercises

1. Find the median from the grouped frequency distribution in the following table.
2. Find mode from the grouped frequency distribution in the following table

Value x	5 – 8	9 – 12	13 – 16	17 – 20	21 – 24
Frequency f	5	7	10	6	2

$f_1 = 7$

$f = 10$, the maximum frequency

$f_2 = 6$

3. Find the mean, median and mode from the following grouped frequency distribution.

Height to the nearest cm x	**Frequency** f
164 – 168	5
169 – 173	15

Height to the nearest cm x	Frequency f
174 – 178	12
179 – 183	10
184 – 188	3

4. For each of the following grouped frequency distribution, find the mean, mediam and mode.

(a)

V	16–20	21–25	26–30	31–35	36–40
f	7	8	5	4	3

(b)

V	4–7	8–11	12–15	16–19	20–23	24–27
f	1	2	4	5	2	1

Cumulative frequency is of two types.

1. **Less than cumulative frequency distribution**
   It is obtained by adding successively the frequencies of all the previous classes including the frequency of the present class. It is accumulated starting from the lowest class size to the height class size.

2. **More than cumulative distribution**
   It is obtained by adding successively the frequencies of all the following classes including the frequencies of the present class. It is accumulated starting from highest class size to the lowest class size.

EXAMPLE 14

The following are cumulative frequency tables for the less than and more than cumulative distribution.

(a)

Value	Frequency	Less than frequency	More than frequency	Also using minus
7	3	3 ↓	29 + 3 = 32	32 – 0 = 32
9	5	3 + 5 = 8	24 + 5 = 29	32 – 3 = 29
12	7	8 + 7 = 15	17 + 7 = 24	29 – 5 = 24
17	2	15 + 2 = 17	15 + 2 = 17	24 – 7 = 17
20	11	17 + 11 = 28	4 + 11 = 15	17 - 2 = 15
35	4	28 + 44 = 32	4 ↑	15 – 11 = 4

(b)

Class limits	Frequency	Less than cumulative frequency	More than cumulative frequency
6 – 10	16	16	64 + 16 = 80
11 – 15	4	16 + 4 = 20	60 + 4 = 64
16 – 20	21	20 + 21 = 41	39 + 21 = 60
21 – 24	39	41 + 39 = 80	39

Measures of central tendency for a grouped frequency distribution

As you know the measure of central tendency is a single number (if it is unimodal in the case of mode) that represents all the data.

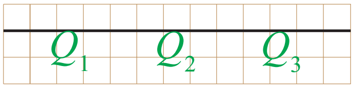
The mean, median, mode, quartiles, deciles and percentiles are the most common measures of central tendency. The quartiles, deciles and percentiles are extension of median.

A data that is arranged in an ascending order (in an increasing) of size can be divided as nearly as possible into 4, 10 and 100 equal parts by the values called quartiles, deciles and percentiles respectively. They are measures of location. Quartiles and percentiles. There are different ways to compute quartiles and percentiles but all of them give approximately the same result. One of the methods is given in this book.

The formulas for quartiles and percentiles

1. Quartiles

The quartiles are three partition values Q_1, Q_2 and Q_3, which divide the set of values that are arranged in an ascending order into 4 equal parts.



The first quartiles Q_1 has 25% of the values at or below it and 75% of the values at or above it.

The second quartile Q_2 is the same as the median.

The third quartile Q_3 has 75% of the values at or below it and 25% of the values at or above it.

The formula for quartiles

Let n be the number of observations. Then,

1. If n is odd

$$Q_1 = \text{Value of } \left(\frac{n+1}{4}\right)^{th} \text{ item}$$

$$Q_2 = \text{Value of } \left(2\left(\frac{n+1}{4}\right)\right)^{th} \text{ item}$$

$$Q_3 = \text{Value of } \left(3\left(\frac{n+1}{4}\right)\right)^{th} \text{ item}$$

In general $Q_k = \text{value of } \left(\frac{k(n+1)}{4}\right)^{th}$ item where $k = 1, 2, 3$.

If n is even

$$Q_k = \text{Value of } \left(\frac{kn+2}{4}\right)^{th} \text{ item; } k = 1, 2, 3$$

$$Q_1 = \text{Value of } \left(\frac{n+2}{4}\right)^{th} \text{ item}$$

$$Q_2 = \text{Value of } \left(\frac{2(n+1)}{4}\right)^{th} \text{ item}$$

$$Q_3 = \text{value of } \left(\frac{3n+2}{4}\right)^{th} \text{ item}$$

Note: If $\frac{k(n+1)}{4}$ is not an integer, then $Q_k = x_j + g(x_{j+1} - x_j)$. where j is the integer part and g is the decimal part; x_j is the j^{th} item.

For example if Q_k is the value of 4.6^{th} item, then $Q_k = x_4 + 0.6\,(x_5 - x_4)$.

EXERCISES

Arrange the data in ascending order and find the quartiles.

(a) 1, 3, 5, 2, 3, 2, 1, 5, 6, 7, 11, 10, 8

(b) 10, 11, 13, 14, 12, 13, 17, 16, 10, 15

2. Percentiles

The percentiles are 99 partition values P_1, P_2, P_3, …, P_{99}, which divide the set of values that are arranged in an ascending order into 100 equal parts.

P_1 has 1% of the values at or below it and 99% of the value at or above it. Generally, for $k = 1, 2, 3, \ldots, 99$, the P percent of the values are at or below it and $(100 - P)$ percent of the values are at or above it.

The formula for percentiles

Let n be the number of observations and P_k be the k^{th} percentile. Then,

(i) If n is odd, P_k = value of $\left(\frac{k(n+1)}{100}\right)^{th}$ item.

(ii) If n is even, P_k = value of $\left(\frac{kn+50}{100}\right)^{th}$ item

Example 15

Find P_{10}, P_{35}, P_{86} and P_{92} for each of the following data.

(a) 1, 1, 2, 2, 3, 3, 5, 5, 6, 7, 8, 10, 11

(b) 10, 10, 11, 12, 13, 13, 14, 15, 16, 17

Solution

(a) $n = 13$. n is odd. Hence we use the formula

$$P_k = \text{Value of } \left(\frac{k(n+1)}{100}\right)^{th} \text{ item}$$

$$P_{10} = \text{Value of } \left(\frac{10(13+1)}{100}\right)^{th} \text{ item} = 1.4^{th} \text{ item}$$
$$= x_1 + 0.4\,(x_2 - x_1)$$
$$= 1 + 0.4\,(1 - 1)$$
$$= 1$$

$$P_{35} = \text{Value of } \left(\frac{35(13+1)}{100}\right)^{th} \text{ item} = 4.9^{th} \text{ item}$$
$$= x_4 + 0.9\,(x_5 - x_4)$$
$$= 2 + 0.9\,(3 - 2)$$
$$= 2.9$$

$$P_{86} = \text{Value of } \left(\frac{86(13+1)}{100}\right)^{th} \text{ item} = \text{Value of } (12.04)^{th} \text{ item}$$
$$= x_{12} + 0.04\,(x_{13} - x_{12})$$
$$= 10 + 0.04\,(11 - 10)$$
$$= 10.04$$

$$P_{92} = \text{Value of } \left(\frac{92(13+10)}{100}\right)^{th} \text{ item} = 12.88^{\text{th}} \text{ item}$$
$$= x_{12} + 0.88\ (x_{13} - x_{12})$$
$$= 10 + 0.88\ (11 - 10)$$
$$= 10.88$$

(b) $n = 10$; n is even

$$\text{Then } P_k = \text{Value of } \left(\frac{kn+50}{100}\right)^{th} \text{ item.}$$

$$P_{10} = \text{Value } \left(\frac{10\times10+50}{100}\right)^{th} \text{ item} = \text{Value of } 1.5^{\text{th}} \text{ item}$$
$$= x_1 + 0.5\ (x_2 - x_1)$$
$$= 10 + 0.5\ (11 - 10) = 10$$

$$P_{35} = \text{Value of } \left(\frac{35\times10+50}{100}\right)^{th} \text{ item} = 4^{\text{th}} \text{ item} = x_4 = 12$$

$$P_{86} = \text{Value of } \left(\frac{86\times10+50}{100}\right)^{th} \text{ item} = 9^{\text{th}} \text{ item.}$$
$$= x_9 + 0.1\ (x_{10} - x_9)$$
$$= 16 + 0.1\ (17 - 16) = 16.1$$

$$P_{92} = \text{Value of } \left(\frac{92\times10+150}{100}\right) = \text{Value of } 9.7^{\text{th}} \text{ item}$$
$$= x_9 + 0.7(x_{10} - x_9)$$
$$= 16 + 0.7(17 - 16)$$
$$= 16.7$$

Relationship between quartiles and percentiles

(i) $Q_2 = P_{50} = \text{Median}$

(ii) $Q_1 = P_{25}$ and $Q_3 = P_{75}$

Example 16

1. Given the following data:

25, 10, 10, 18, 13, 11, 30, 22, 28, 28, 17, 11, 18, 24, 19, 16, 13, 25, 32, 27

Find

(a) The quartiles

(b) The percentiles P_{12}, P_{38}, P_{50} and P_{79}

Solution

The first step is to arrange the data in ascending order as follows

10, 10, 11, 11, 13, 13, 16, 17, 18, 18, 19, 22, 24, 25, 25, 27, 28, 30, 32

$n = 19$; there are odd number of observations

Hence, $P_k = \text{Value of} \left(\frac{k(n+1)}{100}\right)^{th}$ item

(a) $Q_1 = P_{25} = \text{Value of} \left(\frac{25(19+1)}{100}\right)^{th}$ item = 5th item = $x_5 = 13$

$Q_2 = P_{50} = \text{Value of} \left(\frac{50(19+1)}{100}\right)^{th}$ item = 10th item = $x_{10} = 18$, median

$Q_3 = P_{75} = \text{Value of} \left(\frac{75(19+1)}{100}\right)^{th}$ item = 15th item = $x_{15} = 25$

(b) $P_{38} = \text{Value of} \left(\frac{38(19+1)}{100}\right)^{th}$ item = 7.6th item

$= x_7 + 0.6\,(x_8 - x_7)$

$= 13 + 0.6\,(17 - 16) = 16.6$

$P_{50} = 18$

$P_{79} = \text{Value of} \left(\frac{79(19+1)}{100}\right)^{th}$ item = 15.8th item

$= x_{15} + 0.8\,(x_{16} - x_{15}) = 25 + 0.8\,(27 - 25)$

$= 26.6$

1. Find the difference $Q_3 - Q_1$ from the following data.

x	9	12	14	18	23	25
f	6	10	12	9	8	5

Solution

The number of observation,

$n = 6 + 10 + 12 + 9 + 8 + 5 = 50$

There are even number of observation

Hence, Q_1 = Value of $\left(\frac{50+2}{4}\right)^{th}$ item = 13^{th} item

= 12, because the first 6 values are equal to 9 and the next 10 values are equal to 12, hence the 13^{th} value is 12.

Q_3 = Value of $\left(\frac{3\times 50+2}{4}\right)^{th}$ item = Value of 38^{th} item = 23

Because 6 + 10 + 12 + 9 = 37, the values of the first 37 items is less than 18. The next term is 23.

$Q_3 - Q_1 = 23 - 12 = 11$

Note: The difference $Q_3 - Q_1$ is said to be inter-quartile range, *IQR*.

The formula for quartile and percentiles for a grouped data

1. Quartiles

Let $k = 1, 2, 3$.

The k^{th} quartile, Q_k for a grouped data of n observations is given by

$$Q_k = L + \left(\frac{kn}{4} - f\right) \times \frac{I}{f_k}; \text{where}$$

L = the lower class boundary of the k^{th} quartile class.

f = the cumulative frequency before the k^{th} quartile class.

f_k = the frequency of the k^{th} quartile class.

I = the size of the k^{th} quartile class.

2. Percentiles

Let $k = 1, 2, 3, ..., 99$

The k^{th} percentile, P_k for a grouped data of n observations is given by

$$P_k = L + \left(\frac{kn}{100} - f\right)\frac{I}{f_k}$$

Example 17

Find the quartiles and P_{72} for the following grouped frequency distribution.

v	1 – 7	8 – 14	15 – 21	22 – 28	29 – 35
f	1	2	2	2	1
$c.f$	1	3	5	7	8

Solution

Look at the cumulative frequencies, f.

There are 8 – observations. $n = 8$ and $I = 7$.

(i) Quartiles

(a) The first quartile when $k = 1$

$$\frac{1\times 8}{4} = 2$$

Look at the cumulative frequencies. The smallest cumulative frequency greater than or equal to 2 is 3.

- The first quartile class is the interval 8 – 14, which is the second class.
- $L = 7.5, f = 1$ and $f_1 = 2$

$$Q_1 = 7.5 + \left(\frac{4\times 8}{4} - 1\right)\times\frac{7}{2} = 7.5 + 3.5 = 11$$

(b) The second quartile, when $k = 2$.

$$\frac{2\times 8}{4} = 4$$

- The second quartils class is the interval 15 – 21 which is the third class. Look at the cumulative frequency.
- $L = 14.5, f = 3, f_2 = 2$
- $Q_2 = 14.5 + (4 - 3)\frac{7}{2}$

$= 14.5 + 3.5$

$= 18$

(c) The third quartile, when $k = 3$.

$$\frac{3\times 8}{4} = 6$$

- The third quartile class is the interval with the smallest cumulative frequency greaterthan or equal to 6 which is the interval 22 – 28
- $L = 81.5, f = 5$ and $f_3 = 2$
- $Q_3 = 21.5 + (6-5)\frac{7}{2}$

  $= 21.5 + 3.5$

  $= 25$

(ii) $P_k = L + \left(\frac{kn}{100} - f\right)\frac{I}{f_k}; n = 8, I = 7$

In P_{72}, $k = 72$. $\frac{72 \times 8}{100} = 5.76$

- The 72^{th} percentile class is 22 – 28 becuase the cumutative frequency 7 is the smallest one which is greater than or equal to 5.76.
- $L = 21.5$
- $P_{72} = 21.5 + (5.76 - 5)\frac{7}{2}$

  $= 21.5 + 8.66$

  $= 24.16$

Example 18

The following table contains the monthly contribution of 80 individuals in L$ for a the Monrovia Rotary Club. Determine the quartiles and the 60,80 and 95 percentiles.

Contribution USD	7–12	13–18	19–24	25–30	31–36	37–42
Frequency (*f*)	5	7	15	20	16	17

Solution

Class Boundary		6.5–12.5	12.5–18.5	18.5–24.5	24.5–30.5	30.5–36.5	36.5–42.5
Frequency (*f*)		5	7	15	20	16	17
Cumulative Frequency	More Than	5	12	27	47	63	80
	Less Than	80	75	68	53	33	17

Clearly, $N = 80$

(i) Quartiles. Using formulas for quartiles, we have:

(a) $\frac{N}{4} = 20 \Rightarrow Q_1 = 18.5 + \left(\frac{20-12}{15}\right) \times 6 = 21.7$

(b) $\frac{N}{2} = 40 \Rightarrow Q_2 = 24.5 + \left(\frac{40-27}{20}\right) \times 6 = 28.4$

(c) $\frac{3N}{4} = 60 \Rightarrow Q_3 = 30.5 + \left(\frac{60-47}{16}\right) \times 6 = 35.375$

(ii) Percentiles. Using formulas for percentiles, we have:

(a) $\frac{60N}{100} = 48 \Rightarrow P_{60} = 30.5 + \left(\frac{48-47}{16}\right) \times 6 = 30.875$

(b) $\frac{80N}{100} = 64 \Rightarrow P_{80} = 36.5 + \left(\frac{64-63}{17}\right) \times 6 = 36.853$

(c) $\frac{95N}{100} = 76 \Rightarrow P_{95} = 36.5 + \left(\frac{76-47}{17}\right) \times 6 = 41.09$

Example 19

Draw the cumulative frequency polygon (or ogive) for the following frequency distribution and determine the quartiles and the 60, 80 and 95 percentiles.

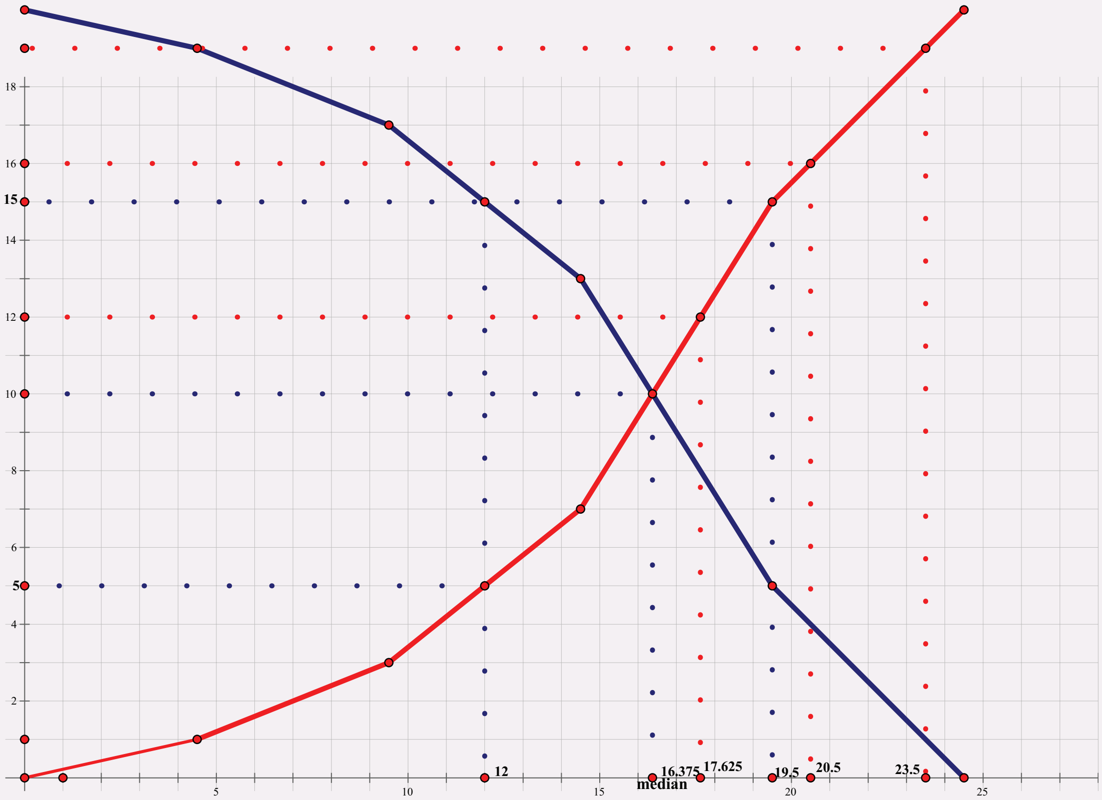
Contribution	0–4	5–9	10–14	15–19	20–24
Frequency (f)	1	2	4	8	5

Solution

If the cumulative frequencies are plotted against the class – boundaries and successive points are joined by line segments we get a graph called frequency polygon (or ogive).

In particular, the cumulative frequencies from below are plotted against the upper class - boundaries and the cumulative frequencies from above are plotted against the lower class – boundaries.

Class – Boundary		0–4.5	4.5–9.5	9.5–14.5	14.5–19.5	19.5–24.5
Frequency (*f*)		1	2	4	8	5
Cumulative Frequency	**More Than**	1	3	7	15	20
	Less Than	20	19	17	13	5



Clearly, $N = 20$

(i) Quartiles. Using frequency polygon, we have:

(a) $\frac{N}{4} = 5$ shows $Q_1 = 12$

(b) $\frac{N}{2} = 10$ shows $Q_2 = 16.375$

(c) $\frac{3N}{4} = 15$ shows $Q_3 = 19.5$

(ii) Percentiles. Using frequency polygon, we have:

(a) $\frac{60N}{100} = 12$ shows $P_{60} = 17.625$

(b) $\frac{80N}{100} = 16$ shows $P_{80} = 20.5$

(c) $\frac{95N}{100} = 19$ shows $P_{95} = 23.5$

Exercises

1. For each of the following distributions find
   (a) The quartiles
   (b) The deciles D_2, D_5 and D_7
   (c) The percentiles P_{28}, P_{69} and P_{85}
   (d) The mean, median and mode
   (e) The range, variance and standard deviation
2. 23, 14, 21, 15, 27, 25, 19, 30, 36
3. 10, 19, 9, 15 , 2, 5, 14, 13, 21, 4, 25, 7
4.

10% marks	3	5	8	10
Number of students	7	8	6	4

5.

Daily reading time in minutes	30	70	90	110	120	160	190	200
Number of students	3	5	11	10	7	6	4	20

6.

100% Marks	40–49	50–59	60–69	70–79	80–89
Number of students	5	8	16	14	2

7.

Age in years	10–14	15–19	20–24	25–29	30–34	35–34	40–44	45–49	50–54
Number of persons	16	20	24	18	22	16	12	5	7

8. Find
   (a) The mean, median and mode.
   (b) The quartiles
   (c) The deciles; D_3, D_7 and D_9
   (d) The percentiles; P_{20}, P_{72} and P_{94}

(e) The range, variance and standard deviation for the following grouped data.

80 – 82	83 – 85	86 – 88	89 – 91	92 – 94	95 – 97	98 –100	101 – 103
9	10	11	20	17	14	13	6

9. Consider the following frequency distribution for the daily income of 40 workers in Liberian Dollar. Find

Income (in LRD)	Number of workers *f*
30 – 39	2
40 – 49	4
50 – 59	13
60 – 69	12
70 – 79	6
80 – 89	3

Find

(a) The mean median and mode

(b) The range, variance and standard deviation

10. Given the following frequency distribution

Age in years	30–33	34–37	38–41	42–45	46–45	50–53
Frequency	18	26	30	13	9	4

Find

(a) The quartile

(b) The deciles D_4, D_7 and D_8

(c) The percentiles P_{19}, P_{62}, P_{65} and P_{98}

(d) The median, mode and mean

(e) The range, variance and standard deviation

Key Terms

- Bar chart
- Commutative frequency
- Grouped data
- Made
- Mean
- Median
- Percentiles
- Pie chart
- Quartiles
- Quantitative data
- Qualitative data
- Un grouped data

Summary

- Quantitative data can be numerically described. Height, weight, age, etc. are quantitative.
- Qualitative data cannot be expressed numerically. Honesty, beauty, sex, love, religion, etc. are qualitative.
- A quantity which assumes different values is said to be a variable. A variable may be
  - continuous, if it can take any numerical value within a certain range. Some examples are height, weight, temperature.
  - discrete, if it takes only discrete or exact values. It is obtained by counting.
- Frequency means the number of times a certain value of a variable is repeated in the given data.
- A grouped frequency distribution is constructed to summarize a large sample of data.

  The appropriate class interval is given by

$$\text{Class interval} = \frac{\begin{pmatrix}\text{Largest value}\\ \text{in ungrouped data}\end{pmatrix} - \begin{pmatrix}\text{Smallest value}\\ \text{in ungrouped data}\end{pmatrix}}{\text{Number of class required}}$$

- A measure of location is a single value that is used to represent a mass of data. The common measures of location are mean, median, mode, quartiles,

$$\text{Mean } (\bar{x}) = \frac{\sum_{i=1}^{n} x_i}{n} \text{ for raw data}$$

$$= \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} \quad \text{for discrete data}$$

$$= \frac{\sum_{i=1}^{n} f_i m_i}{\sum_{i=1}^{n} f_i} \quad \text{for grouped data } (m = \text{class mark})$$

- Median of ungrouped data is given by

$$\left.\begin{aligned} M_d &= \left(\frac{(n+1)^{th}}{2}\text{item}\right), \text{ if } n \text{ is odd} \\ &= \frac{\left(\frac{n}{2}\right)^{th}\text{item} + \left(\frac{n}{2}+1\right)^{th}}{2}\text{item}, \quad \text{if } n \text{ is even} \end{aligned}\right\}$$

(After data is arranged in increasing or decreasing order of magnitude.)

- Median for a grouped data is given by $M_d = B_L + \left(\frac{\frac{n}{2} - cf_b}{f_c}\right)i$

- Mode is the value with the highest frequency.
- If a distribution has a single mode it is "unimodal". If it has two modes, it is "bimodal". If it has more than two modes, it is called "multimodal".

- For grouped frequency distributions, the mode is given by

$$M_o = B_L + \left(\frac{d_1}{d_1 + d_2}\right)i$$

- Quartiles for grouped frequency distributions are given by $Q_R = B_L + \left(\frac{\frac{tn}{4} - cf_b}{f}\right)i$

Exercises

1. Construct a grouped frequency distribution table for the following data:

13	1	18	21	2	5	15	17	3	20
15	5	16	12	4	2	1	5	12	10
22	13	18	16	15	9	8	7	6	12
24	16	3	13	17	15	15	4	3	12

Hint: Use 8 classes.

2. Find the mode(s) of each of the following scores

(a) 10, 4, 3, 6, 4, 2, 3, 4, 5, 6, 8, 10, 2, 1, 4, 3

(b) 4, 3, 2, 4, 6, 5, 5, 7, 6, 5, 7, 3, 1, 7, 2

(c)

x	20–40	40–60	60–80	80–100	100–120	120–140	140–160	160–180	180–200
f	6	9	11	14	20	15	10	8	7

3. Find the median of each of the following scores

(a) 2, 3, 16, 5, 15, 38, 18, 17, 12

(b) 3, 2, 6, 8, 12, 4, 3, 2, 1, 6

(c)

x	300–309	310–319	320–329	330–339	340–349	350–359	360–369	370–379
f	9	20	24	38	48	27	17	6

4. Find the mean of each of the following scores

(a) 12, 8, 7, 10, 6, 14, 7, 6, 12, 9

(b) 2.1, 6.3, 7.1, 4.8, 3.2

(c)

x	12	13	14	15	16	17	18	20
f	4	11	32	21	15	8	5	4

(d) Find the mean score of 30 students with the following scores in mathematics

Score	Number of students
40 – 49	2
50 – 59	0

Score	Number of students
60 – 69	6
70 – 79	12
80 – 89	8
90 – 99	2

5. Find Q_1, Q_2 and Q_3 of the following.

x	2.5	7.5	12.5	17.5	22.5
f	7	18	25	30	20

M12CH05

CHAPTER 5

STANDARD DEVIATION

Chapter Contents

Chapter Outcomes

Upon completion of this chapter, learners will:

- define, discuss and identify dispersion;
- define, discuss and calculate deviation;
- define and calculate standard deviation.

The average (mean, median or mode) gives a general idea of the size of the data, but two sets of numbers can have the same mean while being very different in other ways. The other main statistic we need to find is a measure of dispersion or spread. There are several ways of measuring dispersion.

Range

The range is the easiest measure of dispersion to calculate. It is defined as **the difference between the highest value and the lowest value**. The range is a crude measure of dispersion since it makes no use of the intermediate values and it can be distorted by one or two extreme values.

Range = the highest value – the lowest value

Inter-quartile range

The inter-quartile range is the difference between the third quartile and the first quartile. Inter-quartile range (IR) = $Q_3 - Q_1$

Quartile Deviation (*QD*)

The Quartile Deviation is half of the inter-quartile range; i.e., $QD = \dfrac{Q_3 - Q_1}{2}$

Example 1

A student's marks in ten subjects in two sets of tests are given below.

Test 1	22	28	20	19	20	24	23	20	24	20
Test 2	13	15	36	11	18	30	23	8	32	34

Find, for each set of tests:

(a) the range

(b) the inter-quartile range

(c) The quartile deviation

Solution

For Test 1:

(a) Highest value = 28 and lowest value = 19. Therefore, range = 28 – 19 = 9.

(b) Arranging the data in increasing order: 19, 20, 20, 20, 20, 22, 23, 24, 24, 28.

$Q_1 = 20$ $\quad Q_2 = 21$ $\quad Q_3 = 24$

Thus the inter-quartile range IR is given by ,

IR = $Q_3 - Q_1 = 24 - 20 = 4$

(c) The quartile-deviation (*QD*) is given by,

(d) $QD = \dfrac{Q_3 - Q_1}{2} = 2$

Solution

For Test 2:

(a) highest value = 36 and lowest value = 8. Therefore, range = 36 – 8 = 28.

(b) Arranging the data in increasing order: 8, 11, 13, 15, 18, 23, 30, 32, 34, 36.

$Q_1 = 13$ $\quad Q_2 = 20.5$ $\quad Q_3 = 32$

Thus the inter-quartile range IR is given by ,

$IR = Q_3 - Q_1 = 32 - 13 = 19$

(c) The quartile-deviation (QD) is given by,

(d) $QD = \dfrac{Q_3 - Q_1}{2} = 9.5$

Mean Deviation is defined as the average or mean of the absolute value of deviations of each value from the mean.

$MD = \dfrac{\sum_{k=1}^{n} |x_k - \bar{x}|}{n}$, where $x_1, x_2 \ldots x_n$ are the n individual values and is the mean which is given by,

$$\bar{x} = \frac{\sum_{k=1}^{n} x_k}{n}$$

Example 2

Find the mean-deviation of 8, 9, 7, 6, 8, and 10.

Solution

$$\bar{x} = \frac{\sum_{k=1}^{n} x_k}{n} = \frac{8+9+7+6+8+10}{6} = 8$$

$$MD = \frac{\sum_{k=1}^{n} |x_k - \bar{x}|}{n} = \frac{|8-8|+|9-8|+|7-8|+|6-8|+|8-8|+|10-8|}{6} = \frac{6}{6} = 1$$

Variance (Var)

$$Var = \frac{\sum_{k=1}^{n} (x_k - \bar{x})^2}{n}$$

where $x_1, x_2, \ldots x_n$ are the n individual values and is the mean.

Standard deviation (σ) is the square root of the variance.

$$\sigma = \sqrt{Var}$$

Exercises

1. A survey recorded the weight of students in a class. The data were as follows: 41, 44, 45, 46, 47, 47, 48, 49, 41, 43, 54, 44, 42, 45, 46, 47, 47, 48, 49, 50, 52. Find the range, and the quartiles connected to the data.
2. Find the range, and the quartiles connected to the following data.
   (a) 7, 5, 10, 20, 13, 8, 2
   (b) 27, 60, 25, 43, 38, 26, 44, 23, 31, 46
   (c) 92, 87, 86, 77, 83, 70, 80, 66, 64, 96, 100, 99, 94, 68
3. Find the range, median, inter-quartile range, and deviation quartile of the following data:
   (a) 88, 67, 64, 76, 86, 85, 82, 81, 68
   (b) 51, 38, 34, 37, 25, 45, 22, 41, 75, 49
4. Find the mean deviation of the data: 3, 2, 1, 2, 2, 1, 4, 5
5. Find the variance and the standard deviation of the data: 4, 5, 6, 7, 8, 6

You remember that the range, variance and standard deviation are used to measure how the values are close to each other or how the values widely spread farther apart. Also in this topic these measures of variation are used to measure the variability in a grouped data

The range, variance and standard deviations

1. Range for a grouped data

For a grouped data, we assume that the lowest value is the lower class boundary of the lowest class and the highest value is the upper class boundary of the highest class.

Range for a grouped data is the difference between the upper class boundary of the highest class and the lower class boundary of the lowest class.

2. Variance for a grouped Data

The variance for a grouped data is the average of the squared deviation of the individual class marks from the grouped data mean i.e The variance for a grouped data is

$$var = \frac{\sum f_1\left(M_i - \bar{x}\right)}{\sum f_1};\ \text{where}$$

M_i the class mark (or midpoint) of the i^{th} class.

$\bar{x}$ = the mean for the grouped data.

f_i = the frequency of the i^{th} class

3. Standard deviation for a grouped data

The standard deviation for a grouped data is the square root of the variance.

The standard deviation, S.d, $S = \sqrt{var}$ or $S^2 = var$.

Example 3

1. Find the range, variance and standard deviation for the following grouped data.

x	1 – 7	8 – 14	15 – 21	22 – 28	29 – 35
f	1	2	2	2	1

Solution

Range = 35.5 – 0.5 = 35

Variance

x	1 – 7	8 – 14	15 – 21	22 – 28	29 – 35
M_i	4	11	18	25	31
f	1	2	2	2	1
f_iM_i	4	22	36	50	32

$$\bar{x} = \frac{4 + 22 + 36 + 50 + 32}{8} = \frac{144}{8} = 18$$

$$var = \frac{\sum f_i\left(M_i - \bar{x}\right)^2}{\sum f_i}$$

$$= \frac{1\times(4-18)^2 + 2\times(11-18)^2 + 2\times(18-18)^2 + 2\times(25-18)^2 + 1\times(32-18)^2}{8}$$

$$= \frac{196+98+0+98+196}{8} = 73.5$$

Standard deviation $= \sqrt{73.5}$

Note: The range depends on the extreme boundary values. If there has no access to the original data, the range is misleading.

KEY TERMS

- Dispersion
- Range
- Mean deviation
- Variance
- Standard Deviation
- Inter - quartile range
- Quartile deviation

SUMMARY

- Dispersion is used to demonstrate the extent to which the individual item in the distribution varies from the average.
- The different measures of dispersion are Range, Inter - quartile range, Quartile deviation, Mean deviation, Variance and Standard Deviation.
  - Range = $x_{max} - x_{min}$
  - Inter-quartile range (IR) = $Q_3 - Q_1$
  - The Quartile deviation is half of the inter-quartile range; $QD = \frac{Q_3 - Q_1}{2}$
  - $MD = \frac{\sum_{k=1}^{n}|x_k - \bar{x}|}{n}$, where $x_1, x_2 \ldots x_n$ are the n individual values and is the mean which is given by, $\bar{x} = \frac{\sum_{k=1}^{n} x_k}{n}$
  - Variance = $\frac{\sum_{i=0}^{n}(x_i - \bar{x})^2}{n}$
  - Standard deviation (S) is the positive square root of variance, $S = \sqrt{\text{Variance}}$

EXERCISES

1. Find the range, quartile deviations, mean deviation, variance and standard deviation for each of the following data.
   (a) 18, 2, 4, 6, 10, 7, 9, 11
   (b) 3, 4, 5, 5, 6, 7, 7, 7

(c)

x	31	35	36	40	42	50
f	7	8	2	12	6	3

(d)

Class	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80	80 – 90
Frequency	8	10	16	14	10	12

2. Why do we study measures of dispersion?
3. If the standard deviation of $x_1, x_2, x_3, \ldots, x_n$ is 3, then what is the standard deviation of $2x_1 + 3, 2x_2 + 3, \ldots, 2x_n + 3$?
4. The standard deviation of the temperature for one week in a certain city is zero. What can you say about the temperature of that week?
5. Two basketball players scored points for their team. The scores were recorded for 9 games as follows:

Player *A*	3	4	5	6	7	8	9	10	11
Player *B*	4	3	5	6	7	8	9	9	1

   (a) Calculate the standard deviation of the points of each player.

   (b) Which player, *A* or *B*, is more consistent in scoring points for his team? How do you know?

6. Consider the following raw data representing yield of Barley (in quintals) of three farmers from their respective hectare of land for consecutive 8 years.

Farmer 1	12	14	11	13	17	18	12	13	11
Farmer 2	14	13	15	13	14	13	15	13	13
Farmer 3	12	5	14	3	17	8	4	12	13

   (a) Determine the range, quartile deviations, mean deviation, variance and standard deviation of each of the three farmers.

   (b) Who of the farmers has higher variation in yield? What does this tell?

   (c) Who of th farmers has lesser variation in yield?

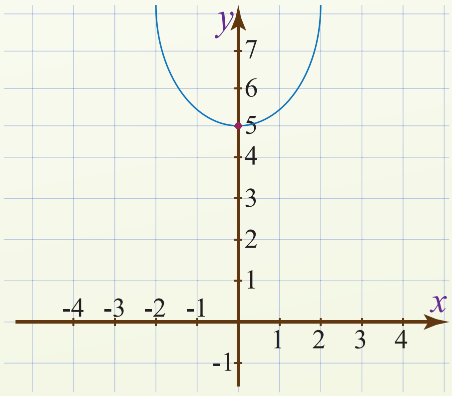
   (d) Who of the farmers has consistent yield?

M12CH06

CHAPTER 6

INTERPRETATION OF LINEAR AND QUADRATIC GRAPHS

Chapter Contents



Chapter Outcomes

Upon completion of this chapter, learners will:

- solve simultaneous equations; one linear and one quadratic, using graphs.
- use a quadratic graph to solve related equations.
- find the range of values of x for which y is increasing or decreasing.
- find the range of values of x for which y is positive or negative.

6.1 Graphing of Simultaneous Equations: One Linear and One Quadratic

Everyone knows how to sketch the graph of linear and quadratic functions at this level. So you are asked to sketch the graph of linear and quadratic functions in the following activities.

Activity 1

1. Sketch the graph of
   (a) $y = 3x + 2$
   (b) $y = x^2$
   (c) (a) and (b) on the same coordinate system.
   (d) Find the coordinates of the intersection point (s) of the two graphs.
   (e) What does the intersection point(s) tell you?
   (f) Substitute the coordinates in both equations. Do they satisfy both equations?
2. Sketch the graph of the following pairs on the same coordinate axes and find the intersection point(s).
   (a) $y = 2x - 3,\quad y = 2x^2 + 3$
   (b) $y = x - 3,\quad y = 2x^2 - 8$
   (c) $y = -2x + 3,\quad y = -2x^2 + 4$
   (d) $y = x + 3,\quad y = x^2 + 5x - 2$

Solving simultaneous equations graphically is the process that allows us to solve two or more algebraic equations that share variables by sketching their graphs.

The point (or points) of intersection give(s) the solution(s) to the simultaneous equations.

This is because at the point of intersection the two equations are equal to one another and therefore, the values of the variables are the same for both equations.

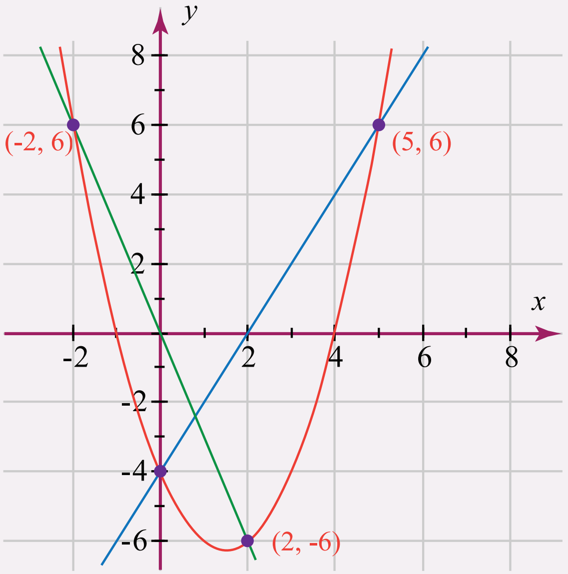
Note on how to solve simultaneous equations graphically

1. Identify if the equations are linear or quadratic (or a mix of both)
2. Draw each equation on the same set of axes
3. Find the coordinates where the curves intersect
4. State the values of the variable where the lines intersect and clearly state your answer *(if you have multiples values of a variable ensure you match the correct pair)*

Example 1

Using the following graph, find the solutions of the following pairs of equations.

(a) $\begin{cases} y = x^2 - 3x - 4 \\ y = 2x - 4 \end{cases}$ (b) $\begin{cases} y = x^2 - 3x - 4 \\ y = -3x \end{cases}$



Solution

The intersection points of the line and the parabola are the solutions of the equations.

(a) The parabola and the line intersect at (0, –4) and (5, 6)

Therefore, the solution set is {(0, –4), (5, 6)}

(b) The parabola and the line intersect at (2, –6) and (–2, 6)

Therefore, the solution set is {(2, –6) and (–2, 6)}

Example 2

Solve the following simultaneous equations

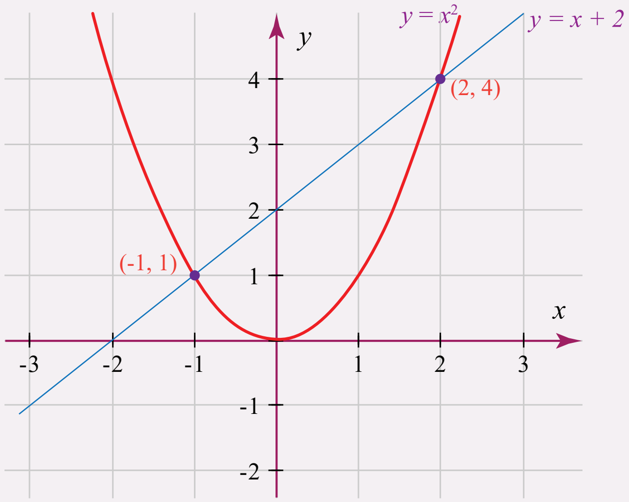
(a) $\begin{cases} y = x^2 \\ y = x + 2 \end{cases}$ (b) $\begin{cases} y = -x^2 + 2x + 1 \\ y = x - 5 \end{cases}$

Solution

The first step is to draw the parabola and the line on the same coordinate plane.

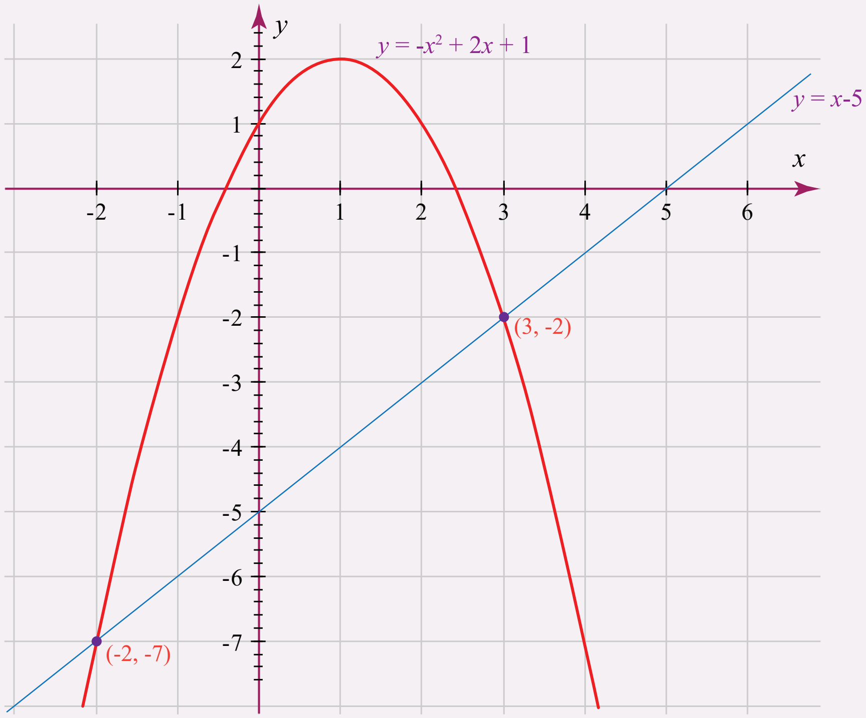
The solution is given by the point or the points of intersections of the line and the parabola representing the equations.

(a)



From the graphs, the line and the parabola intersect at (–1, 1) and (2, 4).
Therefore, the solution set is {(–1, 1), (2, 4)}.

(b)



Clearly, the graphs intersect at (–2, –7) and (3, –2).
Therefore, the solution set is {(–2, –7), (3, –2)}.

Linear and quadratic equation appears in several application problems. You can see the following problems as an application of quadratic equations.

1. A ball is thrown straight up, from 3 m above the ground, with a velocity of 14 m/s. When does it hit the ground? (Hint: ignore air resistance and take acceleration due to gravity equal to 9.8 m/s^2 approximately 10 m/s^2)

Solution

Let t be the time required in seconds. Then the height h of the ball is given by,

$$h(\text{t}) = 3 + 14t - 5t^2$$

Observe that the height starts at 3 m, it travels at 14 m/s and gravity pulls it down, changing its position by about 5 m/s^2.

The ball hits the ground when the height is zero. Hence, solving $3 + 14t - 5t^2 = 0$, you get

$$t = -0.2 \text{ or } t = 3$$

Since negative time is impossible, the time required is $t = 3$ s.

Therefore, the ball hits the ground after 3 seconds!

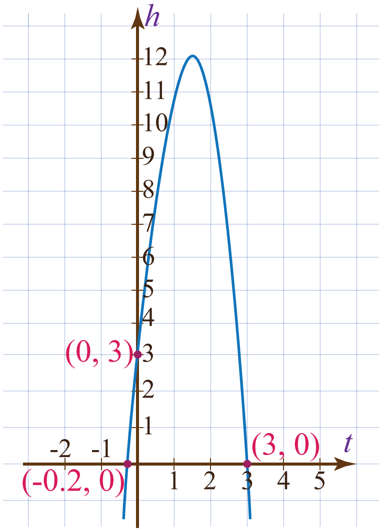


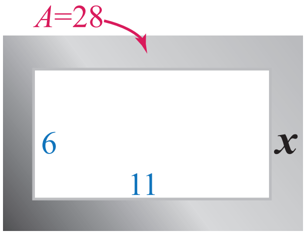
Figure 1.

2. A company is going to make frames as part of a new product they are launching. The frame will be cut out of a piece of steel, and to keep the

weight down, the final area should be 28 cm^2. The inside of the frame has to be 11 cm by 6 cm. What should the width x of the metal be?

Solution

Consider the following as small steel frame:



Area of steel before cutting is given by,

Area = $(11 + 2x) \times (6 + 2x)$ cm^2

Area = $66 + 22x + 12x + 4x^2$

Area = $4x^2 + 34x + 66$

Area of steel after cutting out the 11×6 middle is given by,

Area = $4x^2 + 34x + 66 - 66$

Area = $4x^2 + 34x$

Since the final area is given to be 28 cm^2, solve, $4x^2 + 34x = 28$, which is equivalent to,

$$4x^2 + 34x - 28 = 0.$$

So approximately x is about -9.3 or 0.8 (use calculator). The negative value of x make no sense, so the answer is $x = 0.8$ cm (approximately) The negative value of x makes no sense, so the answer is:

$x = 0.8$ cm (approx.)

Finding the range of values of *x* for which *y* is increasing or decreasing/ Finding the range of values of *x* for which *y* is positive or negative

Domain: The function $f(x) = x^2 + 5$ is a quadratic function. It is defined for all values of x since there is no restriction on the value of x. Therefore, its domain is "all real values of x".

Range: Since x^2 is never negative, the function is never less than 5. Therefore, its range is "all real numbers greater than or equal to 5."

As we see from the graph the value of the function y is decreasing as x increases from left to right up to zero; and as x increases from zero to the right y is increasing. So for the value of x

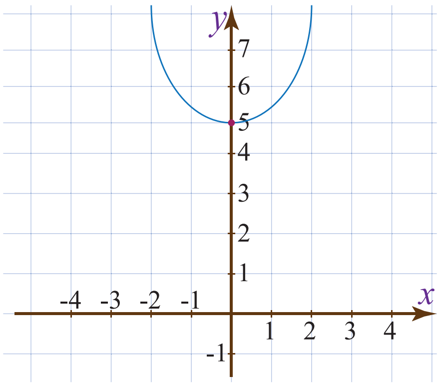


Figure 2.

in the interval, y is decreasing and for the value of x in the interval the value of y is increasing. In all cases the value of y is positive.

Domain: The function is a quadratic function. It is defined for all values of x since there is no restriction on the value of x. Therefore, its domain is "all real values of x".

Range: Since is negative for each real value of x, the function is never greater than 9. Therefore, its range is "all real numbers less than or equal to 9."

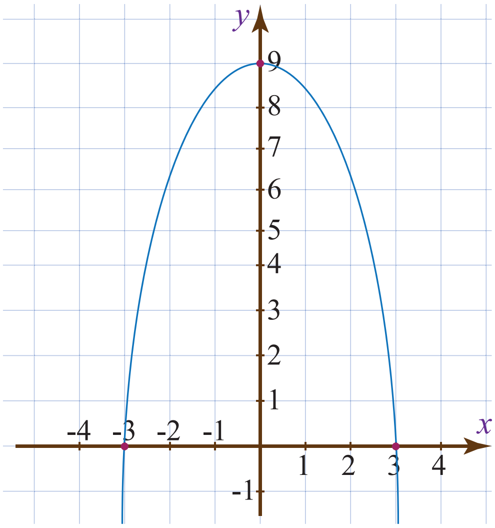


Figure 3

As we see from the graph the value of the function y is increasing as x increases from left to right up to zero; and as x increases from zero to the right y is decreasing. So for the value of x in the interval, y is increasing and for the value of x in the interval the value of y is decreasing. Observe that the value of y is nonnegative for the real value of x in the interval [–3, 3] and negative outside this interval.

Exercises

1. Solve graphically the simultaneous equations:

   (a) $y = x^2$ and $y = 2x + 15$

   (b) $y = 2x^2$ and $x = 20 - 3x$

2. Solve algebraically the simultaneous equations:

   (a) $y = 5x - 1$ and $y = (x + 1)^2$

   (b) $2x + y = 6$ and $x^2 + y^2 = 20$

3. A 3 hour river goes 15 km upstream and then back again. The river has a current of 2 km an hour. What is the boat's speed and how long was the upstream journey?
4. Sketch the graph of the following quadratic functions and find the range of values of x for which y is increasing or decreasing. Find also the range of values of x for which y is positive or negative.
   (a) $y = x^2 + 5x + 6$
   (b) $y = -x^2 + 2x + 10$

Key Terms

- Graph
- Linear equation
- Parabola
- Quadratic equation
- Simultaneous equation
- Solution set

Summary

- The graph of a linear function is a straight line.
- The graph of a quadratic function is a parabola.
- The intersection point of two graphs is the solution of the simultaneous equation.
- If the two graphs has no intersection point, the solution set of the simultaneous equation is empty set.

Exercises

1. Solve graphically the simultaneous equations:
   (a) $y = 3x - 1$ and $x^2 + y^2 = 5$
   (b) $x^2 + y^2 = 20$ and $y = 10 - 2x$
2. Solve algebraically the simultaneous equations:
   $2x - y = 7$ and $x^2 + y^2 = 34$
3. Two resistors are in parallel, like in the following diagram:
   The total resistance has been measured at 2 Ohms, and one of the resistors is known to be 3 Ohms more than the other. What are the values of the two resistors?

4. Sketch the graph of the following quadratic functions and find the range of values of x for which y is increasing or decreasing. Find also the range of values of x for which y is positive or negative.

   (a) $y = x^2 - 7x + 12$

   (b) $y = -x^2 - 8x + 15$

M12CH07

CHAPTER 7

MENSURATION

Chapter Contents

Chapter Outcomes

Upon completion of this chapter, learners will:

- calculate the surface area of prisms;
- calculate the volume of prisms;
- calculate the total surface area of a cone;
- calculate the volume of a cone;
- calculate total surface area of pyramids;
- calculate the volume of pyramids;
- calculate the surface area of sphere;
- calculate the volume of sphere;
- calculate the distance along a given latitude and longitude.

Mensuration is about the length, volume or area of different geometric shapes. These shapes exist in 2 dimension or 3 dimensions. This unit is concerned only about mensuration in three dimensional. The three dimensional (3D) shapes can be described in terms of their faces, vertices and edges. Face is a flat curved surface; edge is a line where two faces meet; and vertex is a point where three or more edges meet. In this unit we consider mensurations of the following three dimensional shapes: Prisms, Cones, pyramids, and Sphere.

Activity 1

1. Describe the following terms by the number of faces, edges vertices and diagonals. Prism, cuboids, cube, cylinder
2. What are the special cases of cuboids?
3. Are cuboids, cubes and cylinders prisms?

A prism is any three dimensional geometric figure which has a uniform cross-section. A cross-section of the figure is parallel to the base such that the resulting figure has the same shape and size as that of the base of the figure. The cross-section can be triangular, rectangular, pentagonal, hexagonal, circular or combinations of shapes.

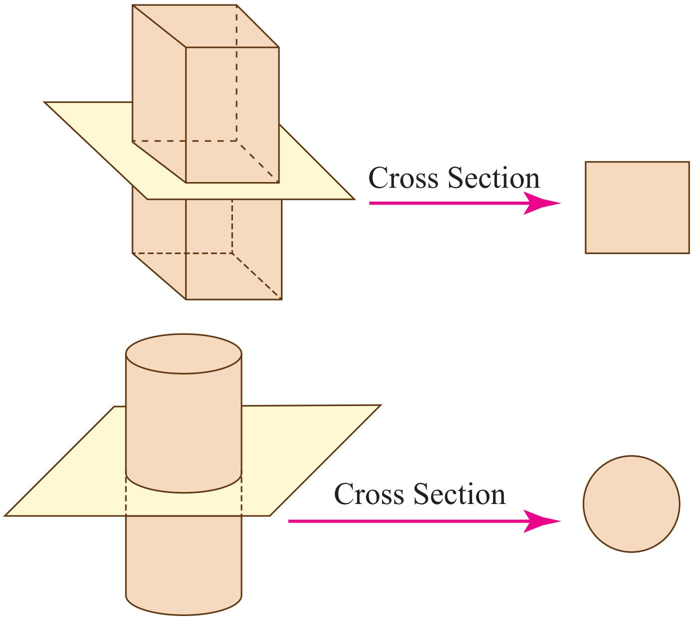


Figure 1.

Prisms are named by the shape of the base. If the base is,

1. Triangular it is called triangular prism.
2. Rectangular it is called rectangular prism (or Cuboids).
3. Parallelogram it is called Parallelepiped.
4. A square it is called a cube or square prism.
5. Circular it is called cylinder.
6. Pentagonal it is called pentagonal prism.
7. Hexagonal it is called hexagonal prism.

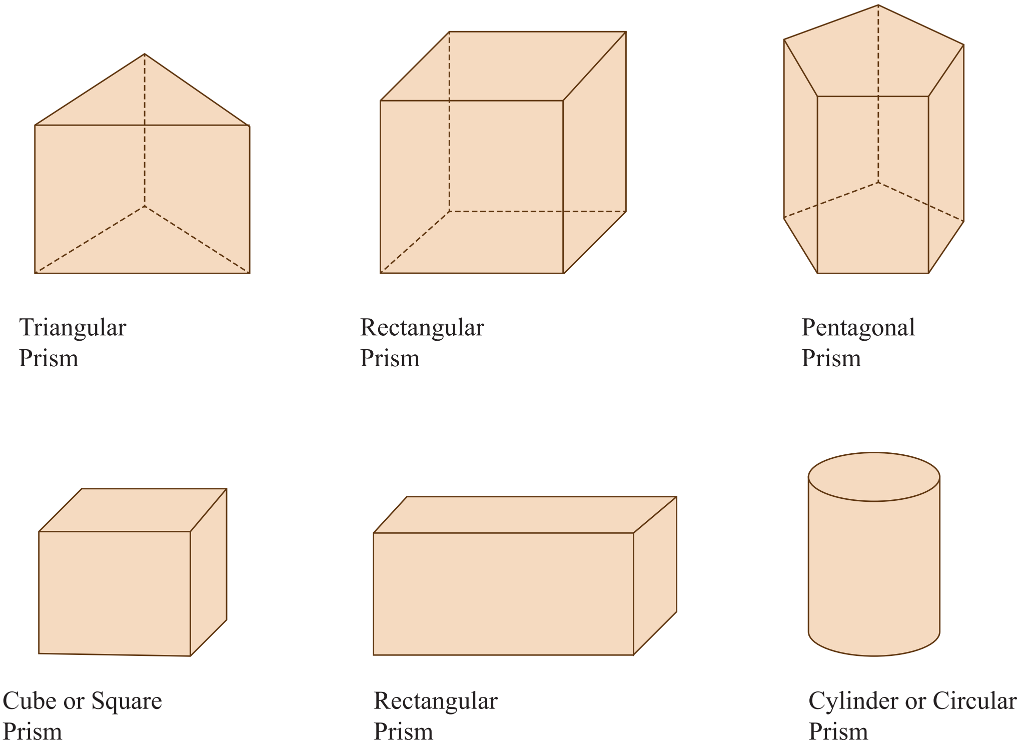


Figure 2.

Prisms: The sides of the base are said to be edges of the base. The lateral edges and edges of the base are said to be edges of the prism. The height of the prism is the distance between the bases. Prisms are classified as right prisms and oblique prisms.

A Right prism is a prism whose lateral edges are perpendicular to the bases.

An oblique prisms is a prism whose lateral edges are not perpendicular to the bases.

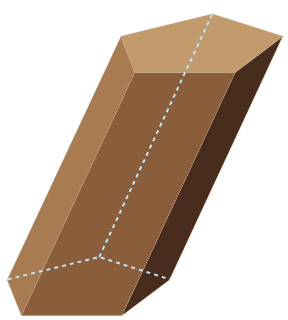


Figure 3.

Base area A_B of a prism is the area of the polygon which is the base of the prism. Lateral surface area A_L of a prism is the sum of the areas of the lateral faces.

Therefore, the total surface area A_T of a prism and its volume V are given by,

$A_T = A_B + A_L$

$V = hA_B$, where h is the height of the prism.

Example 1

In a triangular prism, let the lengths of sides of its bases be 3 cm, 4 cm, and 5 cm. Find the volume of the prism if it has an altitude of 6 cm.

Solution

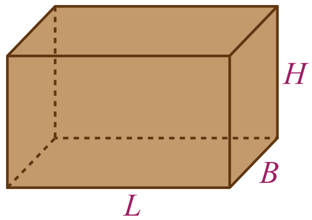
The area of the base of the prism is the area of a right triangle whose legs are 3 cm and 4 cm. Which is $\frac{1}{2} . 3.4 \text{ cm}^2 = 6$.

$\therefore\ V = A_B \times h = 6 \times 6 \text{ cm}^3 = 36 \text{ cm}^3$

Cuboid

Let the length, breadth and height of the cuboid be 'L', 'B' and 'H' respectively.

- Volume = $L \times B \times H$
- Curved Surface area = $2\, H\, (L + B)$
- Total surface area = $2\, (L\, B + B\, H + H\, \text{L})$
- Length of diagonal = $(L^2 + B^2 + H^2)^{1/2}$



Example 2

Find the length of the largest rod that can be kept in a cuboidal room of dimensions $10 \times 15 \times 6$ m.

Solution

Largest rod would lie along the diagonal.

$\Rightarrow$ Length of largest rod = Length of diagonal of the room = $(L^2 + B^2 + H^2)^{1/2}$

$\Rightarrow$ Length of the largest rod = $(10^2 + 15^2 + 6^2)^{1/2} = (100 + 225 + 36)^{1/2} = (361)^{1/2}$

$\Rightarrow$ Length of the largest rod = 19 m

Example 3

Find the number of bricks of dimension 24 cm × 12 cm × 8 cm each that would be required to make a wall 24 m long, 8 m high and 60 cm thick.

Solution

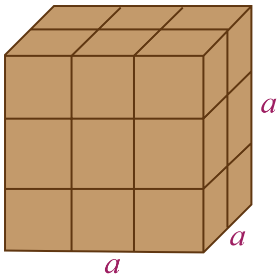
Volume of 1 brick = $24 \times 12 \times 8 = 2304$ cm^3

Volume of wall = $2400 \times 800 \times 60 = 115200000$ cm^3

Therefore, number of bricks required = 115200000 / 2304 = 50000

Cube

Let the side of the cube be 'a'



- Volume = a^3
- Curved Surface area = 4 a^2
- Total surface area = 6 a^2
- Length of diagonal = a

Example 4

If the diagonal of cube is $\sqrt{18}$ cm, then its volume is

Solution

Let the side of the cube is x cm. We know diagonal of cube = $a\sqrt{3}$ cm. Put equal both

$a\sqrt{3} = \sqrt{18}$ Squaring both sides $a^2(3) = 18$

$a^2 = 6$

$a = \sqrt{6}$

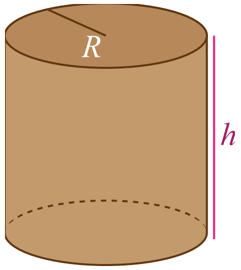
Volume of the cube = a^3

$= \left(\sqrt{6}\right)^3$

$= 6\sqrt{6}\ \text{cm}^3$

Cylinder (Right circular cylinder)

Let the radius of the base and height of the right circular cylinder be 'R' and 'H' respectively.



- Volume = $\pi R^2 H$
- Curved Surface area = $2 \pi R H$
- Total surface area = $2 \pi R H + 2 \pi R^2$

EXAMPLE 5

A rectangular sheet of paper measuring 22 cm × 7 cm is rolled along the longer side to make a cylinder. Find the volume of the cylinder formed.

Solution

Let the radius of the cylinder be 'R'. The sheet is rolled along the longer side.

$\Rightarrow 2 \pi R = 22$

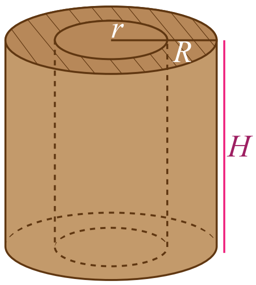
$\Rightarrow R = 3.5$ cm

Also, height = 7 cm

Therefore, volume of the cylinder = $\pi R^2 H = \pi (3.5)^2\, 7 = 269.5\ \text{cm}^3$

Hollow cylinder (Hollow right circular cylinder)

Let the inner radius of the base, outer radius of the base and height of the hollow right circular cylinder be 'r', 'R' and 'H' respectively.



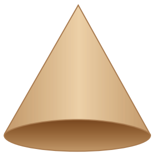
- Volume = $\pi H (R^2 - r^2)$
- Curved Surface area = $2 \pi R H + 2 \pi r \text{H}$

  $= 2 \pi H (R + r)$
- Total surface area = $2 \pi H (R + r) + 2 \pi (R^2 - r^2)$

Exercisess

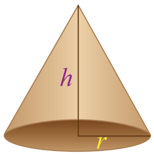
1. The areas of the three non-parallel faces of a right rectangular prism are in the ratio 15:9:4 and its volume is 300. Determine the lengths of its sides.
2. A water tank is supplied by a pipe of diameter 50 cm. The pipe is replaced by K pipes each of which has diameter 10 cm. Find K.
3. The dimension of a right rectangular prism are x, $x - 1$, and $x + 1$. Find x if its volume is 24 cm^3.
4. The circumference of the base of a right circular cylinder is 20π cm. If the altitude of the cylinder is 12 cm, find its volume and total surface area.
5. The volume and total surface area of a right circular cylinder is 324π cm^3 and 180π cm^2, respectively. Find its height and radius of the base.

A cone is a distinctive three dimensional geometric figure that has a flat surface and a curved surface, pointed towards the top. The pointed end of the cone is called the apex, whereas the flat surface is called the base. This is what a cone looks like:

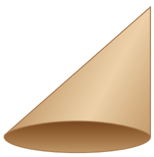


A cone can be of two categories, depending upon the position of the vertex on the base:

- A right circular cone is one whose apex is perpendicular to the base, which means that the perpendicular line falls exactly on the center of the circular base of the cone. In the image below, h represents the height of the cone, and r is the radius.

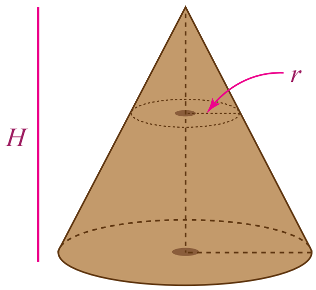


- If the position of the vertex is anywhere apart from the center of the base, then it is an oblique cone.



Cone

Let the radius of the base, slant height and height of the cone be 'R', 'L' and 'H' respectively.



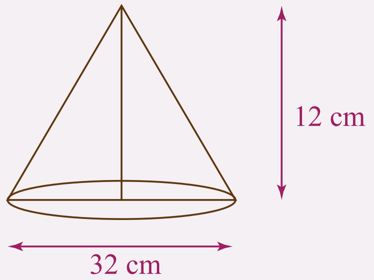
- $L^2 = R^2 + H^2$
- Volume $= \frac{1}{3}\pi R^2 H$
- Curved Surface area $= \pi R L$
- Total surface area $= \pi R L + \pi R^2$

Example 6

The height of a conical tank is 12 cm and the diameter of its base is 32 cm. The cost of painting if from outside at the rate of 21 LRD per sq. m. is

Solution

We have to find slant height(l) of the cone. Radius of cone = 32/2 = 16 cm



$l = \sqrt{(r^2 + h^2)} = \sqrt{(16^2 + 12^2)} = \sqrt{400} = 20$ cm

Cost of painting = Surface area of cone × 21 = πrl × 21

= 22/7 × 16 × 20 × 21 = 21120

Hence, the cost of painting is 21120 LRD.

Exercises

1. Find the volume and total surface area of a cone of height 5 cm and diameter of the base 24 cm.
2. Find the volume and total surface area of a cone with altitude 8 cm and area of the base 36π cm^2.
3. The lateral area of a right circular cone is 65π cm^2. Find the volume of the cone if its base area is 25π cm^2.
4. The slant height of a right circular cone is 13 and the base radius is 12, then find the lateral area is of the cone.
5. The volume of a right circular cone is $256\pi\ u^3$. If the radius of the base is $3u$, then find the altitude is of the cone.
6. Find the lateral area of a right circular cone whose volume is $20\pi\ u^3$ and whose radius is $2u$.

A pyramid is a three-dimensional shape, with a polygonal base and flat triangular faces, which join at a common point called the apex. It is formed by connecting the bases to an apex. Each edge of the base is connected to the apex, and forms the triangular face, called the lateral face. If a pyramid has an n-sided base, then it has $n + 1$ faces, $n + 1$vertices, and $2n$ edges.

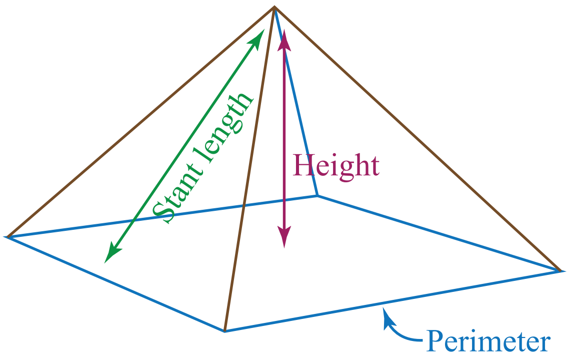
Based on the shape of the base of the pyramid, the pyramid is classified as a triangular pyramid, square pyramid, pentagonal pyramid, and so on.

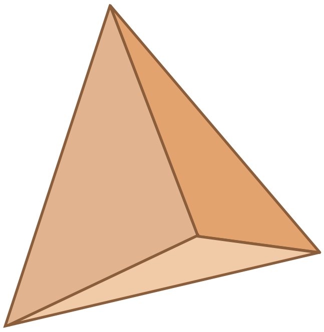
If the apex of the pyramid is directly above the centre of the base, it is called the right pyramid. Otherwise, the pyramid is called an oblique pyramid.

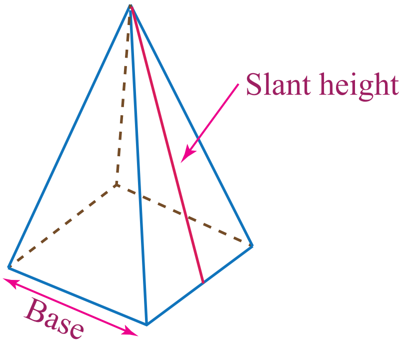
If the base of the pyramid is a regular polygon, the pyramid is a regular pyramid. Otherwise, it is called an irregular pyramid.

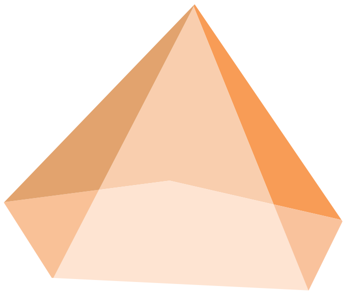
The volume of a pyramid is one-third of the product of the base area and the height of the pyramid.

$$\text{Volume} = 1/3\ (\text{Base Area})(\text{height})$$



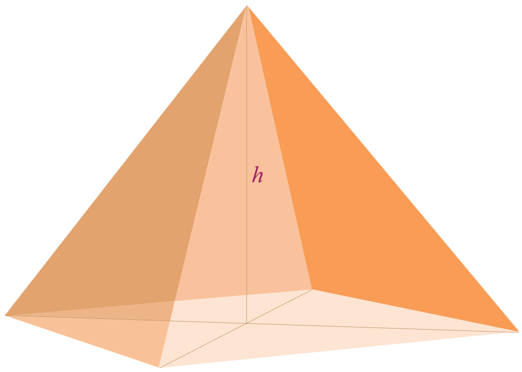






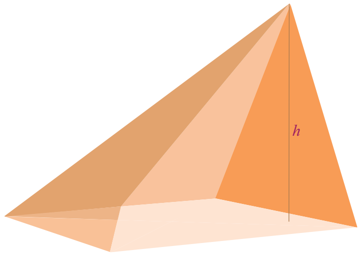
Right pyramid

The apex of this pyramid is exactly over the middle of the base, hence named as Right Pyramid.



Oblique pyramid

The apex of this pyramid is not exactly over the middle of its base and named as Oblique Pyramid.



The total surface area of a pyramid is the sum of the base area and half the product of the base perimeter and the slant height.

The total surface area of pyramid = $(½)Pl + A_B$ square units

Where,

"P" is the perimeter of the base

"l" is the slant height

"A_B" is the base area.

$$V = \frac{1}{3} A_B h$$

The general form to find the volume of the pyramid is one-third of the base area and the height of the pyramid.

The volume of the pyramid = (⅓) × (Base Area) × (Height) Cubic units.

Example 7

Find the volume of the square pyramid, if its base area is 56 cm^2 and its height is 9 cm.

Solution

The base area of the square pyramid = 56 cm^2, Height = 9 cm

Thus, the volume of the square pyramid = $\left(\frac{1}{3}\right)$(Base area)(Height) cubic units

Now, substituting the values in the formula, we get

The volume of a square pyramid = $\left(\frac{1}{3}\right)(56)(9)$

$V = (56)(3) = 168$ cm^3

Example 8

Find the total surface area of the square pyramid if each side of the base measures 16 cm, and the slant height is 17 cm, and the altitude is 15 cm.

Solution

As the base is a square, the perimeter of the base is 4 times 16 cm, $P = 4(16) = 64$ cm

The area of the base $= a^2 = 16^2 = 256\ \text{cm}^2$

We know, that the total surface area of the square pyramid is $(½)Pl + B$ square units

Substituting the values in the given formula, we get

The total surface area of the square pyramid $= [(½)(64)(17)] + (256)$

$$A_T = 544 + 256 = 800\ \text{cm}^2$$

EXERCISES

1. Find the volume and total surface area of a regular square pyramid if its slant height is 13 cm and the edge of the base is 10 cm.
2. Find the lateral area of regular square pyramid of volume $\frac{2}{3}x^3$ and altitude $2x$.
3. A regular square pyramid is dropped into a cube of side s which is half filled with water and the pyramid sank with vertex upward. If the surface of the water rose to $\frac{3}{4}$ of the altitude of the cube and the vertex of the pyramid reached the water surface, find the volume of the pyramid.

ACTIVITY 2

Describe the shapes of each of the following.

Earth	Balloon	Football
egg	Basketball	Orange

A sphere is a three dimensional figure which has a round shape. Any point on the sphere is equidistant from the fixed point. The fixed point is called the center of the sphere. The distance from the center to any point on the sphere is called the radius. It looks like the following.

If the radius of a sphere is R, then its surface area S_A and its volume V are given as follows:

$$S_A = 4\pi R^2$$

$$V = \frac{4}{3}\pi R^3$$

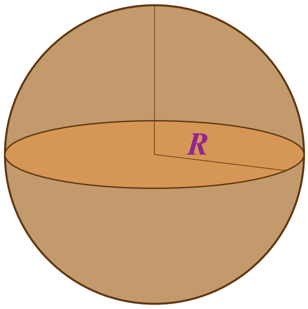


Figure 4. Sphere

If the radius of a hemisphere is R, then its surface area SA and its volume V are given as follows:

$$S_A = 2\pi R^2$$

$$V = \frac{2}{3}\pi R^3$$

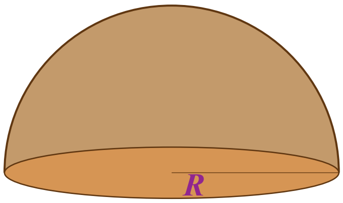


Figure 5. Hemisphere

Example 9

By melting a solid lead sphere of radius 6 cm, three small spheres are made whose radii are in ratio 3:4:5. The radius of the largest sphere is

Solution

Let the radius of small spheres is $3a$, $4a$ and $5a$.

$$V = \frac{4}{3}\pi R^3$$

Acc . to question

$$\frac{4}{3}\pi 6^3 = \frac{4}{3}\pi\ ((3a)^2 + (4a)^2 + (5a)^2)$$

$$216 = 27a^3 + 64a^3 + 125a^3$$

$$216 = 216a^3$$

$$a = 1$$

Hence, Radius of largest sphere is $5\times1 = 5$ cm.

Example 10

A sphere and a cylinder have equal volume and equal radius. The ratio of the curved surface area of the cylinder to that of the sphere is?

Solution

Let the radius of sphere is r.

Let the height of the cylinder is h.

Given volume of sphere = Volume of cylinder

$$\frac{4}{3}\pi r^3 = \pi r^2 h$$

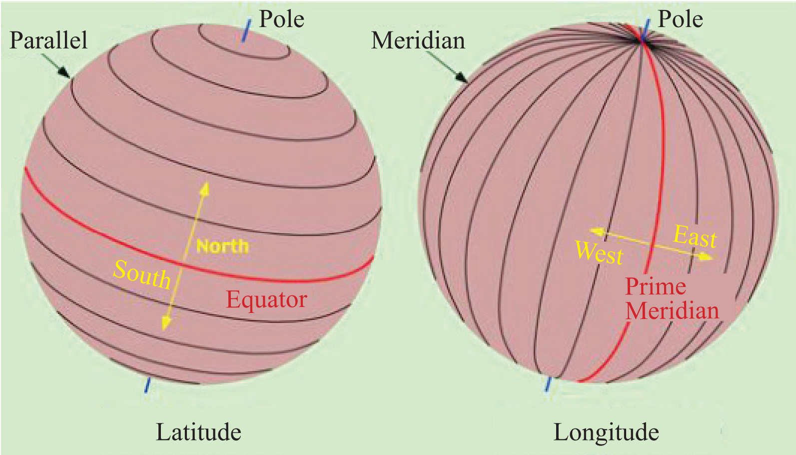
$$\Rightarrow \frac{4}{3} r = h$$

Distance on the Earth's surface can be found using the imaginary lines called lines of latitude and longitude. These lines are used to describe the position of any point on the Earth's surface.

Activity 3

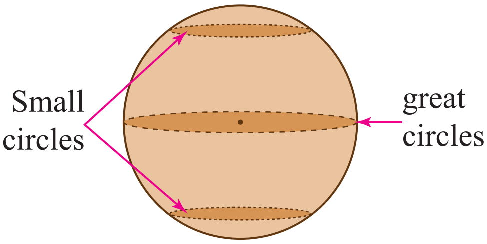
1. Describe the following terms.
   (a) Latitude
   (b) Longitude
   (c) Great circles on a sphere
   (d) Small circles on sphere
2. How can we measure distance on the earth's surface?

The lines of latitude are drawn from west (W) to east (E) and the lines of longitude are drawn from north (N) to south (S) as shown in figure below.



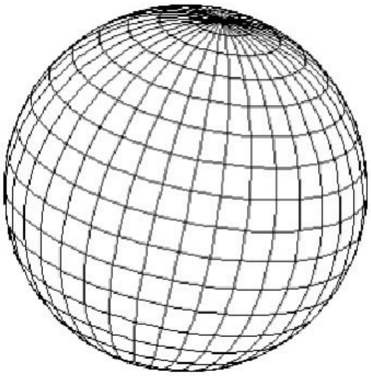
The earth can be approximated as a sphere with a radius of 6,400 km.

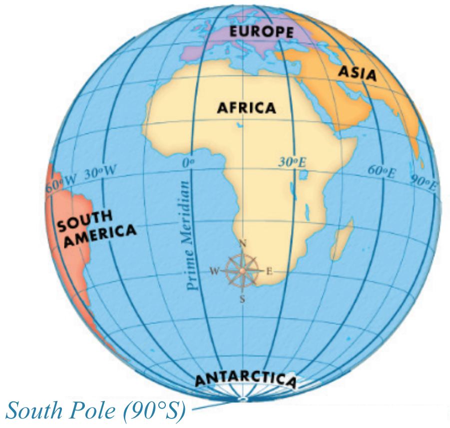
A great circle on a sphere has the same diameter as the sphere itself. The centre of a great circle is the centre of the sphere. The equator is a great circle. A small circle has a diameter less than the sphere's diameter, and the centre of a small circle is not the centre of the sphere. Parallels of latitude are small



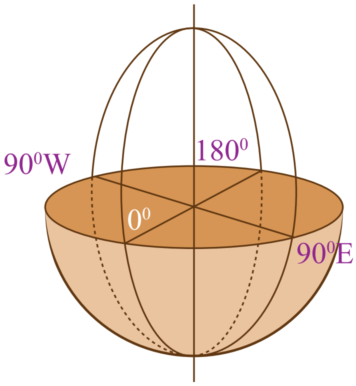
circles that lie in an orientation parallel to the equator. The meridians of longitude are half great circles that pass through the north and south poles.

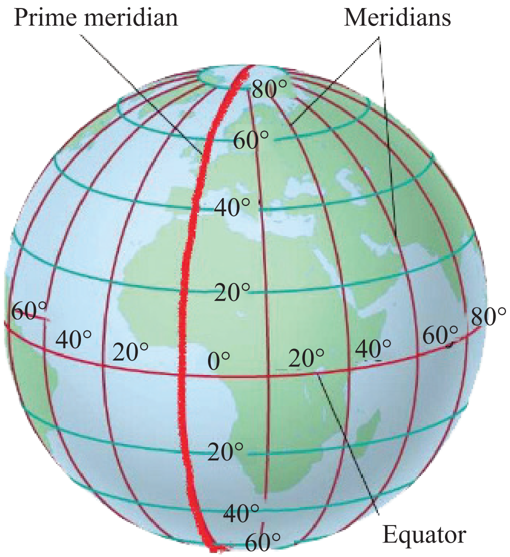
Any position on the Earth's surface can be specified by co-ordinates:
(latitude, longitude). The meridians and the parallels of latitude cover the earth in a grid:





The meridians and the parallels of latitude cover the earth in a grid: If the earth is cut in half vertically the parallels of latitude would look like this: The prime meridian (Greenwich meridian) is the meridian that passes through Greenwich, London.





The co-ordinates of every point on the earth are always expressed as (Latitude N or S, Longitude E or W), Latitude always goes first and longitude second.

Shortest distance

On a flat surface, the shortest distance between two points is a straight line. On the curved surface of a sphere, the shortest distance between two points is the arc of a great circle.

On the earth, every great circle has a radius R of 6,400 km (R = 6,400 km). All the lines of latitude except the equator are small circle, since they do not divide the Earth into two equal parts. Thus the radii of small circles are not the same as that of the Earth. Let r be the radius of a small circle, A any point on the Earth's surface whose latitude α° N. Then R and r can be related as follows:

$$r = R \cos \alpha.$$

It means that the radius r of any small circle (parallel of latitude) is given by the product of the radius R of the Earth and the cosine of the angle of the latitude.

Example 11

Find the radius of each of the given latitudes.

(a) 30° N (b) 60° N

Solution

(a) $r = R \cos \alpha = 6400 \cos 30 = 3200\sqrt{3}$ km.

(b) $r = R \cos \alpha = 6400 \cos 60 = 3200$ km.

Distance along a line of longitudes

To find the distance along a line of longitude, one needs to know the following:

The angle between the two latitudes that cut the given longitude.

The radius of the parallel of longitude.

If the two latitudes lie on the same side of the equator, then the angle between them is the difference of the latitudes.

If the two latitudes lie on different sides of the equator, then the angle between them is the sum of the latitudes.

The longitudes divide the Earth into two halves. Therefore, the radius of the parallel of longitude is equal to the radius R of the Earth which is approximately equal to 6,400 km.

Suppose two points A and B is lying along the same longitude but different latitudes. Assume θ is the angle between the latitudes. Then the distance AB between the two points A and B is given by,

$$AB = \frac{\theta}{360} 2\pi R.$$

EXAMPLE 12

Find the shortest distance between the two towns A(20°S,35°E) and B(40°N,35°E)

Solution

The two towns have common longitude, and therefore, the distance is on that longitude which is a great circle with radius R = 6400 km. The angle between the latitudes is the sum of the latitudes. Why? Hence, θ = 40° + 20° = 60°.

Therefore, the distance AB is given by,

$$AB = \frac{\theta}{360} 2\pi R = \frac{60}{360} (2)(\pi)6400 \text{ km} = 2{,}133.33\pi \text{ km}.$$

EXAMPLE 13

Find the shortest distance between the two towns A(40° N, 35° E) and B(70° N, 35° E)

Solution

The two towns have common longitude, and therefore, the distance is on that longitude which is a great circle with radius R = 6400 km. The angle between the latitudes is the difference of the latitudes. Why? Hence, θ = 70 − 40° = 30°.

Therefore, the distance AB is given by,

$$AB = \frac{\theta}{360} 2\pi R = \frac{30}{360} (2)(\pi)6400 \text{ km} = 1{,}066.67\pi \text{ km}.$$

Distance along a line of latitudes

The theory is the same as that of distances on great circles, but the difference is that the radius r of a small circle is used instead of R; that is; $r = R \cos \alpha$ as given in the previous discussion α is the common latitude.

Therefore, to find the distance along a line of latitude, one needs to know the following:

1. The angle between the two longitudes that cut the given latitude.
2. The radius of the parallel of latitude.

If the two longitudes lie on the same side of longitude, 0°, then the angle between them is the difference of the longitudes.

If the two longitudes lie on different sides of longitude, 0°, then the angle between them is the sum of the longitudes.

Suppose two points A and B is lying along the same latitude α but in different longitudes. Assume θ is the angle between the longitudes. Then the distance AB between the two points A and B is given by,

$$AB = \frac{\theta}{360} 2\pi R \cos\alpha.$$

Example 14

Find the shortest distance between the two towns $A(60°$ N, 40° E) and $B(60°$ N, 50° W)

Solution

The two towns have common latitude which is 60° and hence the distance is on that of latitude which is a small circle with radius $r = 6400 \cos 60°$ km. The angle between the longitudes is the sum of the longitudes. Why? Hence, $\theta = 40° + 50° = 90°$.

Therefore, the distance AB is given by,

$$AB = \frac{\theta}{360} 2\pi r = \frac{90}{360} (2)(\pi)6400\ (0.5) \text{ km} = 1600\pi \text{ km}.$$

Example 15

Find the shortest distance between the two towns $A(60°$ N, 35° E) and $B(60°$ S, 95° E)

Solution

The two towns have common latitude which is 60° and hence the distance is on that of latitude which is a small circle with radius $r = 6400 \cos 60°$ km. The angle between the longitudes is the difference of the longitudes. Why? Hence, $\theta = 95° - 35° = 60°$.

Therefore, the distance AB is given by,

$$AB = \frac{\theta}{360} 2\pi r = \frac{60}{360} (2)(\pi)6400\ (0.5) \text{ km} = 1066.67\pi \text{ km}.$$

Exercises

1. Find the radius of the parallel of each of the given latitude.

   (a) 30° N   (c) 90° N

   (b) 45° N   (d) 55° N

2. What if the north N is replaced by south S?
3. Find the angle between each of the pairs of latitudes.
   (a) (40° N, 85° S)
   (b) (60° N, 15° S)
   (c) (40° N, 85° N)
   (d) (40° S, 85° S)
4. Find the angle between each of the pairs of longitudes.
   (a) (20° E, 65° W)
   (b) (40° E, 55° E)
   (c) (40° W, 20° W)
   (d) (10° E, 70° W)
5. Find the distance along parallel of latitudes for each pairs of the following points.
   (a) *Q*(40° N, 25° E) and *P*(40° N, 85° E)
   (b) *Q*(40° N, 25° W) and *P*(40° N, 35° E)
   (c) *Q*(20° S, 25° E) and *P*(20° S, 45° E)
6. Find the distance along latitudes for each pairs of the following.
   (a) *A*(15° N, 25° E) and *B*(30° S, 25° E)
   (b) *A*(20° N, 35° E) and *B*(80° N, 35° E)
   (c) *A*(40° N, 85° E) and *B*(20° S, 85° E)
7. Find the distance along parallel of longitudes for each pairs of the following.
   (a) *Q*(40° N, 25° E) and *P*(40° N, 85° E
   (b) *Q*(40° N, 25° W) and *P*(40° N, 35° E
   (c) *Q*(20° S, 25° E) and *P*(20° S, 45° E)

Key Terms

- Apex
- Base
- Cone
- Cube
- Cylinder
- Edges
- Lateral faces
- Parallelepiped
- Prism
- Polyhedrons
- Pyramid
- Regular Polyhedron
- Solid figure
- Surface area
- Tetrahedron
- Vertex
- Volume

Summary

- **Solid**

  A solid object is a three dimensional shape formed by a combination of planes and curved surfaces.

- **Prism**

  A prism is a three dimensional geometric figure in which the two ends called bases are congruent polygons on two parallel planes.

  The lateral faces of a prims are parallelograms.

- **Types of Prisms**
  - Prisms can be classified is right prism or oblique prism.
  - In aright prism, the lateral faces are rectangles. Prisms can be classified according to their bases as triangular prism, rectangular prism, …

- **Lateral Area Formula**

  The lateral surface area of a right prism with perimeter p and height h is ph.

- **Surface Area Formula**

  The surface area of a right prism with lateral area ph and base area A_B is $ph + 2A_B$.

- **Cylinder**

  A cylinder is a three dimensional figure formed by a combination of two equal circles called bases that are on two parallel planes and all line segments with end points on the bases that are parallel to the line segment joining the centers of their bases.

  Cylinder can be classified as right cylinder and oblique cylinder.

- **Surface Area Formula**

  The surface area of right circular cylinder of height h and radius of the base r is $2\pi r^2 + 2\pi rh$.

- **Volume Formula**

  The volume of a right circular cylinder of height h and radius of the base r is $\pi r^2 h$.

- **Cone**

  A circular cone is a three dimensional geometric figure composed of a circular region called base and a point not on the plane of the base called apex or vertex together with all line segments joining every point of the base to the apex.

  Circular cones can be classified as right circular cone and oblique circular cone.

- **Surface Area of Formula**

  The surface area of a right circular cone with radius of the base r, height h and slant height l is $\pi rl + \pi r^2$

- **Volume Formula**

  The volume of a right circular cone with height h and radius of the base r is

  $$\frac{1}{3}\pi r^2 h$$

- **Pyramid**

  A pyramid is a polyhedron formed by all line segments whose end points are on a polygonal region called base to a point called apex or vertex which is not on the plane of the base.

- **Surface area formula**

  The surface area of a pyramid is the sum of the area of the base and the area of the lateral faces.

  $$A_B + A_L$$

- **Volume Formula**

  The volume of a pyramid with base area AB and height h is $\frac{1}{3}A_B h$.

Exercises

Answer the following. Be sure to include units.

1. A right circular cylinder has a radius of 10 cm and a height of 4 cm. Then find,
   (a) the lateral area.
   (b) the total surface area.
   (c) the volume.

2. A right circular cylinder has a diameter of 14 cm and a height of 9 cm. Then find,
   (a) the lateral area.
   (b) the total surface area.
   (c) the volume.
3. A right circular cylinder has a diameter of 5 cm and a height of 6 cm. Then find,
   (a) the lateral area.
   (b) the total surface area.
   (c) the volume.
4. A right circular cylinder has a height of 5 cm and a volume of 245 cm^3. Then find,
   (a) the radius.
   (b) the lateral area.
   (c) the total surface area.
5. A right circular cylinder has a diameter of 16 cm and a lateral area of 192 cm^2. Then find,
   (a) the height
   (b) the total surface area
   (c) the volume
6. A right circular cone has a radius of 7 cm and a slant height of 25 cm. Then find,
   (a) the lateral area
   (b) the total surface area.
   (c) the volume.
7. A right circular cone has a radius of 5 cm a lateral area of 35 cm^2. Then find,
   (a) the slant height.
   (b) the height.
   (c) the total surface area.
   (d) the volume
8. Find the volume of a regular square pyramid whose lateral faces are equilateral triangle of side $\sqrt{2}$ cm
9. If a right pyramid has a square base with side 10 cm and a vertical height of 5 $\sqrt{3}$ cm, find its total surface area.
10. A right circular conical vessel of altitude 72cm and base radius 17cm kept with its vertex downwards. If the volume of water in the vessel is $\frac{2312\,\pi}{9}$ cubic cm, find the height of water in the vessel .

M12CH08

CHAPTER 8

LOGICAL REASONING

Chapter Contents

Chapter Outcomes

Upon completion of this chapter, learners will:

- identify & form true or false statements;
- form the negation of simple statements;
- draw conclusions using the implication signs;
- deduce an equivalent implication from a given implication;
- use Vena Diagrams to determine the validity or otherwise of implication or Conclusions.

Introduction

Under this topic we study about arguments. We use logical reasoning to check whether or not a given argument is valid. Recall that Logic is the Science of how to evaluate arguments for its validity and reasoning.

The word logic comes from the Greek word logos. Logic describes the use of valid reasoning in some activities.

Note: From activity 1 what one can see is that, we can assign truth value for some sentences and we cannot assign truth value for some other sentences.

Activity 1

Which of the following sentences are true, false or neither true nor false?

The integer next to 4 is 5.

(i) The day after Monday is Wednesday.

(ii) It is now 10 o'clock.

(iii) What is the time?

(iv) I wish every student is good at mathematics.

Definition

A statement, or a proposition is a declarative sentence which is either true or false but not both.

Example 1

(a) Addis Ababa is the capital city of Liberia.

This sentence is a statement. The truth value of this statement is true.

(b) $7 + 3 = 16$. This is a statement; its truth value is false.

(c) I wish I score 100% in Mathematics test. This sentence is not a statement. We cannot assign the truth value.

Notation

- A statement or proposition is denoted by small letters: p, q, r, t, etc...
- The symbol " : " or "≡" is used in the representation of a statement and we write "p:" or "p≡" to mean statement "p".

Example 2

(a) $q : 5 + 9 = 14$. This is called statement q or proposition q.

(b) p ≡Today is Monday. This is called statement p or proposition p. The truth value of p is true if today is Monday and false if today is not Monday.

Remark: Since a statement is either true or false, but not both, there are only two possible truth values, true or false. We use F for false and T for true.

(c) q : For all real number x, $x^2 < 0$. The truth value of q is F.

(d) p : The product of two negative integers is positive.

(e) Here the truth value of p is T.

Exercises

Identity the sentence which is a statement and assign a truth value if the sentence is a statement.

1. t : 85 is a multiple of 17.
2. q : I like number 2 very much.
3. s : 6 is the smallest integer.
4. r : I wish all student of grade 9 are good at logic.
5. $p : x + 3 = 10; x \in \mathbb{N}$.

Logical connectives (or logical operators)

There are several "connectives" that make new statements from old ones, just as connectives in English word like "and" makes compound sentences. Also in arithmetic, we have the operators such as: "+", "-", " × ", "÷" that connect numbers as: $2 + 3$, $5 – 9$, 4×6 , $9 \div 3$ etc.

In logic, a logical connective (also called a logical operator) is a symbol or word used to connect two or more statements.

The following are the most common logical connectives or propositional connectives. "Negation" , "and" (conjunction), "or" (disjunction) "if ... then" (implication) and " if and only if "(biconditional). The statement so formed by using the statement connectives is called compound or a complex statement.

Complex or compound propositions

DEFINITION

- If two or more statements are combined by one or more statement connectives the resulting statement is called complex or compound statement while the individual statements are called component statements.
- The truth or falsity of the compound statement depends only on the truth or falacity of the components.

Now, we are to study logical connectives, negation and implication.

The most common logical connectives are binary connectives (also called dyadic connectives) that join two statements.

Negation

Consider the statements,

p : The day next to Monday is Tuesday.

r : The day next to Monday is not Tuesday.

r is called the negation of p or p is called the negation of r.

Observe that if r is true, then p is false and if r is false, then p is true.

Notation

- If p is a statement, then its negation is denoted as:
- $\neg p$ or $\sim p$, i.e. $\neg$p or $\sim p$ (read as "not p") is the negation of p.

Note: The negation connective does not connect two statements, but produces the opposite of the statement. Negation is considered to be a unary connective.

The following table illustrates the relation between the statement p and its negation $\neg p$.

Example 3

Statement	Negation
a : It is snowing	$\neg a$: It is not snowing
p : All donkeys have white mouth	$\neg p$: some donkeys do not have white mouth. or It is not true that all donkeys have white mouth.
r : 13 is a prime number	$\neg r$: 13 is not a prime number

Remark: From the examples we can see that either the statement, or its negation is true.

In particular: If p is true, then $\neg p$ is false.

If p is false, then $\neg p$ is true.

The following table summarizes the relationship between p and $\neg p$.

Truth table for "NEGATION"

p	$\neg p$
T	F
F	T

This is a table introducing the rule assigning truth value for the negation of a proposition. We call such a table as truth table.

Definition

Table presenting rules of the assignment of the truth values to the complex proposition are called truth tables.

Remark: Each of the following are the English expressions of $\neg p$:

not p, deny p, negative p, it is false that p, opposite of p.

Exercises

Find the negation of each of the following statements

1. p : 9 is a prime number.
2. q : $15 = 14.9$
3. t : For a real number x, $x = 5 + x$.
4. u : Liberia is the source of white Nile.
5. $m \equiv 7 > 6.0\dot{9}$

6. Find truth value of $\neg p$ if p is:
7. p : There are infinitely many prime numbers.
8. p : The sum of two odd integers is odd.

Or (Disjunction)

The connective "or" connects two statements. The truth value of "p or q" is true if either one or both statements are true. If both statements are flase, the compound statement is false.

Notation

- $\vee$ is a symbol for "and" we write $p \vee q$ to mean p and q.
- The two statements p and q are called the conjuncts and the compound statement $p \wedge q$ is called a conjunction.

ACTIVITY 2

1. Who do we mean by a conditional statement?
2. Write five conditional statements.
3. Do you know how to determine its truth value?

DEFINITION

A conditional statement is an if ... then statement, where the if part is the hypothesis (antecedent) and the then part is conclusion (consequent).

As its name indicate, this connective needs a condition. Consider the statements:

If x is an odd integer, then x^2 is an odd integer

If $3 + 5 = 9$, then $4 + 6 = 11$

A statement of the above forms is known as an implication or conditional statement.

Notation

$\Rightarrow$ is a symbol for implication. Read "$p \Rightarrow q$" as: p implies q.

Based on the above notation of p and q each of the following statements are the same as $p \Rightarrow q$; where x is an integer.

$x = 2$ implies that $x^2 = 4$

$x = 2$ therefore $x^2 = 4$

$x = 2$ is a sufficient condition for $x^2 = 4$

$x = 2$ only if $x^2 = 4$

In the conditional statement $p \Rightarrow q$

- p is called hypothesis or antecedent.
- q is called conclusion or consequent.

The implication $p \Rightarrow q$ will be false if a false conclusion is drawn from a true hypothesis. A statement of the form $p \Rightarrow q$ is false if p is true and q is false. A statement of the form $p \Rightarrow q$ is true if either p is false or both p and q are true.

- If p is false, then $p \Rightarrow q$ is true no matter whether q is true or false.

Truth table for "$p \Rightarrow q$"

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Let p : It is raining.

q : There are clouds in the sky.

(i) Give a linguistic equivalent for the following statements.

(a) $p \Rightarrow q$ (c) $\neg p \Rightarrow \neg q$ (e) $\neg p \vee q$

(b) $q \Rightarrow p$ (d) $\neg q \Rightarrow \neg p$

(ii) Construct truth table for the statements in question (i) above.

Solution

(a) $p \Rightarrow q$: If it is raining, then there are clouds in the sky.

(b) $q \Rightarrow p$: If there are clouds in the sky, then it is raining.

(c) $\neg p \Rightarrow \neg q$: If it is **not** raining, then there are **no** clouds in the sky.

(d) $\neg q \Rightarrow \neg p$: If there are **no** clouds in the sky, then it is **not** raining.

(e) $\neg p \vee q$: It is not raining or there are clouds in the sky.

p	q	$\neg p$	$\neg q$	$p \Rightarrow q$	$\neg p \Rightarrow \neg q$	$\neg q \Rightarrow \neg p$	$\neg p \vee q$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	F	T	F	T	T
F	F	T	T	T	T	T	T

From the table, observe that $p \Rightarrow q$, $\neg q \Rightarrow \neg p$, and $\neg p \vee q$ have the same truth values.

Remark: Each of the following is the English expression of $p \Rightarrow q$

- If p then q
- p implies q
- q if p
- q provided that p
- p is sufficient for q
- p only if q, q is necessary for p

Exercises

(i) Consider the statements p and q where;

p : It is cold.

q : There are clouds in the sky.

(ii) Write the following statements in symbolic form

1. If it is cold, then there are clouds in the sky.
2. If there are clouds in the sky, then it is cold.
3. It is cold because there are clouds in the sky.
4. If it is not cold, then there are no clouds in the sky.

(iii) Write the truth values of the following statements; where $x \in \mathbb{N}$

5. If $x = 3$, then $x + 5 = 8$ If $x + 5 = 8$, then $x = 3$ If $x \neq 3$, then $x + 5 = 8$

Inverse

Given a conditional statement $p \Rightarrow q$, one can analyze also the conditional statements, $q \Rightarrow p$, $\neg p \Rightarrow \neg q$ and $\neg q \Rightarrow \neg p$.

Definition

$\neg p \Rightarrow \neg q$ is called the inverse of $p \Rightarrow q$.

Note: $p \Rightarrow q$ and its inverse $(\neg p \Rightarrow \neg q)$ are not necessarily both true and both false.

Converse

Definition

- $q \Rightarrow p$ is called the converse of $p \Rightarrow q$.
- $p \Rightarrow q$ is not the same as $q \Rightarrow p$.

Example 4

The converse of the statement

"If $x = 5$, then $x^2 = 25$" is

"If $x^2 = 25$, then $x = 5$".

Example 5

Let p : Today is Monday.

q : Tomorrow is Tuesday.

$p \Rightarrow q$: If today is Monday, then tomorrow is Tuesday ($p \Rightarrow q$ is true).

The converse of $p \Rightarrow q$ is

$q \Rightarrow p$: If tomorrow is Tuesday, then today is Monday. (The converse of $p \Rightarrow q$ is also true)

Example 6

The converse of $(x = 2) \Rightarrow (x^2 = 4)$ is $(x^2 = 4) \Rightarrow (x = 2)$.

Here the statement is true but the converse is false (note $(-2)^2 = 4$)

Example 7

What is the converse of the following statement?

“If a triangle is equilateral, then it is isosceles.”

Solution

“If a triangle is isosceles, then it is equilateral.” Here the statement is true but the converse is false.

Example 8

Let $p : x$ is an even whole number and

$q : x$ is a composite number.

Then the truth value of $p \Rightarrow q$ is false and the truth value of its converse

$q \Rightarrow p$ is also false.

Contrapositive

In logic, the contra positive of a conditional statement is formed by negating both terms and reversing the direction of inferences.

Definition

- The contrapositive of " if p, then q" is "if not q, then not p".
- A statement and its contra positive are logically equivalent.
- $\neg q \Rightarrow \neg p$ is called the contra positive of $p \Rightarrow q$, and

  $p \Rightarrow q$ is the contra positive of $\neg q \Rightarrow \neg p$.

Example 9

Let $p \equiv$ It rained this morning.

$q \equiv$ The road is dirtied with mud.

$p \Rightarrow q \equiv$ If it rained this morning, then the road is dirtied with mud.

$\neg q \Rightarrow \neg p \equiv$ If the road is not dirtied with mud, then it did not rain this morning.

Note:

- A statement and its contrapositive are both either true or false. i.e If a statement is true, then its contrapositive is true, and vice versa.

Exercises

1. For the following statements write

   (i) Inverse (iii) Contrapositive

   (ii) Converse (iv) Negation

   (a) p : If a natural number is even, then it is divisible by 2.

   (b) q : If a triangle is equilateral, then it is isosceles.

   (c) r : If $x = 2$, then $x^3 = 8$; where $x \in \mathbb{R}$.

2. Consider the propositions

   p : Triangle ABC is isosceles.

   q : Triangle ABC is equilateral.

   r : $ABCD$ is a square.

   s : $ABCD$ is a parallelogram.

3. Find the truth value of the following compound propositions.

   (a) $p \Rightarrow q$

   (b) Converse of $p \Rightarrow q$

   (c) Contra positive of $p \Rightarrow q$

   (d) Inverse of $p \Rightarrow q$

   (e) Negation of $p \Rightarrow q$

   (f) $r \Rightarrow s$

   (g) $s \Rightarrow r$

4. If p, q and r are statements with truth values T, F and T respectively, then find truth values of the following compound propositions.

   (a) $p \Rightarrow q$

   (b) Converse of $p \Rightarrow r$

   (c) Contrapositive of $p \Rightarrow q$

   (d) Negation of $(\neg p \Rightarrow q)$

   (e) $(p \Rightarrow q) \vee r$

   (f) $(\neg p \Rightarrow \neg q) \vee r$

   (g) Inverse of $p \Rightarrow r$

   (h) Converse of $q \Rightarrow p$

Equivalent propositions

Definition

- Two propositions involving the same component propositions are said to be equivalent if and only if all truth-value assignments to the statements making up them result in the same resulting truth values for the whole statements.
- "≡" is symbol for equivalence. We write $p \equiv q$ to mean ''p is equivalent to q''.

Example 10

p and $\neg(\neg p)$ are equivalent

p	$\neg p$	$\neg(\neg p)$
T	F	T
F	T	F

Let's observe the negation of a conditional statement.

The negation of $p \Rightarrow q$ is $\neg(p \Rightarrow q)$.

EXAMPLE 11

The negation of "$x^2 = 4 \Rightarrow x = 2$" is

"It is not true to say that $(x^2 = 4) \Rightarrow (x = 2)$".

Consider the following truth table.

p	q	$\neg q$	$p \Rightarrow q$	$\neg (p \Rightarrow q)$	$p \wedge \neg q$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

From the truth table you can see that $\neg (p \Rightarrow q) \equiv (p \wedge \neg q)$

In fact $(p \Rightarrow q) \equiv (\neg p \vee q)$ Hence $\neg(p \Rightarrow q) \equiv \neg(\neg p \vee q) \equiv (p \wedge \neg q)$.

A logical deduction (or argument form) is an assertion that a given set of statements $P_1, P_2, P_3, ..., P_n$ called the premise (or hypothesis) yields another statement Q, called the conclusion (or consequent).

Notation

- Let statements $P_2, P_2, P_3, \ldots, P_n$ be premises or hypothesis and Q be the conclusion, we represent this logical deduction or Argument by either of the following two forms.

  (i) $P_1, P_2, P_3, \ldots, P_n \quad \vdash Q$

- The symbol " $\vdash$ " is spoken as Turnstile. The turnstile is used to separate the premises from the conclusion

  (ii) P_1

  P_2

  P_3

  P_4

  .

  .

  .

Definition

- An argument form is valid if and only if the conclusion is true under all interpretations of that argument in which the premises are true. A valid argument cannot have true premises and a false conclusion.
- An argument $P_1, P_2, P_3, \ldots P_n \vdash Q$ is said to be valid if Q is true whenever $P_1 \wedge P_2 \wedge P_3 \wedge \ldots \wedge P_n$ is True.
- An Argument $P_1, P_2, P_3, \ldots P_n \vdash Q$ is said to be invalid if $P_1 \wedge P_1 \wedge P_1 \wedge \ldots \wedge P_n$ is True but there is a case that Q is false.

An argument can be shown to be invalid by giving a counter example of the same argument with premises that are true under a given interpretation, but a conclusion that is false under that interpretation.

Note:

$p_1, p_2, p_3, \ldots, p_n \vdash Q$ is valid if $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \Rightarrow Q$ *is a tautology.*

Example 12

1. Using a Truth table check the validity of $p \Rightarrow q, (p \Rightarrow q) \wedge p \vdash q$.

Solution

p	q	$p \Rightarrow q$	$(p \Rightarrow q) \wedge p$	$[(p \Rightarrow q) \wedge p] \Rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

$[(p \Rightarrow q) \wedge p] \Rightarrow q$ is a tautology

Therefore, the Argument is valid.

2. Using a Truth Table, test the validity of $p \Rightarrow q, (p \wedge \neg q) \Rightarrow r \vdash q$

Solution

p	q	r	$p \Rightarrow q$	$p \wedge \Rightarrow q$	$(p \wedge \neg q) \Rightarrow r$	$(p \Rightarrow q) \wedge ((p \wedge \neg q) \Rightarrow r)$	$((p \Rightarrow q) \wedge ((p \wedge \neg q) \Rightarrow r)) \Rightarrow q$
T	T	T	T	F	T	T	T
T	T	F	T	F	T	T	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T

p	q	r	$p \Rightarrow q$	$p \wedge \Rightarrow q$	$(p \wedge \neg q) \Rightarrow r$	$(p \Rightarrow q) \wedge ((p \wedge \neg q) \Rightarrow r)$	$((p \Rightarrow q) \wedge ((p \wedge \neg q) \Rightarrow r)) \Rightarrow q$
F	T	T	T	F	T	T	T
F	T	F	T	F	T	T	T
F	F	T	T	F	T	T	F
F	F	F	T	F	T	T	F

$((p \Rightarrow q) \wedge (p \wedge \neg q) \Rightarrow r) \Rightarrow q$ is not tautology

Therefore, the argument is not valid.

Also, observe that at the 7th and 8th rows, both of the premises $p \Rightarrow q$ and $(p \wedge \neg q) \Rightarrow r$ are true but the conclusion q is false.

EXERCISES

1. Prove that $p, p \Rightarrow q \vdash q$ is valid
2. Prove that $\neg q, p \Rightarrow q \vdash \neg p$ is valid.
3. Prove that $p \Rightarrow q, q \Rightarrow r \vdash p \Rightarrow r$ is valid.
4. Show that $\neg p \Rightarrow \neg q, p \Rightarrow r \vdash \neg r \Rightarrow \neg q$ is valid
5. Test the validity of the following reasoning.
6. Check the validity of

   $\neg p \wedge \neg q$

   $(q \vee r) \Rightarrow p$

   $\neg r$

Rule of inference

A rule of inference is a rule justifing a logical step from hypothesis to conclusion. There are several rules of inferences. The following are some of the rules of inferences.

1. Modus Ponens (MP) :

   $p \Rightarrow q, p \vdash q$

2. Modus Tollen (MT)

   $p \Rightarrow q, \neg q \vdash \neg p$

3. Law of syllogism(LS)

   The law of syllogism takes two conditional statements and forms conclusion by combining the hypothesis of one statement with the conclusion of another.

   (LS)$p \Rightarrow q, q \Rightarrow r \vdash p \Rightarrow r$

4.
   (a) $p, q \vdash p \wedge q$
   (b) $p, q \vdash p \vee q$

5. $p \wedge q \vdash p$

6. $p \vee q, \neg p \vdash q$

7. $\neg (p \wedge q), p \vdash \neg q$

8.
   (a) $p \Rightarrow q, p \vdash q$
   (b) $p \Leftrightarrow q, \neg p \vdash \neg q$

9. $p \vdash q \Rightarrow p$

10.
   (a) $p \vdash \neg(\neg p)$
   (b) $\neg(\neg p) \vdash p$

11.
   (a) $p \wedge q \vdash q \wedge p$
   (b) $p \vee q \vdash q \vee p$

EXERCISES

Show that the following arguments are valid

1. $q \vdash p \Rightarrow q$
2. $p \vee q, \neg q \vdash q$
3. $p \vdash p \vee q$
4. $p, p \Rightarrow q \vdash q$
5. $p \Rightarrow q, q \Rightarrow r \vdash p \Rightarrow r$
6. $p \wedge q, p \Rightarrow q \vdash p$
7. $\neg p \vee \neg q, p \vdash \neg q$
8. $p, q \vdash p \wedge q$
9. Disprove by counter example that
   (a) $2n - 1$ is prime $\forall n \in \mathbb{N}$.
   (b) $5n - 2$ is prime $\forall n \in \mathbb{N}$.
10. Prove by contradiction that the set of irrational numbers is not closed under
   (a) Addition
   (b) Subtraction
   (c) Multiplication
   (d) Division
11. Prove by contradiction that the set of prime numbers is infinite.
12. Prove by contra positive that if in ΔABC, $\angle B \cong \angle C$, then $AB = AC$.
13. Prove by contradiction that $\sqrt{2}$ is an irrational number.
14. Prove directly that if $x^2 - 2x - 3 = 0$, then $x = 3$ or $x = -1$.
15. Give a counter example to disprove that for a sequence $\{a_n\}$, $a_1 + a_2 + a_3 + \ldots$ Converges if $a_n \rightarrow 0$ as $n \rightarrow \infty$.
16. Prove directly that if a number is perfect number then it is not prime.
17. Prove by cases that if $n \in \mathbb{N}$, then $n^2 - n$ in even.
18. Prove directly the sum of a positive number and its reciprocal is greater than or equal to 2.

Key Terms

- Argument
- Contradiction
- Contrapositive
- Converse
- Implication
- Inverse
- Proposition
- Statement
- Tautology
- Truth Value
- Valid argument

Summary

- Statement (or proposition) is a sentence which is either true or false but not both
- For the conditional statement $p \Rightarrow q$,
  $q \Rightarrow p$ is called the converse
  $\neg p \Rightarrow \neg q$ is called the inverse
  $\neg q \Rightarrow \neg p$ is called the contra positive
- Truth table is one of the important tools to show whether a given statement is tautology or a contradiction.
- A logical deduction (argument form) is an assertion that a given set of statements $P_1, P_2, .., P_n$, called hypotheses or premises, yield another statement Q, called the conclusion.
- Truth table is also one of the important tools to check the validity of a certain argument.

Exercises

I. Fill in the truth table

1.

p	q	r	$q \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r)$	$((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$
T	T	T			
T	T	F			
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F			

2.

p	q	$p \Rightarrow q$	converse $q \Rightarrow p$	Contra positive $\neg p \Rightarrow \neg q$	Inverse $\neg p \Rightarrow \neg q$
T	T				
T	F				
F	T				
F	F				

II. Workout the following

3. Let $p :\equiv$ Teferi is brainy

   $q :\equiv$ Teferi likes maths.

   Give linguistic equivalents for the following statements.

   (a) $p \Rightarrow q$

   (b) $\neg p \Rightarrow q$

   (c) $\neg(p \Rightarrow q)$

4. If the truth value of $\neg q$, p, r and s are all false, then write the truth values of the following compound propositions.

   (a) $p \Rightarrow q$

   (b) $q \Rightarrow p$

   (c) Contrapositive of $p \Rightarrow q$

   (d) The converse of $p \Rightarrow q$

5. Decide whether each of the following argument forms is valid or invalid.

   (a) $\neg p \Rightarrow q,\ q \vdash p$

   (b) $p \Rightarrow \neg q,\ p,\ r \Rightarrow q \vdash \neg r$

   (c) $p \Rightarrow q,\ \neg r \Rightarrow \neg q \vdash \neg r \Rightarrow \neg p$

   (d) $p \Rightarrow q,\ q \vdash p$

6. For the following argument forms given in (I) and (II) below:

   (a) Identify the premises and the conclusion.

   (b) Use appropriate symbols to represent the statements in the argument.

   (c) Write the argument forms using symbols.

   (d) Check the validity.

      (i) If the rain does not come, then the crops are ruined and the people will starve. The crops are not ruined or the people will not starve.

      Therefore, the rain comes.

      (ii) If the team is late, then it cannot play the game. If the referee is here, then the team can play the game. The team is late.

      Therefore, the referee is not here.

M12CH09

CHAPTER 9

PERCENTAGES

Chapter Contents

Chapter Outcomes

Upon completion of this chapter, learners will:

- identify business partnerships and the way they function;
- calculate share interest or profit in a given ratio;
- calculate interest on savings and loans;
- calculate payment using hire purchase;
- calculate taxes paid on goods and services;
- calculate and explain the value added tax (VAT);
- calculate electricity, water and telephone bills.

Introduction

It is very common that in our daily life, we are using the idea of percentage. A percentage of some quantity is a part of the quantity out of a total of 100. In this unit, you learn about percentages and some applications of percentage in calculating taxes.

A percent is a fraction whose denominator is 100, for example, $\frac{35}{100}$ is read as thirty five percent and the symbol "%" is used for the term percent. So the fraction, $\frac{35}{100}$ can be denoted using percentage as 35%. The word "percentage" means "per hundred" or "out of hundred".

Activity 1

1. If a student scored 45 out of 60 in an examination and that mark has to be considered out of 100, how much did the students scored out of 100?
2. If a student scored 90 out of 120 in an examination and that mark has to be considered out of 100, how much did the students scored out of 100?

From your responses to Activity 1, observe that to convert the marks of the students out of 100, first write the marks in fraction form and multiply the fraction by 100.

In general to change a fraction into percent, multiply the fraction by 100, simplify it and attach the percent sign " %" with it.

Example 1

Convert $\frac{17}{20}$ to a percent form.

Solution

Multiply the fraction by 100 and attach the percent sign (%) with it.

$$\frac{17}{20}\times 100\% = 0.85\times 100\% = 85\%.$$

Thus, $\frac{17}{20}=0.85$

Note

To convert a percent to fraction or to write a percent as a fraction, drop the % symbol, multiply the number by $\frac{1}{100}$ (or divide the number by 100) and simplify it.

Example 2

A student scored 21 marks out of 30 marks in a given test. What was the percentage of marks of the student's score?

Solution

The total marks were 30 and the marks the student scored were 21. The student scored 21 out of 30 and this information in fraction form is $\frac{21}{30}$.

Therefore, marks scored by the student in percent is $\frac{21}{30} \times 100\% = 0.7 \times 100\% = 70\%$.

This can be interpreted as: the student scored 70 marks out of the 100 marks.

Example 3

Convert each of the following percentages to fraction form.

(a) 74% (b) 15% (c) 150%

Solution

(a) $74\% = 74 \times \frac{1}{100} = \frac{74}{100} = \frac{37}{50}$.

(b) $15\% = 15 \times \frac{1}{100} = \frac{15}{100} = \frac{3}{20}$.

(c) $250\% = 250 \times \frac{1}{100} = \frac{250}{100} = \frac{5}{2}$.

Sometimes the information may be given in decimal form. To write a decimal as a percent, multiply the decimal by 100 and attach the % symbol with it

Example 4

Convert each of the following decimals into percentages.

(a) 0.25 (b) 0.015 (c) 1.35

Solution

(a) $0.25 = 0.25 \times 100\% = 25\%$

(b) $0.015 = 0.015 \times 100\% = 1.5\%$

(c) $1.35 = 1.35 \times 100\% = 135\%$

Percent of a number or a quantity

It is very common that people want to determine a certain specified percent of a given number or quantity. To determine a specified percent of a number or quantity, first change the percent to a fraction or a decimal and then multiply it with the number or quantity.

Example 5

Find 35% of 500.

Solution

Find 35% of 500 is given by:

$$\frac{35}{100} \times 500 = 175.$$

Thus, 35% of 500 is 175.

Example 6

What percent of 4000 is 2500?

Solution

Let $y\%$ of 4000 be 2500.

Therefore, $\frac{y}{100} \times 4000 = 2500.$

This implies $y = \frac{2500}{4000} \times 100\% = 0.625 \times 100\% = 62.5\%.$.

Therefore, 62.5% of 4000 is 2500.

Example 7

Alphia earns L$ 3800 per month. She uses 50% for household expenses, 15% for her personal expenses, 20% for expenses on her children and the rest for saving. What amount does she save per month?

Solution

Expenses for household= 50%

Expenses for herself = 15%

Expenses on children = 20%

Total expenditure = (50 + 15 + 20)% = 85%

Therefore, her saving per month is (100 − 85)% = 15%.

Thus, her saving is 15% of 3800 she earns per month and this is equal to $\frac{15}{100} \times 3800 = 570$

Therefore, Alphia was saving L$ 570 per month.

Exercises

1. 24 % of what number is 72?
2. 27 is what percent of 18?
3. There are 160 students going out of their school for a field trip. Of those students, 45% are boys and the rest are girls. Find the number of boys and the number of girls participating in a field trip.
4. 1200 students at a certain primary school were asked to name their favorite clubs in their school. The table below shows the result of the survey.

Name of the club	% of students
Mathematics and Science	30%
Environmental protection	24%
Red cross	28%
Others	18%

What is the number of students who favorite

(a) Mathematics and Science club?

(b) Environmental protection?

(c) Red cross?

In all countries in the world, governments collect money in different forms for the government spending and expenditure. For the purpose of collecting money, governments impose compulsory financial charge or some other type of levy on individuals or legal entities and one of such cases is called tax. Individuals or legal entities that pay taxes are called tax payers.

Most countries have a tax system in place and the tax systems are different for different countries. In Liberia, there are different taxation systems and the basic taxes are given below.

Personal taxes

In Liberia, residents pay tax on their worldwide income and non-residents pay taxes on income sourced in Liberia. The tax rates are progressive as they are given as follows:

Income	Tax
Income up to LRD (Liberian dollar) 70,000	0%
From LRD 70,001 to 200,000	5%
From LRD 200,001 to 800,000	15%
Over LRD 800,000	25%
The income tax of non-residents	15%

Corporate income tax

Corporate income tax is levied at the rate of 25%.

Withholding tax

Withholding tax on dividends, interest, and royalties is usually levied at the rate of 15%.

Property tax

The tax is levied at the rate of 0.25% on residential property and at the rate of 1.5% on commercial property. A land that is vacant and within the city boundaries attracts a tax of 5%.

Value Added Tax (VAT)

The tax that has to be paid on good with value added on them is 10%.

EXAMPLE 8

If the price of a certain item is 120000 LRD, what is the price of the item after a VAT of 10%?

Solution

$$\text{VAT} = 120000 \times 10\% = 120000 \times \frac{10}{100} = 12000 \text{ LRD}.$$

Thus, the total price of the item is the sum of the price of the item and VAT.

So, 120000 LRD + 12000 LRD = 132000 LRD.

Example 9

George is a resident in Liberia and George's income is 750000 LRD. What is the amount of tax that George has to pay to the government?

Solution

The income 750000 LRD is between 200,001 and 800,000. The tax for an income from 200,001 to 800,000 is 15%. Therefore, the tax George has to pay to the government is:

$$750000 \times 15\% = 750000 \times \frac{15}{100} = 750000 \times 0.15 = 112500 \text{ LRD}.$$

Example 10

Sofia is a non-resident in Liberia and she earns 850000 LRD after tax. What is the gross income of Sofia?

Solution

Let x be the gross income of Sofia.

As Sofia is a non-resident in Liberia, her income tax is 15%.

Thus, the gross income of Sofia can be calculated as follows.

$$x - \frac{15}{100} \times x = 850000 \text{ or } \frac{85x}{100} = 850000$$

Therefore, $x = \dfrac{850000 \times 100}{85} = 1000000$ LRD.

Therefore, the gross income of Sofia was 1000000 LRD.

Exercises

1. A person paid a price of 10000 LRD including 10% VAT for a certain item. What was the price of the item before VAT?
2. A corporate generated an income of 12000000000 LRD. What is the amount of tax that the corporate has to pay to the government?
3. A man invested a certain amount of money at 5 % simple interest and gained 300 LRD in 6 years' time. How much money did he invest?
4. A man earned 2000000 LRD as dividends from a share that he has in a certain share company.
   (a) What is the amount of tax that the person has to pay to the government?
   (b) What is the amount that the man gets after he paid tax?

In our day to day life, we come across a number of situations wherein we use the concept of percent. Next we discuss the application of percentage in different fields, like problems in profit and loss, discount, simple interest, etc.

Profit and loss

Let us recall the terms and formulae related to profit and loss.

Cost Price (C.P.): The price at which an article is purchased, is called its cost price.

Selling Price (S.P.): The price at which an article is sold, is called its selling price.

Profit(Gain): When S.P. > C.P., then ther is profit, and Profit = S.P. − C.P..

Loss: When C.P. > S.P., then there is loss, and Loss = C.P. − S.P.

Formulae: **Profit** % = $\frac{\text{S.P.} - \text{C.P}}{\text{C.P.}}$ and **Loss** % = $\frac{\text{C.P.} - \text{S.P.}}{\text{C.P.}}$ × 100%.

Example 11

A shopkeeper buys an article for L$ 360 and sells it for L$ 270. Find his gain or loss percent.

Solution

Here C.P. = 360 and S.P. = 270. Since C.P. > S.P., there is a loss and the loss is 360 − 270 = L$ 90. Thus,

$$\% \text{ Loss } = \frac{90}{360} \times 100\% = 25\% .$$

Example 12

Ahmed purchased a house for LRD 452,000 and spent LRD 28,000 on its repairs. He had to sell it for LRD 492,000. Find his gain or loss percent.

Solution

Here C.P. = 452,000 + 28,000 = 480,000 LRD and S.P. = 492,000 LRD. Since S.P. > C.P., there is a gain and the gain is 492,000 − 480,000 = 12,000.

$$\% \text{ Profit } = \frac{12000}{480000} \times 100\% = 2.5\%$$

EXAMPLE 13

By selling a book for 258 LRD, a publisher gains 20%. For how much should he sell it to gain 30%?

Solution

S.P. = 258 LRD, Profit = 20 %. Then,

$$\text{C.P.} = \frac{\text{S.P.} \times 100}{100 + \%\text{Gain}} = \frac{258 \times 100}{120} = 215 \text{ LRD}.$$

Now, if profit = 30% and C.P. = 215 LRD, the selling price must be

$$\text{S.P.} = \frac{\text{C.P.} \times (100 + \%\text{Gain})}{100} = \frac{215 \times 130}{100} = 279.5 \text{ LRD}.$$

Banking (Simple interest)

When a person has to borrow some money as a loan from a bank etc. he promises to return it after a specified time period along with some extra money for using the money of the lender.

The money borrowed is called the **Principal**, usually denoted by P, and the extra money paid is called the **Interest**, usually denoted by I. The total money paid back, that is, the sum of the principal and the interest is called the **Amount**, and denoted by A. Thus,

$$A = P + I.$$

The interest is mostly expressed as a rate percent per year (per annum). Interest depends on, how much money (P) has been borrowed and the duration of time (T) for which it is used. Interest is calculated according to a mutually agreed rate percent per annum (R). $\left(\text{i.e. } R = R\% = \frac{R}{100}\right)$. Thus,

$$\text{Interest} = (\text{Principal}) \times (\text{Rate \% per annum}) \times \text{Time}$$

Therefore, $I = \frac{P \times R \times T}{100}$.

Interest calculated as above is called **simple interest**. Let us take some examples, involving simple interest.

Example 14

Find the simple interest in each of the following cases.

	P	R	T (in Years)
(a)	8000	5%	2
(b)	20000	15%	1.5

Solution

(a) $I = P \times R\% \times T = 8000 \times \frac{5}{100} \times 2 = 800.$

(b) $I = P \times R\% \times T = 20000 \times \frac{15}{100} \times \frac{3}{2} = 4500.$

Example 15

At what rate of simple interest per annum will LRD 5000 amount be 6050 in 3 years?

Solution

Here $A = 6050$, $P = 5000$, and $T = 3$ years. Thus, $I = 6050 - 5000 = 1050$.

Since $I = P \times R\% \times T$, then we have $R\% = \frac{I}{P \times T}$ and hence

$$R = \frac{I \times 100}{P \times T} = \frac{1050 \times 100}{500 \times 3} = 7.$$

Thus, $R = 7\%.$.

Example 16

A sum amounts to LRD 4875 at 12.5% simple interest per annum after 4 years. Find the sum.

Solution

Here $A = 4875$, $R = 12.5\%$, $T = 4$ years. Thus, the interest is

$$I = P \times R\% \times T = P \times \frac{12.5}{100} \times 4 = P \times \frac{1}{2} = \frac{P}{2}.$$

Thus, $A = P + I = P + \frac{P}{2} = \frac{3P}{2}$ and hence $P = \frac{2}{3}A = \frac{2}{3} \times 4875 = 3250$ LRD.

Therefore, the sum is 3250 LRD.

Example 17

In how many years will an amount of 2000 LRD yield an interest (Simple) of 560 LRD at the rate of 14 % per annum.

Solution

We have,

Principal (P) = 2000, Interest (I) = 560, Rate (R) = 14%

$$I = P \times R\% \times T \text{ or } 560 = 2000 \times \frac{14}{199} \times T$$

Therefore, $T = \frac{560 \times 100}{2000 \times 14} = 2$ years.

Thus, in 2 years, a sum of 2000 LRD will yield an interest of 560 LRD at a rate of 14 % per annum.

Compound interest

Compound interest is the interest calculated on the initial principal at the end of each period and the principal includes all of the accumulated interest from previous periods on a given deposit or loan. Banks and other financial institutions mostly use compound interest for both savings and loans.

If P amount is invested or loaned at a rate of r% compounded annually, the amount at the end of the n^{th} year is computed by

$$A = P\left(1 + \frac{r}{n}\right)^{nt},$$

where A is the total amount accumulated at the end of the n^{th} year, P is the principal amount at the beginning, r is the interest rate, t is the time and n is number of times that the interest is compounded per unit time.

Example 18

Find the amount when 20000 LRD is invested for 3 years at 10% compounded annually.

Solution

We use the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$, where A is the total amount accumulated at the end of the 3rd year, P is the principal amount at the beginning, which is 20000 LRD in this case, r is interest rate, which is 10% in our case, t is the time, which is 3 years and n is number of times that the interest is compounded per unit time and it is 1 in this case.

Therefore $A = 20000\left(1+\frac{10}{100\times 1}\right)^{1\times 3}$ LRD $= 20000(1.1)^3$ LRD = 26620 LRD.

Example 19

Find the amount at the end of the 5th year if 100000 LRD is borrowed at a rate of 5% compounded annually quarterly.

Solution

We use the formula $A = P\left(1+\frac{r}{n}\right)^{nt}$, where A is the total amount accumulated at the end of the 5th year, P is the principal amount at the beginning, which is 100000 LRD in our case, r is interest rate, which is 5% in our case, t is the time, which is 5 years for this case and n is number of times that the interest is compounded per unit time and it is 4 in this case.

$$A = P\left(1+\frac{r}{n}\right)^{nt} = 100000\left(1+\frac{5}{100\times 4}\right)^{4\times 5} \text{ LRD } = 100000(1.0125) \text{ LRD.}$$

$$= 128203.723 \text{ LRD.}$$

Therefore, the amount at the end of the 5th year is 128203.723 LRD.

Exercises

1. A company increases its capital by 15% each year. If the company started with a capital of 1000,000 LRD, what will be the capital of the company at the beginning of the fifth year?
2. Find the amount at the end of the 10th year if 200,000 LRD is invested at a rate of 10% compounded semi-annually.

Percentage increase and percentage decrease

Here we need to find the percentage change when a quantity increases or decreases. We use the formula's below to calculate percentage increase and percentage decrease. That is,

$$\text{Percentage increase} = \frac{\text{actual change}}{\text{original quantity}} \times 100\%.$$

$$\text{Percentage decrease} = \frac{\text{actual change}}{\text{original quantity}} \times 100\%.$$

Example 20

If 120 is reduced to 96, what is the reduction percent?

Solution

Here, the actual change is 120 − 96 = 24.

Therefore, Percentage decrease = $\frac{24}{120} \times 100\% = 20\%$.

Example 21

The cost of an article has increased from 450 LRD to 495 LRD. By what percent did the cost increased?

Solution

The increase in cost price = 495 − 450 = 45 LRD. Thus,

increase percent = $\frac{45}{450} \times 100 = 10\%$

Example 22

Find the amount that when increased by 15% becomes 19320.

Solution

Let the amount be x. Thus,

$$x + \frac{15}{100} \times x = 19320 \text{ or } \frac{115x}{100} = 19320.$$

Therefore, $x = \frac{19320 \times 100}{115} = 16800.$

Hence, the required amount is 16800.

Discount

You must have seen advertisements of the following types, especially during the festival seasons.

SALE, Discount up to 50% and 20% discount on all items, etc

A discount is a reduction in the marked (or list) price of an article. "20% discount" means a reduction of 20% in the marked price of an article.

Marked price (or List price): The marked price (M.P.) of an article is the price at which the article is listed for sale. Since this price is written (marked) on an article, so it is called the marked price.

Discount: The discount is the reduction from the marked price of the article.

Net selling Price (S.P.): In case of discount selling, the price of the article obtained by subtracting discount from the marked price is called the Net Selling Price of Selling Price (S.P.).

Example 23

A material is marked at LRD 2400. Find its selling price if a discount of 12 % is offered.

Solution

Here, Marked Price (M.P.) of the material = 2400 LRD.

Discount = 12%

Net Selling Price(S.P.) = M.P. – Discount

$$= 2400 - \frac{12}{100} \times 2400$$

$$= 2400 - 288$$

$$= 2112 \text{ LRD.}$$

Thus, the net selling price of the material is 2112 LRD.

Example 24

A machine listed at LRD 8400 is available for LRD 6300. Find the rate of discount offered.

Solution

We have

Marked Price (M.P.) = 8400 LRD

Selling Price (S.P.) = 6300 LRD

Discount Offered = 8400 LRD − 6300 LRD

= 2100 LRD

Discount % = $\frac{2100}{8400} \times 100\% = 25\%$

Thus, the net selling price of the material is 2112 LRD.

Example 25

A wholesaler's marked price of an article is 1250 LRD and is available at a discount of 20 %. For how much should the retailer sell it, to earn a profit of 15%.

Solution

M.P. = 1250 LRD

Discount = 20% of 1250

$= \frac{20}{100} \times 1250 = 250.$

Therefore, Cost price of the retailer = 1250 – 250 = 1000 LRD.

Profit = 15 %

Thus, S.P. = $\text{C.P.}\left(\frac{15}{100}+1\right) = \frac{1000 \times 115}{100} = 1150 \text{ LRD.}$

Hire purchase

Hire purchase is an arrangement for buying consumer goods usually for goods that people cannot afford immediate cash payment and the buyer makes an initial down payment and pays the balance plus interest in installments, usually simple interest.

Example 26

A car company was advertising to sell a certain model of car with a manufacturing cost of 10000 LRD for a price of 40000 LRD under the following hire purchase terms: An initial payment of 40% of the price and the balance paid at 15% simple interest per annum in twelve monthly equal installments. Determine

(a) the amount that has to be paid every month;

(b) the total amount the buyer has to pay for the car;

(c) the percentage profit the car company made on the cost price of the car.

Solution

(a) Cost price of car is 10000 LRD and selling price of the car is 40000 LRD.

The Initial amount to be paid by buyer is 20% of selling price, which is equal to:

$\frac{40}{100} \times 40000 \text{ LRD} = 16000 \text{ LRD}.$

The amount remaining to be paid by the buyer is $40000 \text{ LRD} - 16000 \text{ LRD} = 24000 \text{ LRD}$.
This remaining amount attracts simple interest at a rate of 15% per annum, that is, one year.

Simple Interest = $PRT = \frac{15}{100} \times 24000 \text{ LRD} = 3600 \text{ LRD}.$

The total amount that has to paid at the end of the year is
24000 LRD + 3600 LRD = 27600 LRD.

Therefore, the amount that has to be paid every month is $\frac{27600}{12}$ LRD = 2300 LRD.

(b) The total amount the buyer has to pay for the car is the initial payment plus total amount paid at the end of the 12 months = 24000 LRD + 27600 LRD = 51000 LRD.

(c) Percentage profit the company made on car is $\frac{11000}{10000} \times 100 = 110\%.$

EXAMPLE 27

Mr George bought a house for 250000 LRD in 2005. He paid 30% of the cost and paid the rest in equal monthly installments, it took him 10 years to make full payment for the house with a simple interest rate of 5%. Calculate:

(a) the monthly installment;

(b) the total amount Mr. George paid for the house;

(c) the percentage increase in the cost the house.

Solution

Amount paid as initial deposit = $\frac{30}{100} \times 250000 \text{ LRD} = 75000 \text{ LRD}$

(a) The Initial amount to be paid by Mr. George is 30% of selling price, which is equal to:

$\frac{30}{100} \times 250000 \text{ LRD} = 75000 \text{ LRD}.$

The amount remaining to be paid by the buyer is

250000 LRD − 75000 LRD = 175000 LRD.

This remaining amount attracts simple interest at a rate of 5% per annum, that is, one year.

Simple Interest = $PRT = \frac{5}{100} \times 17500 \text{ LRD} \times 10 = 87500 \text{ LRD}$

The total amount that has to paid at the end of the year is:

175000 LRD + 87500 LRD = 262500 LRD.

Therefore, the amount that has to be paid every month is

$$\frac{262500}{12 \times 10}\text{LRD} = 2187.50 \text{ LRD}.$$

(b) The total amount the buyer has to pay for the house is the initial payment plus total amount paid at the end of the 10 years which is
75000 LRD + 262500 LRD = 337500 LRD.

(c) Percentage profit the company made on house is $\frac{87500}{250000} \times 100 = 35\%$

Exercises

1. A trader bought a TV set for 1000 LRD and sold it at a loss of 5.5%, what was the selling price?
2. The salary of a man last year was 2500 LRD per month. This year he got 15% increment. What is the salary of the man at present?
3. If the volume of a certain liquid decreases from 300 ml to 240 ml, what is the percent change of the volume?
4. A merchant gained 15% by selling an article for 2300 LRD. What was the cost price of the article?
5. How long will it take 300 LRD to double itself if it is invested at the rate of 5% simple interest per annum?
6. A man bought a house for 350000 LRD in 2012. He paid 25% of the cost and paid the rest in equal monthly installments, it took him 7 years to make full payment for the house with a simple interest rate of 6%. Calculate:
   (a) the monthly installment;
   (b) the total amount the man paid for the house;
   (c) the percentage increase in the cost the house.

Key Terms

- Corporate income Tax
- Fraction
- Income tax,
- Personal Taxes
- Percentage
- Proportion
- Property Tax
- Value added tax(VAT)
- Withholding Tax

Summary

- A percent is a fraction whose denominator is 100.
- The word "percentage" means "per hundred" or "out of hundred".
- For the purpose of collecting money, governments impose compulsory financial charge or some other type of levy on individuals or legal entities and one of such cases is called tax.
- Individuals or legal entities that pay taxes are called tax payers.
- There are different taxation systems and in Liberia, the following are the different tax systems.
    - **Personal Taxes**

      In Liberia, residents pay tax on their worldwide income and non-residents pay taxes on income sourced in Liberia. The tax rates are progressive as they are given as follows:

Income	Tax
Income up to LRD (Liberian dollar) 70,000	0%
From LRD 70,001 to 200,000	5%
From LRD 200,001 to 800,000	15%
Over LRD 800,000	25%
The income tax of non-residents	15%

    - **Corporate income Tax**

      Corporate income tax is levied at the rate of 25%.
    - **Withholding Tax**

      Withholding tax on dividends, interest, and royalties is usually levied at the rate of 15%.
    - **Property Tax**

      The tax is levied at the rate of 0.25% on residential property and at the rate of 1.5% on commercial property.

      A land that is vacant and within the city boundaries attracts a tax of 5%.
    - **Value Added Tax (VAT)**

      The tax that has to be paid on good with value added on them in 10%.

Exercises

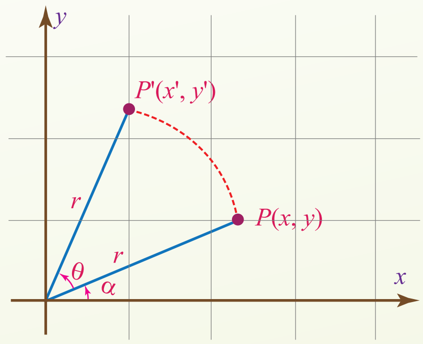
1. Compute each of the following.
   (a) 45% of 200
   (b) 15% of 1800
   (c) 50% of 400
   (d) 15% of 4050
2. In a school, 60% of the total number of students is boys. If the number of girls is 800, then
   (a) Find the number of bays in the school.
   (b) Find the total number of students in the school.
3. A 10% tax on a pair of shoes amounts LRD 4000. What is the cost of the pair of shoes?
4. A VAT of 10% of the cost of a car was added to the purchase price of 600,000 LRD.
   (a) What is VAT of the price of the item?
   (b) What is the total cost of the item including VAT?
5. How long will it take for 12000 LRD to give 36000 LRD with simple interest at the rate of 5% per annum?
6. A man invested a certain amount of money at 5 % simple interest and gained 3000 LRD in 6 years' time. How much money did he invest?
7. A woman invested 5000 LRD at 6% simple interest per annum and gained 1200 LRD. For how long did she invest the money?
8. A man deposited 20000 LRD at a certain bank at the rate of 10% simple interest for 3 years. What is the total interest over the three years?

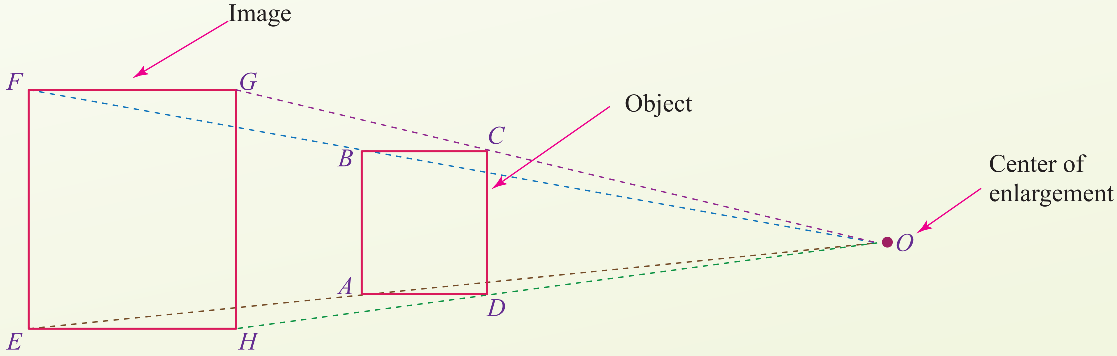
M12CH10

CHAPTER 10

RIGID MOTION AND ENLARGEMENT

Chapter Contents





Chapter Outcomes

Upon completion of this chapter, learners will:

- find the image of an object under rotation;
- carry out an enlargement of a plane shape given a scale factor;
- identify a scale drawing as an enlargement/reduction of a plane figure (shape);
- establish the relationship between the areas and volumes of plane figures and solids and their images.

Introduction

Rotation is a type of transformation of figures which turn around a point called the centre of rotation.

DEFINITION

A rotation R about a point O through an angle θ is a transformation of the plane onto itself which carries every point P of the plane into the point P' of the plane such that $OP = OP'$ and m $(\angle POP') = \theta$. O is called the centre of rotation and θ is called the angle of rotation.

A rotation by angle θ is in the counter clockwise direction, if $\theta > 0$ and in the clockwise direction if $\theta < 0$.

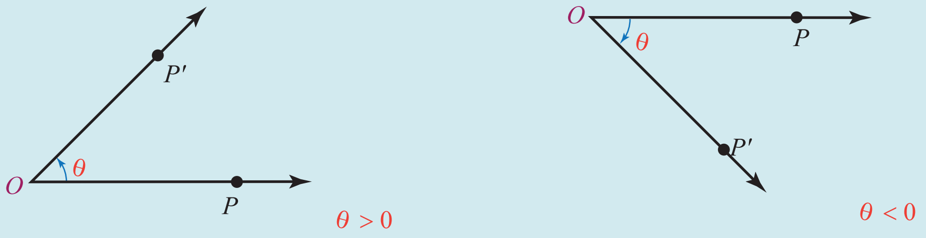


Figure 1.

Note: Rotation is a rigid motion, that is, after a given rotation, the shape and size of the image of the given figure is the same as the shape and size of the original figure

EXAMPLE 1

Find the image of point A (1, 0) when it is rotated through 30° about the origin.

Solution

Let the image of A (1, 0) be A' (a, b) as shown in the figure.

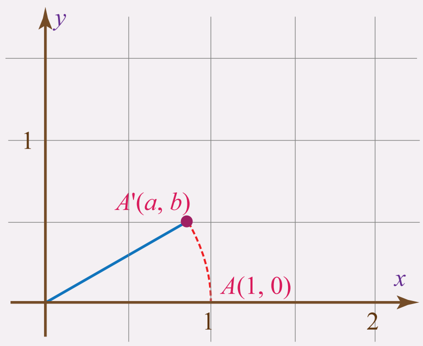


Figure 2.

Then $OA = OA'$ and from your knowledge in trigonometry, $(a, b) = (r\cos\theta, r\sin\theta)$ where $r = 1$ and $\theta = 30°$. Therefore, the image of A (1, 0) is $A'\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Notation

If R is rotation through an angle θ, then the image of $P(x, y)$ is denoted by $R_\theta(x, y)$.

In the above example, $R_{30°}(1, 0) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

Theorem

Let R be a rotation through angle θ about the origin. If $R_\theta(x, y) = (x', y')$. *Then* $x' = x\cos\theta - y\sin\theta$ and $y' = x\sin\theta + y\cos\theta$.

Proof

Let $P'(x', y')$ be the image of $P(x, y)$ after a rotation by an angle θ about the origin, as shown in the figure below.

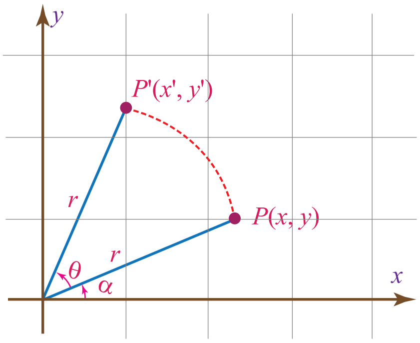


Figure 3.

Let α be the angle that $\overrightarrow{OP}$ with the positive x-axis. Then from your knowledge trigonometry you have:

$(x, y) = (r\cos\alpha, r\sin\alpha)$ and $(x', y') = (r\cos(\alpha + \theta), r\sin(\alpha + \theta))$.

This implies, $r\cos(\alpha + \theta) = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta$

$= x\cos\theta - y\sin\theta$

and $r\sin(\alpha + \theta) = r\sin\alpha\cos\theta + r\cos\alpha\sin\theta$

$= y\cos\theta + x\sin\theta$

Therefore, $R_\theta(x, y) = (x\cos\theta - y\sin\theta, y\cos\theta + x\sin\theta)$.

Example 2

Using the formula, find the images of the following points in rotation about the origin through the indicated angle.

(a) (4, 0) ; 60° (b) (1, 1); $-\frac{\pi}{6}$ (c) (1, 2); 90°

Solution

(a) If (x', y') is the image of (4, 1) after a rotation about the origin by an angle of $\theta = 60^\circ$, then

$x' = x\cos q - y\sin q$; where $x=4$, $y=0$; $\theta=60^\circ$, $x' = 4\cos 60^\circ - 0\times\sin 60^\circ = 2$.

$y' = x\sin\theta + y\cos\theta = 4\sin 60^\circ + 0\times\cos 60^\circ = 2\sqrt{3}$

Therefore $R_{60^\circ}(4, 0) = (2, 2\sqrt{3})$

(b) If (x', y') is the image of $(1,1)$ after a rotation about the origin by an angle of $\theta = -\frac{\pi}{6}$,

then, $x' = 1\times\cos\left(-\frac{\pi}{6}\right) - 1\times\sin\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{1+\sqrt{3}}{2}$.

$y' = x\sin\theta + y\cos\theta = 1\times\sin\left(\frac{-\pi}{6}\right) + 1\times\cos\left(\frac{-\pi}{6}\right) = \frac{-\sqrt{3}}{2} + \frac{1}{2} = \frac{1-\sqrt{3}}{2}$.

Therefore, $R_{-\frac{\pi}{6}}(1, 1) = \left(\frac{1+\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}\right)$.

(c) If (x', y') is the image of (1, 2) after a rotation about the origin by an angle of $\theta = 90^\circ$, then

$x' = 1\times\cos(90^\circ) - 2\times\sin(90^\circ) = 1(0) - 2(1) = -2$.

$y' = 1\times\sin(90^\circ) + 2\times\cos(90^\circ) = 1(1) - 2(0) = 1$.

$R_{90^\circ}(1,2) = (-2, 1)$.

Rotation by some special angles

Let R be a rotation through an angle θ about the origin. Then

(i) If $\theta = \frac{\pi}{2}$, then $R_{\frac{\pi}{2}}(x, y) = (-y, x)$.

(ii) If $\theta = \pi$, then $R_{\pi}(x, y) = (-x, -y)$.

(iii) If $\theta = \frac{3\pi}{2}$, then $R_{\frac{3\pi}{2}}(x, y) = (y, -x)$.

(iv) If $\theta = 2n\pi$ for $n \in \mathbb{Z}$, then $R_{2n\pi}(x, y) = (x, y)$.

Image of a triangle after a rotation

To find the image of a triangle after a rotation by a certain angle, first find the images of the vertices of the triangle. Then the triangle whose vertices are the images of the vertices of the given triangle is the image of the given triangle.

Example 3

Find the image of the triangle with vertices $A(1,0)$, $B(1,1)$ and $C(0,1)$ after a rotation about the origin through each of the following angles.

(a) 90°

(b) 180°

(c) 270°

Solution

(a) The image of a point (x, y) after rotation about the origin through an angle of 90° is $R_{90^\circ}(x,y) = (-y, x)$.

Thus, $R_{90^\circ}(1,0) = (0,1)$, $R_{90^\circ}(1,1) = (-1,1)$ and $R_{90^\circ}(0,1) = (-1,0)$.

That is, the image of the triangle with vertices $A(1,0)$, $B(1,1)$ and $C(0,1)$ is the triangle with vertices $A'(0,1)$, $B'(-1,1)$ and $C'(-1,0)$.

(b) The image of a point (x, y) after rotation about the origin through an angle of 180° is $R_{180^\circ}(x,y) = (-x,-y)$.

Thus, $R_{180^\circ}(1,0) = (-1,0)$, $R_{180^\circ}(1,1) = (-1,-1)$ and $R_{180^\circ}(0,1) = (0,-1)$.

That is, the image of the triangle with vertices $A(1,0)$, $B(1,1)$ and $C(0,1)$ is the triangle with vertices $A'(-1,0)$, $B'(-1,-1)$ and $C'(0,-1)$.

(c) The image of a point (x, y) after rotation about the origin through an angle of 270° is $R_{270^\circ}(x,y) = (y,-x)$.

Thus, $R_{270^\circ}(1,0) = (0,-1)$, $R_{180^\circ}(1,1) = (1,-1)$ and $R_{180^\circ}(0,1) = (1,0)$.

That is, the image of the triangle with vertices $A(1,0)$, $B(1,1)$ and $C(0,1)$ is the triangle with vertices $A'(0,-1)$, $B'(1,-1)$ and $C'(1,0)$.

Rotation when the centre of rotation is (x_o, y_o)

So far you have seen rotation about the origin. The next activity introduces rotation about an arbitrary point (x_o, y_o).

Activity 1

If R is a rotation through $\frac{\pi}{4}$ about (3, 2), discuss how to determine the image of the point P (2, 0).

Theorem: (Rotation about an arbitrary Point)

If $P'(x', y')$ is the image of $P(x, y)$, after it has been rotated through an angle θ about (x_o, y_o), then

$$x' = x_o + (x - x_o)\cos\theta - (y - y_o)\sin\theta$$
$$y' = y_o + (x - x_o)\sin\theta + (y - y_o)\cos\theta$$

Example 4

Find the image of the triangle with vertices $A(1,0)$, $B(1,1)$ and $C(0,1)$ after a rotation about (4, –3).through each of the following angles.

(a) 90° (b) 180°

Solution

(a) Here $(x_o, y_o) = (4, -3)$ and $\theta = 90°$ The image of a point (x, y) after rotation about the origin through an angle of 90° is:

$$R_{90^\circ}(x,y) = \left(4 + (x-4)\cos 90^\circ - (y+3)\sin 90^\circ, -3 + (x-4)\sin 90^\circ + (y+3)\cos 90^\circ\right)$$
$$= (1 - y, x - 7)$$

$R_{90^\circ}(1, 0) = (1, -6)$, $R_{90^\circ}(1, 1) = (0, -6)$, $R_{90^\circ}(0, 1) = (0, -7)$.

That is, the image of the triangle with vertices $A(1,0)$, $B(1,1)$ and $C(0,1)$ is the triangle with vertices $A'(1,-6)$, $B'(0,-6)$ and $C'(0,-7)$.

(b) The image of a point (x, y) after rotation about the origin through an angle of 180° is:

$$R_{180^\circ}(4,y) = (x + (x-4)\cos 180^\circ - (y+3)\sin 180^\circ, -3 + (x-4)\sin 180^\circ$$
$$+ (y+1)\cos 180^\circ).$$

This implies, $R_{180^\circ}(x,y) = (8 - x, -y - 1)$.

Thus, $R_{180^\circ}(1,0) = (7,-1)$, $R_{180^\circ}(1,1) = (7,-2)$ and $R_{180^\circ}(0,1) = (8,-2)$.

That is, the image of the triangle with vertices $A(1,0)$, $B(1,1)$ and $C(0,1)$ is the triangle with vertices $A'(7,-1)$, $B'(7,-2)$ and $C'(8,-2)$.

EXERCISES

1. Find the point into which the given point is transformed by a rotation of the axes through the indicated angles, about the origin.
   (a) (–3, 4); 90°
   (b) (–2, 0); 60°
   (c) $(0, -1); \frac{\pi}{4}$
   (d) (–1, 2); 30°
2. Triangle *ABC* has vertices *A* (1, 2), *B* (4, 2) and *C* (4, –1). Find the images of the vertices of the triangle when the axes are rotated about the origin through an angle
   (a) $\theta = 90°$
   (b) $\theta = 180°$
3. Triangle *ABC* has vertices *A* (1, 2), *B* (4, 2) and *C* (4, –1). Find the images of the vertices of the triangle when the axes are rotated about the point (1,1) through an angle
   (a) $\theta = 90°$
   (b) $\theta = 180°$
4. Find the image of (1, 0) after it has been rotated -60^0 about (3, 2).
5. In a rotation *R*, the image of *A*(6, 2) is *A*′(3, 5) and the image of *B*(7, 3) is *B*′(2, 6). Find the image of (0, 0).

Similarity

Geometric figures that have the same shape are called similar figures, so similar geometric figures are figures which have exactly the same shape, but only size may be different. If two figures are similar, one of them can be obtained from the other by one of the two uniformly scaling techniques: reduction (also called shrinking) or enlarging.

DEFINITION

Two polygons are said to be **similar** if

1. The polygons have the same number of sides;
2. The ratios of the measures of corresponding sides are equal, that is, the corresponding sides are proportional and
3. Their corresponding angles are congruent.

Example 5

Any two equilateral triangles are similar.

Solution

If the two triangles ΔABC and ΔDEF are both equilateral triangles,

(i) They have the same number of sides, each three sides;

(ii) All the three angles of each triangle equal degree measure of $60°$.

(iii) For triangle ABC, $AB = AC = BC$ and for triangle DEF, $DE = DF = EF$. This implies

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}.$$

Hence, the two triangles, ΔABC and ΔDEF, are similar.

Example 6

Any two squares are similar.

Solution

Consider the two squares $ABCD$ and $EFGH$.

Then both squares have the same number of sides and each angle of the two squares measures $90°$.

Each of the four the sides of each of the two squares are equal.

That is;

$$AB = BC = CD = DA \text{ and } EF = FG = GH = HE.$$

This implies,

$$\frac{AB}{EF} = \frac{BC}{FG} = \frac{CD}{GH} = \frac{DA}{HE}$$

and hence the two squares are similar.

Magnification and reduction

Scale factors and proportionality

Scale factor (or constant of proportionality) of similar figures is the ratio of their corresponding sides and it is usually denoted by k.

Example 7

If ΔABC and ΔDEF are two equilateral triangles with 4 cm and 6 cm as the lengths of their respective sides, then the two triangles are similar and the scale factor (or constant of proportionality) of ΔABC and ΔDEF is $k = \frac{4}{6} = \frac{2}{3}$.

Central enlargement is a mapping which transform (either an increasing or decreasing) size of figures without affecting their shapes.

Under central enlargement

(a) Lines and their images are parallel.

(b) Angles remain the same.

(c) All lengths are increased or decreased by the same ratio, called the constant of proportionality or scale factor.

Steps for central enlargement

Given a plane figure with vertices A, B, … , constant of proportionality k and center of enlargement O, the following steps can be used to enlarge the given figure.

Step 1: Mark any point O in the same plane as the given figure. This point O is called centre of the central enlargement.

Step 2: Fix a positive number k, which is called the constant of proportionality.

Step 3: Determine the image of each point A such a A' such that $A'O = kAO$.

Step 4: The image of point O is itself.

In the process of the central enlargement,

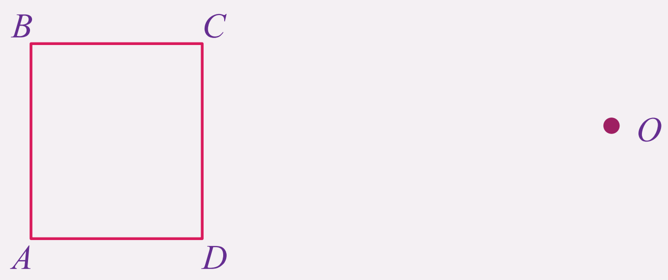
(i) if $k > 1$, the image is larger than the original and this process is called magnifications figure;

(ii) if $0 < k < 1$, the image is smaller than the original figure and this process is called reduction and

(iii) if $k = 1$, the image is has the same size as the original figure.

Example 8

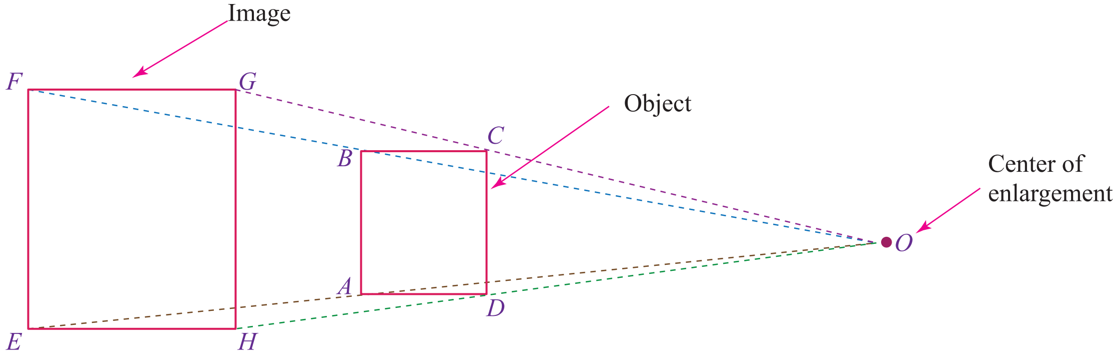
Draw the image of the rectangle *ABCD* shown below after an enlargement by scale factor 4 with centre *O* as shown below and label the image by *EFGH*.



Solution

Mark the points *E*, *F*, *G* and *H* on the lines $\overleftrightarrow{OA}$, $\overleftrightarrow{OB}$, $\overleftrightarrow{OC}$ and $\overleftrightarrow{OD}$, and such that $OE = 4OA$, $OF = 4OB$, $OG = 4OC$ and $OH = 4OD$ and as shown in the figure given below.

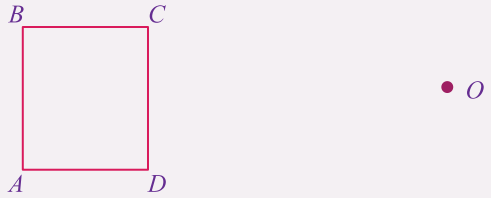
Then join the points *E*, *F*, *G* and *H* with line segments to obtain the rectangle *EFGH*, which is the image of *ABCD* with the given enlargement.
The two rectangles *ABCD* and *EFGH* are similar.



The image is larger in size than the object.

Example 9

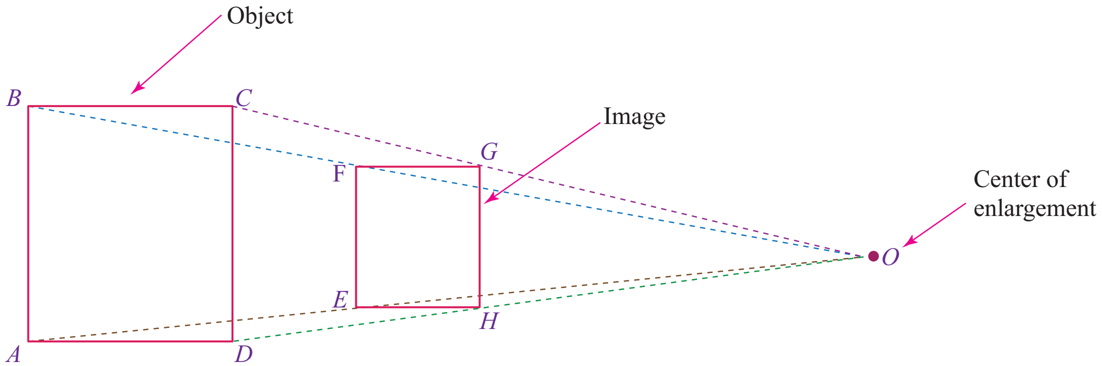
Draw the image of the rectangle *ABCD* shown below after an enlargement by scale factor $\frac{1}{2}$ with centre *O* as shown below and label the image by *EFGH*.



Solution

Mark the points E, F, G and H on the line segments $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ and $\overline{OD}$ such that $OE = \frac{1}{2}OA$, $OF = \frac{1}{2}OB$, $OG = \frac{1}{2}OC$ and $OH = \frac{1}{2}OD$ as shown in the figure given below.

Then join the points E, F, G and H with line segments to obtain the rectangle $EFGH$, which is the image of $ABCD$ with the given enlargement and he two rectangles $ABCD$ and $EFGH$ are similar.



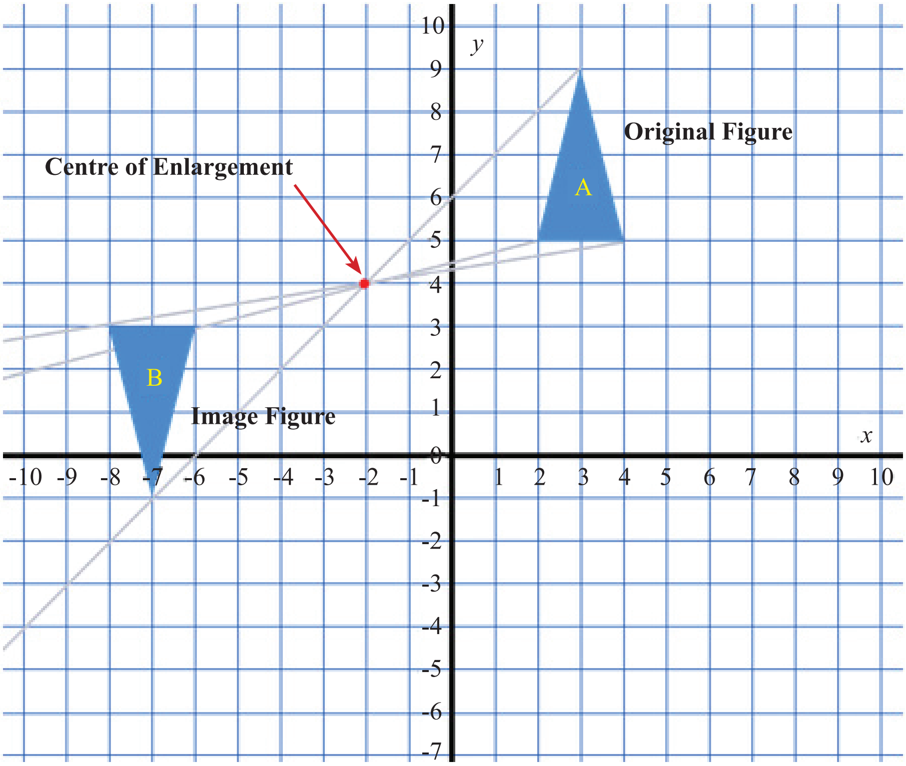
The image is smaller in size than the object.

These are positive enlargements, since both the object and the image are on the same side of the centre of enlargement O.

Furthermore, $\overline{AB} \parallel \overline{EF}$, $\overline{BC} \parallel \overline{FG}$, $\overline{CD} \parallel \overline{GH}$ and $\overline{AD} \parallel \overline{EH}$ and also for the angles we have $\angle A \equiv \angle E$, $\angle B \equiv \angle F$, $\angle C \equiv \angle G$ and $\angle D \equiv \angle H$

Negative enlargement

An enlargement using a negative scale factor will cause the enlargement to appear on the other side of the centre of enlargement; and will be inverted (upside down). The shape will also change size depending on the value of the enlargement.



Negative Enlargement

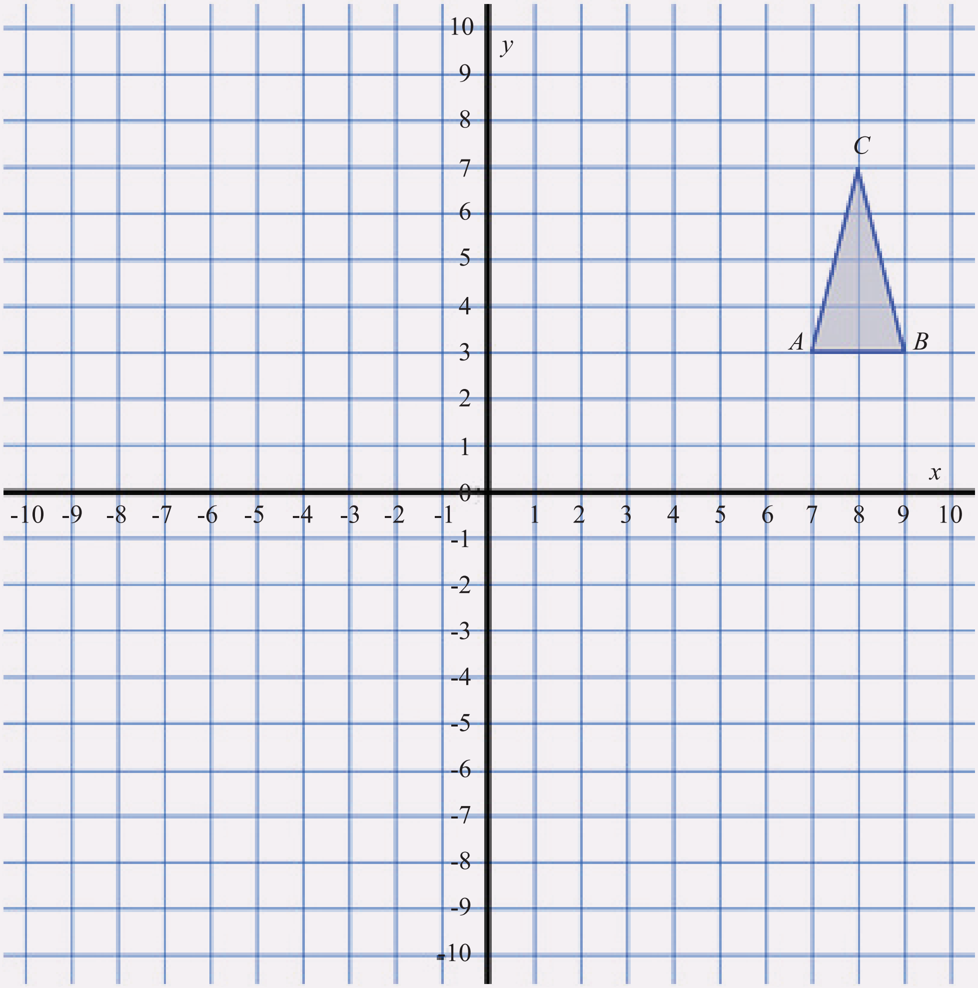
In Figure 1 shape A was mapped onto shape B with a scale factor of -1 From the figure observe that:

1. The original figure and the image are on opposite side of the centre of enlargement and
2. The ray lines pass through centre of enlargement and the figure is inverted.

Example 10

Let $A(7, 3)$, $B(9, 3)$ and $C(8, 7)$. Suppose ΔABC is mapped onto ΔDEF with a scale factor of $-\frac{1}{2}$, with a centre of enlargement at (5, 5). Then determine the coordinates of D, E and F

Solution

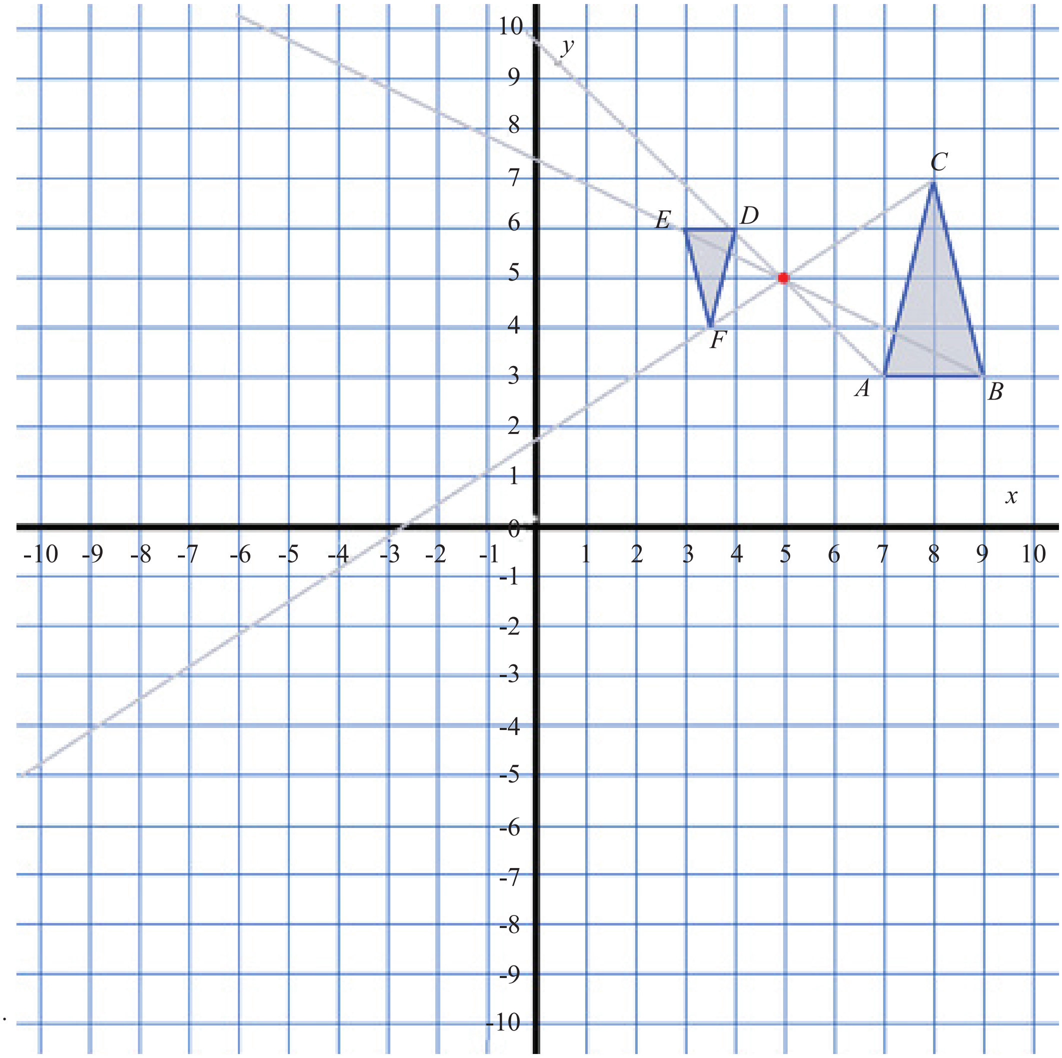


Plot the position of the centre of enlargement (5, 5).

Draw ray lines from the vertices through the centre of enlargement.

Determine the distances, using the grid lines, from the vertices of ΔABC to the centre of enlargement. Take half of these distances and determine the position of the vertices on the ray lines projected through the centre.

Then the coordinates of the vertices of ΔDEF are $D(4,6)$, $E(3,6)$ and $F(3.5,4)$



Two triangles ΔABC and ΔDEF are said to be similar, written as $\Delta ABC \sim \Delta DEF$, if

(i) Their corresponding angles are congruent, that is,

$\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle EFD$ and $\angle CAB \cong \angle FDE$ and

(ii) Their corresponding sides are proportional, that is

$$\frac{AB}{DE} = \frac{BC}{EF}, \ \frac{BC}{EF} = \frac{AC}{DF} \text{ and } \frac{AB}{DE} = \frac{AC}{DF}.$$

EXAMPLE 11

If ΔABC and ΔDEF are similar and AB =10 cm, AC = 12 cm, DF = 6 cm and EF = 5 cm, then find the lengths of the remaining sides of the two triangles.

Solution

If the two triangles ΔABC and ΔDEF are similar, then the corresponding sides are proportional and the corresponding angles are congruent.

This implies $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$ and $\frac{10}{DE} = \frac{BC}{5} = \frac{12}{6} = 2.$

Then

(i) $\frac{10 \text{ cm}}{DE} = 2$ implies $DE = \frac{10 \text{ cm}}{2} = 5 \text{ cm}$ and

(ii) $\frac{BC}{5 \text{ cm}} = 2$ implies $BC = 2 \times 5 \text{ cm} = 10 \text{ cm}.$

Theorem

If the ratio of the corresponding sides of two similar figures is k, then:

1. If the figures are polygons, the ratio of their perimeters is given by $\frac{P_1}{P_2} = \frac{S_1}{S_2} = k$ where P_1 is the perimeter of one polygon with one side S_1 and P_2 is the perimeter of the other polygon with corresponding side S_2
2. If the figures are polygons, the ratio of their areas is given by $\frac{A_1}{A_2} = \left(\frac{S_1}{S_2}\right)^2 = k^2$ where A_1 is the area of one polygon with one side S_1 and A_2 is the area of the other polygon with corresponding side S_2

3. If the figures are solids, the ratio of their volumes, is given by $\frac{V_1}{V_2} = \left(\frac{S_1}{S_2}\right)^3 = k^3$ where V_1 is the volume of one solid with one side S_1 and V_2 is the volume of the other solid with corresponding side S_2.

Example 12

Suppose two triangles are similar and the length of a side of one of the triangles is 5 times that of the corresponding side of the other. If the area of the smaller triangle is 36 sq cm, find the area of the larger triangle.

Solution

Let x be the length of a side of the triangle with A_1= 36 cm². If y is the length of the corresponding side of the other triangle, then $y = 5x$.

Since the two triangles are similar, then $\frac{y}{x} = 5$ is the constant of proportionality of the two similar triangles. If is the area of the second triangle is A_2 then $\frac{A_2}{A_1} = k^2 = 5^2 = 25$.

This implies $A_2 = A_1 k^2 = 36\text{ cm}^2 \times 25 = 900\text{ cm}^2$.

Example 13

The lengths sides of a hexagon are 2 cm, 4 cm, 5 cm,7 cm, 8 cm and 10 cm and the perimeter of a similar hexagon is 72 cm. Find the corresponding sides of the second hexagon.

Solution

For similar polygons, the ratio of the corresponding sides is the same as the ratio of their perimeters. If P_1 is the perimeter of the first hexagon, then

$$P_1 = 2\text{ cm} + 4\text{ cm} + 5\text{ cm} + 7\text{ cm} + 8\text{ cm} + 10\text{ cm} = 36\text{ cm}.$$

It is given that the perimeter P_2 of the second polygon is P_2 = 72 cm.

If we let a, b, c, d and e be the lengths of the sides of the second hexagon that corresponds to the respective sides of lengths 2 cm, 4 cm, 5 cm,7 cm, 8 cm and 10 cm for the second pentagon, then you have:

$$\frac{P_1}{P_2} = \frac{36}{72} = \frac{1}{2} = \frac{2}{a} = \frac{4}{b} = \frac{5}{c} = \frac{7}{d} = \frac{8}{e} = \frac{10}{f}.$$

This implies

$\frac{1}{2}=\frac{2}{a}$ implies $a=4$ cm, $\frac{1}{2}=\frac{4}{b}$ implies $b=8$, $\frac{1}{2}=\frac{5}{c}$ implies $c=10$ cm,

$\frac{1}{2}=\frac{7}{d}$ implies $d=14$cm, $\frac{1}{2}=\frac{8}{e}$ implies $e=16$ cm and $\frac{1}{2}=\frac{10}{f}$ implies $f=20$ cm.

Therefore, the corresponding sides of the given hexagon are:

4 cm, 8 cm, 10 cm, 14 cm, 16 cm and 20 cm.

Example 14

The length of the edge of a cube is 4 cm and the volume of a similar cube is 1728 cm^3. Find the length of the edge of the second cube.

Solution

If V_1 is the volume of the first cube with edge length S_1 and V_2 is the volume of the second cube with edge length S_2, then

$$\frac{V_1}{V_2}=\left(\frac{S_1}{S_2}\right)^3$$

$$V_1=(S_1)^3=(4\text{ cm})^3=64\text{ cm}^3 \text{ and } V_2=1728\text{ cm}^3.$$

Therefore, $\frac{64}{1728}=\left(\frac{4}{S_2}\right)^3=\frac{1}{27}$ implies $S_2=\sqrt[3]{4^3\times 27}=4\times 3=12$ cm.

Exercises

1. If ΔABC and ΔDBE are similar, $AB = 8$ cm, $BC = 6$ cm and $CE = 9$ cm, what is the length of $\overline{DE}$?
2. Two triangles are similar. The length of a side of one triangle is 5 times that of the length of the corresponding side of the other.
   (a) What is the ratio of the perimeter of the larger to the perimeter of the smaller triangle?
   (b) What is the ratio of the area of the larger to the area of the smaller triangle?
3. Two triangles ate similar. The ratio of the length of a side of one triangle to a corresponding side of the other is 2:3. If the perimeter of the larger triangle is 12 cm, what is the perimeter of the smaller triangle?

Key Terms

- Constant of proportionality
- Corresponding angles
- Corresponding sides
- Enlargement
- Proportionality
- Ratio of perimeters
- Ratio of areas
- Ratio of volumes
- Reduction
- Rotation
- Scale Factors
- Similar figures

Summary

- A rotation R about a point O through an angle θ is a transformation of the plane onto itself which maps every point P of the plane into the point P' of the plane such that $OP = OP'$ and $m(\angle POP') = \theta$

  Rotation formulae

  $x' = x \cos\theta - y \sin\theta$

  $y' = x \sin\theta + y \cos\theta$
- Two geometric figures are similar, if they have exactly the same shape, but not necessarily the same size, that is, one is an enlargement or reduction of the other without affecting its shape.
- The ratio of the perimeters of two similar figures is the same as the ratio of the lengths their corresponding sides.
- The ratio of areas of two similar figures is the same as the square of the ratio of the corresponding sides.
- The ratio of volumes of two similar solid figures is the same as the cube of the ratio of the corresponding sides.

Exercises

1. If the plane is rotated 30° about (1, 4) find the image of

   (a) The point (–3, 2) (b) The point (0, 1)
2. Two triangles are similar. The length of a side of one triangle is 3 times that of the length of the corresponding side of the other.

   (a) What is the ratio of the area of the larger to the area of the smaller triangle?

(b) What is the ratio of the perimeter of the larger to the area of the smaller triangle?

3. *ABCD* and *PQRS* are rectangles such that $ABCD \sim PQRS$. If $AB = 3$ cm, $PQ = 6$ cm and

(c) the area of *ABCD* is 15 cm^2, then find the area of *PQRS*;

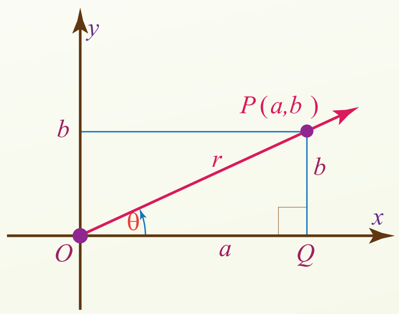
(d) the perimeter of *ABCD* is 20 cm, find the perimeter of *PQRS*.

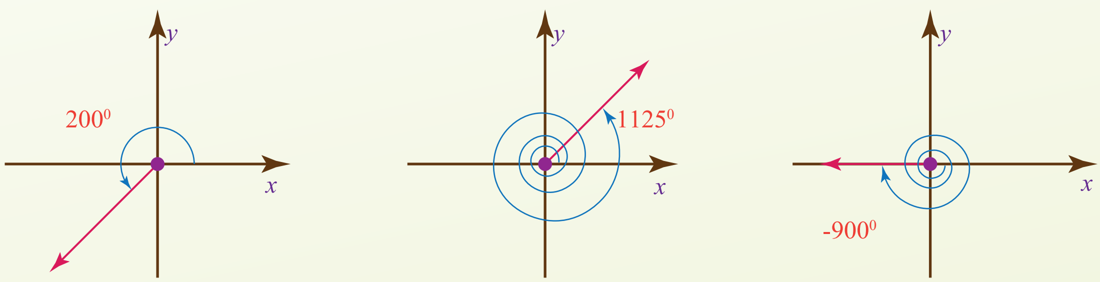
M12CH11

CHAPTER 11

TRIGONOMETRY 2

Chapter Contents





Chapter Outcomes

Upon completion of this chapter, learners will:

- draw the graphs of sin θ and cos θ, for $0° \leq \theta \leq 360°$;
- identify the difference between the graphs of sin θ and cos θ, for $0° \leq \theta \leq 360°$;
- identify the maximum and minimum values of the graphs of sin θ and cos θ, for $0° \leq \theta \leq 360°$;
- interpret the graphs of sin θ and cos θ, for $0° \leq \theta \leq 360°$.

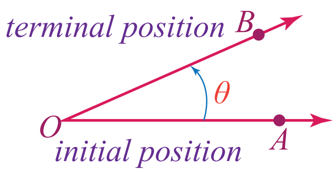
Introduction

In Mathematics, the trigonometric functions (also called circular functions) are functions of angles. They were originally used to relate the angles of a triangle to the lengths of the sides of a triangle. Trigonometric functions are useful in the study of triangles and also in many different phenomena in real life.

The most familiar trigonometric functions are sine and cosine functions. In this unit, you will be studying the properties of these functions in detail, including their graphs and some practical applications of the two basic trigonometric functions.

Basic terminology

If a given ray OA (written as $\overrightarrow{OA}$) rotates around a point O from its initial position to a new position, it forms an angle θ as shown below.



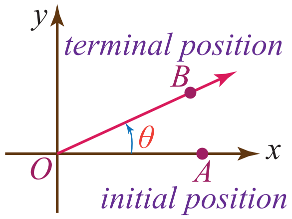


Figure 1. *a.* *b.*

$\overrightarrow{OA}$ (initial position) is called the initial side of θ .

$\overrightarrow{OB}$ (terminal position) is called the terminal side of θ.

The angle formed by a ray rotating anticlockwise is taken to be a positive angle.

An angle formed by a ray rotating clockwise is taken to be a negative angle.

Example 1

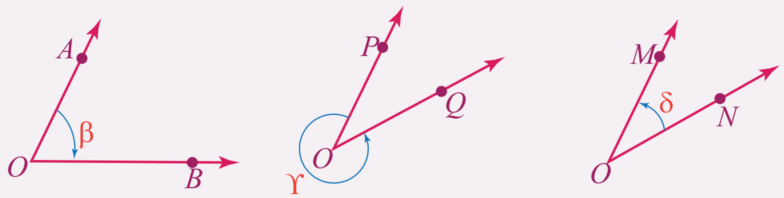


Figure 2. *a.* *b.* *c.*

- Angle β in Figure 2a is a negative angle with initial side $\overrightarrow{OA}$ and terminal side $\overrightarrow{OB}$.
- Angle ϒ in Figure 2b is a positive angle with initial side $\overrightarrow{OP}$ and terminal side $\overrightarrow{OQ}$.
- Angle δ in Figure 2c is a positive angle with initial side $\overrightarrow{ON}$ and terminal side $\overrightarrow{OM}$.

Angles in standard position

An angle in the coordinate plane is said to be in standard position, if

1. Its vertex is at the origin of the coordinate plane, and
2. Its initial side lies on the positive x-axis of the coordinate plane.

Example 2

The following angles are all in standard position:

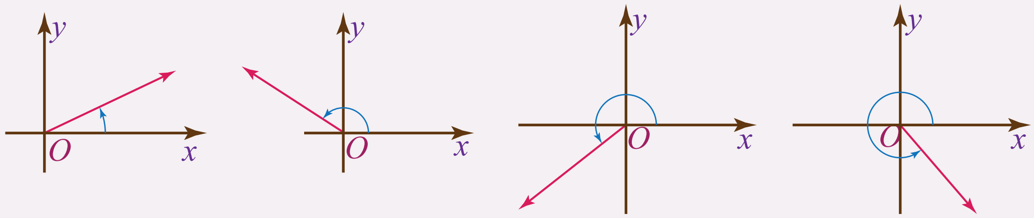


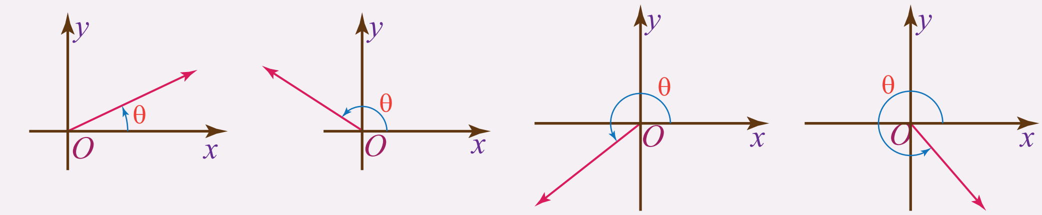
Figure 3. a. b. c. d.

First, second, third and fourth quadrant angles

- If the terminal side of an angle in standard position lies in the first quadrant, then it is called a first quadrant angle.
- If the terminal side of an angle in standard position lies in the second quadrant, then it is called a second quadrant angle.
- If the terminal side of an angle in standard position lies in the third quadrant, then it is called a third quadrant angle.
- If the terminal side of an angle in standard position lies in the fourth quadrant, then it is called a fourth quadrant angle.

Example 3

The following are angles in different quadrants:



θ is a 1st quadrant angle	θ is a 2nd quadrant angle	θ is a 3rd quadrant angle	θ is a 4th quadrant angle

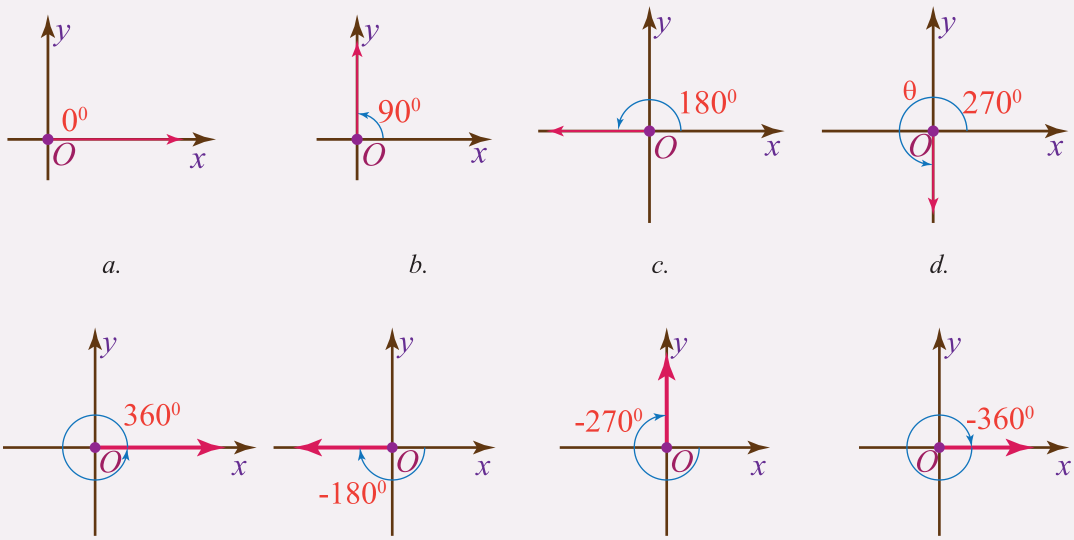
Figure 4. *a.* *b.* *c.* *d.*

Quadrantal angles

If the terminal side of an angle in standard position lies along the x-axis or the y-axis, then the angle is called a quadrantal angle.

Example 4

The following are all quadrantal angles.



a. *b.* *c.* *d.*

e. *f.* *g.* *h.*

Figure 5.

Angles with measures of –360°, –270°, –180°, –90°, 0°, 90°, 180°, 270°, 360° are examples of quadrantal angles because their terminal sides lie along the x-axis or the y-axis.

Example 5

The following are measures of different angles. Put the angles in standard position and indicate to which quadrant they belong:

(a) 200°

(b) 1225°

(c) –900°

Solution

(a) $200° = 180° + 20°$

Therefore, an angle with measure of 200° is a third quadrant angle.

(b) $1125° = 3(360)° + 45°$

1125° is a measure of a first quadrant angle.

(c) $–900° = 2(–360)° + (–180°)$

– 900° is a measure of a quadrantal angle.

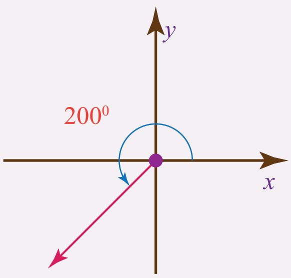


Figure 6.

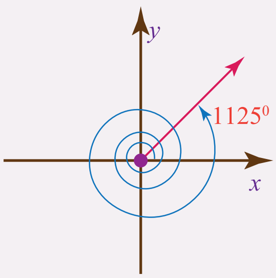


Figure 7.

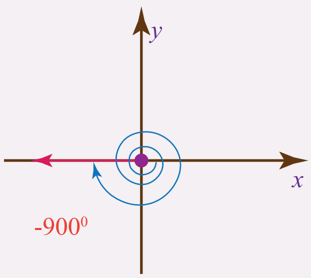


Figure 8.

Exercises

1. The following are measures of different angles. Put the angles in standard position and indicate to which quadrant they belong:

(a) 240°	(c) 620°	(e) –350°	(g) 550°
(b) 350°	(d) 666°	(f) –480°	(h) –1080°

Radian measures of angles

The angle θ subtended at the centre of a circle by an arc equal in length to the radius is 1 *radian*. That is $\theta \ \frac{r}{r} = 1$ *radian*. (*See* Figure 9a.)

Figure 9. *a.* *b.*

In general, if the length of the arc is *s* units and the radius is *r* units, then $\theta \ \frac{s}{r}$ radians. (See Figure 9b.) This indicates that the size of the angle is the ratio of the arc length to the length of the radius.

Example 6

If $s = 3$ cm and $r = 2$ cm, calculate θ in radians.

Solution

$$\theta = \frac{s}{r} = \frac{3}{2} = 1.5 \text{ radians}$$

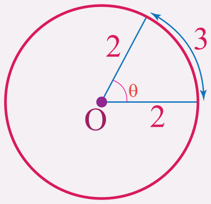


Figure 10.

Example 7

Convert 360° to radians.

Solution

A circle with radius r units has circumference $2\pi r$.

In this case $\theta = \frac{s}{r}$ becomes $\theta = \frac{2\pi r}{r} \Rightarrow \theta = 2\pi$

i.e., $3600 = 2\pi$ radians.

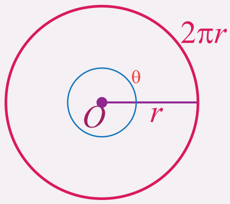


Figure 11.

EXAMPLE 8

Convert 180° to radian measure?

Solution

Since $360° = 2\pi$ radians, $180° = \pi$ rad, because $180° = \dfrac{360°}{2}$

It follows that 1 rad $= \dfrac{180°}{\pi} \cong 57.3°$

Rule 1

To convert degrees to radians, multiply by $\dfrac{\pi}{180°}$

i.e., radians = degrees $\times \dfrac{\pi}{180°}$.

EXAMPLE 9

(a) Convert 30° to radians. (b) Convert 240° to radians.

Solution

(a) $30° = 30° \times \dfrac{\pi}{180°} = \dfrac{\pi}{6}$ radians .

(b) $240° = 240° \times \dfrac{\pi}{180°} = \dfrac{4}{3}\pi$ radians.

Rule 2

To convert radians to degrees, multiply by $\dfrac{180°}{\pi}$.

i.e., degrees = radians $\times \dfrac{180°}{\pi}$.

EXAMPLE 10

(a) $\dfrac{\pi}{2}$ rad $= \dfrac{\pi}{2} \times \dfrac{180°}{\pi} = 90°$

(b) -4π rad $= -4\pi \times \dfrac{180°}{\pi} = -720°$

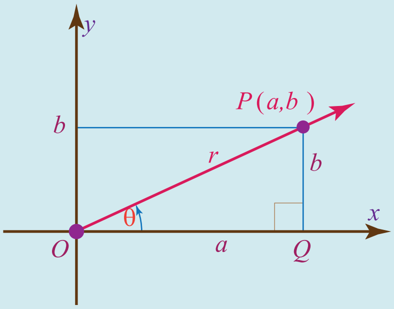
Exercises

1. Convert each of the following *degrees* to *radians*:
   (a) 60° (d) 90°
   (b) 45° (e) –270°
   (c) –150° (f) 135°
2. Convert each of the following *radians* to *degrees*:
   (a) $\frac{\pi}{12}$ (d) $\frac{5\pi}{6}$
   (b) $-\frac{\pi}{6}$ (e) $-\frac{10\pi}{3}$
   (c) $\frac{2\pi}{3}$ (f) 3

Definition of the sine and cosine functions

Definition

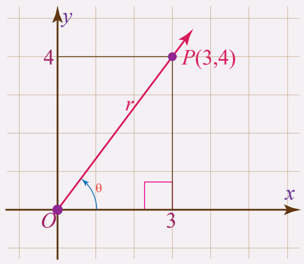
If θ is an angle in standard position and $P(a, b)$ is a point on the terminal side of θ, other than the origin $O(0, 0)$, and r is the distance of point P from the origin O, then



- The sine of angle θ, denoted by sin θ, is defined by $\sin\theta = \frac{b}{r}$.
- The cosine of angle θ, denoted by cos θ, is defined by $\cos\theta = \frac{a}{r}$.

Example 11

If θ is an angle in standard position and P (3, 4) is a point on the terminal side of θ, then evaluate the sine and cosine of θ.



Solution

The distance $r = \sqrt{3^2 + 4^2} = 5$ units

So $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$.

Figure 12.

Exercises

Evaluate the sine and cosine functions of θ, if θ is in standard position and its terminal side contains the given point $P(x, y)$:

(a) $P(3, -4)$

(b) $P(-6, -8)$

(c) $P(1, -1)$

(d) $P\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

(e) $P\left(4\sqrt{5}, -2\sqrt{5}\right)$

(f) $P(1, 0)$

Example 12

Find the values of the sine and cosine of θ ; if θ = 90°, 180°, 270°.

Solution

As shown in the Figure 3, (0, 1) lies on the terminal side of the angle with degree measure 90°.

Then, $a = 0$, $b = 1$ and $r = \sqrt{0^2 + 1^2} = 1$

Hence, $\sin 90° = b = 1$ and $\cos 90° = a = 0$

The point (–1, 0), lies on the terminal side of the angle with degree measure 180°.

Thus, $\sin 180° = y = 0$ and $\cos 180° = x = -1$.

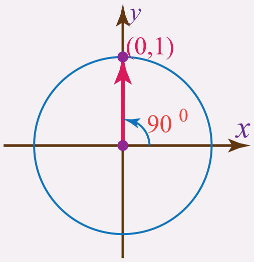


Figure 13.

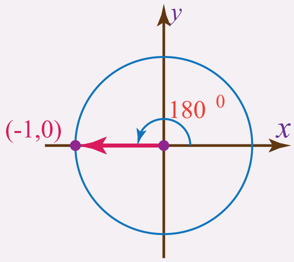


Figure 14.

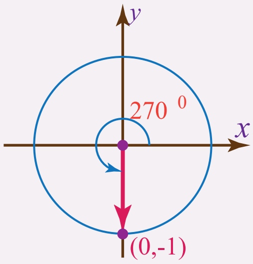


Figure 15.

The point (0, –1) lies on the terminal side of the angle with degree measure 270°.

Thus, $\sin 270° = y = -1$ and $\cos 270° = x = 0$.

Example 13

Find the values of the sine and cosine functions of each of the following quadrantal angles:

(a) 0°

(b) 360°

Activity 1

1. Find the values of the sine and cosine functions of θ and complete the following two tables:

θ	**0°**	**30°**	**45°**	**60°**	**90°**	**180°**	**270°**	**360°**
sin θ	0				1		–1	
cos θ						–1		
θ	–30°	–45°	–60°	–90°	–180°	–270°	–360°	
sin θ	$-\frac{1}{2}$		$-\frac{\sqrt{3}}{2}$					
cos θ	$\frac{\sqrt{3}}{2}$		$\frac{1}{2}$	0				

2. Find the values of the sine and cosine functions of θ and complete the following two tables:
   (a) sin (–30°) and sin (30°)
   (b) cos (–30°) and cos (30°)
   (c) sin (–45°) and sin (45°)
   (d) cos (–45°) and cos (45°)
   (e) sin (–60°) and sin (60°)
   (f) cos (–60°) and cos (60°)

From Activity 1 observe the following.

If θ is any angle, then sin (–θ) = –sin θ and cos (–θ) = cos θ.

Let us refer to Figure 16 to justify the above.

$\sin\theta = \frac{y}{r}$, $\sin(-\theta) = \frac{-y}{r} = -\left|\left(\frac{y}{r}\right)\right|$.

Therefore sin (– θ) = –sin θ

$\cos\theta = \frac{x}{r}$, $\cos(-\theta) = \frac{x}{r}$. Therefore cos (–θ) = cos θ.

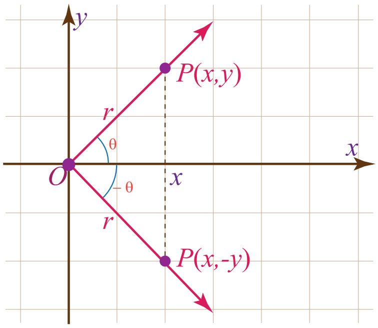


Figure 16.

The signs of sine and cosine functions

In this part you will consider whether the sign for each of the trigonometric functions of an angle is positive or negative.

The sign (whether sin θ and cos θ are positive or negative) depends on the quadrant to which θ belongs.

Example 14

Consider an angle θ in the first and second quadrants.

If θ is a first quadrant angle, then the sign of

$\sin\theta = \dfrac{y}{r}$ is positive

$\cos\theta = \dfrac{x}{r}$ is positive

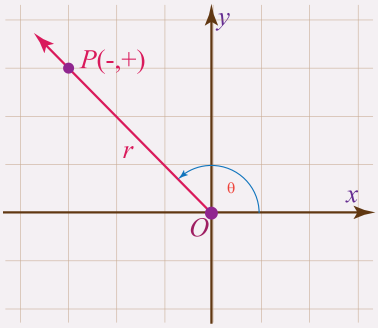


Figure 17.

If θ is a second quadrant angle then, the sign of

$\sin\theta = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{y}{r}$ is positive

$\cos\theta = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{x}{r}$ is negative since x is negative

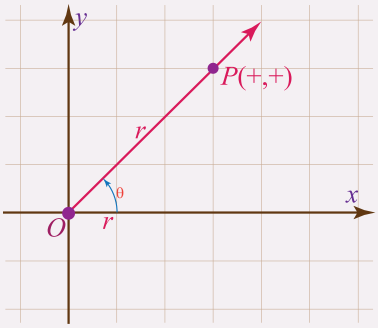


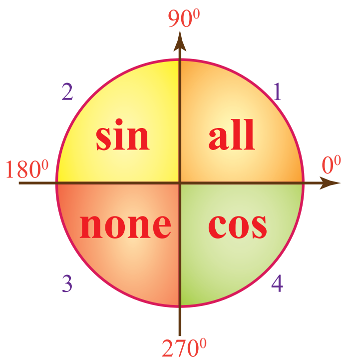
Figure 18.

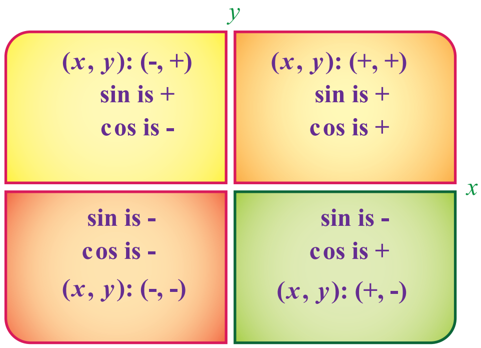
Activity 2

1. Determine whether the signs of sin θ and cos θ are positive or negative:
   (a) if θ is a third quadrant angle
   (b) if θ is a fourth quadrant angle
2. Decide whether the two trigonometric functions are positive or negative and complete the following table:

	θ has terminal side in quadrant			
	I	II	III	IV
sin θ	+			–
cos θ		–		

In general, the signs of the sine and cosine functions in all of the four quadrants can be summarized as below:





- In the first quadrant both trigonometric functions are positive.
- In the second quadrant only sine is positive.
- In the third quadrant both are negative.
- In the fourth quadrant only cosine is positive.

Example 15

Determine the sign of:

(a) $\sin 195°$

(b) $\cos 336°$

Solution

(a) Observe that $180° < 195° < 270°$. So angle $195°$ is a third quadrant angle. In the third quadrant the sine function is negative.

Therefore $\sin 195°$ is negative

(b) Since $270° < 336° < 360°$, the angle whose measure is $336°$ is a fourth quadrant angle. In the fourth quadrant the cosine function is positive.

Hence $\cos 336°$ is positive.

Reference angle(θ_R)

If θ is an angle in standard position whose terminal side does not lie on either coordinate axis, then a reference angle θ_R for θ is the acute angle formed by the terminal side of θ and the x-axis as shown in the following figures:

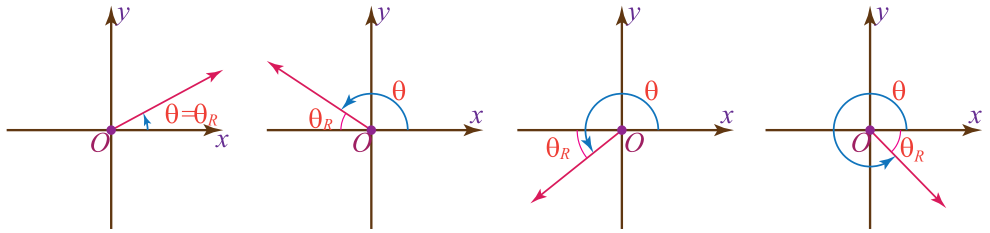


Figure 19. *a.* *b.* *c.* *d.*

Example 16

Find the reference angle θ_R for θ if:

(a) $\theta = 110°$ (b) $\theta = 212°$ (c) $\theta = 280°$

Solution

(a) Since $\theta = 110°$ is a second quadrant angle,

$\theta_R = 180 - 110° = 70°$

(b) Since $\theta = 212°$ is a third quadrant angle,

$\theta_R = 212° - 180° = 32°$

(c) Since $\theta = 280°$ is a fourth quadrant angle,

$\theta_R = 360° - 280° = 80°$

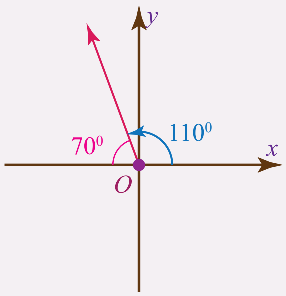


Figure 20.

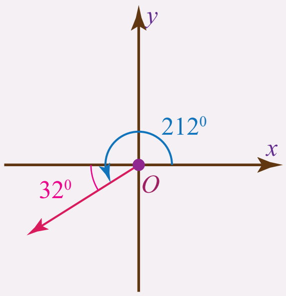


Figure21.

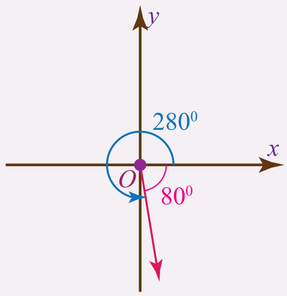


Figure 22.

Example 17

Find the reference angle θ_R for θ if:

(a) $\theta = 150°$ (d) $\theta = 320°$ (g) $\theta = 315°$

(b) $\theta = 170°$ (e) $\theta = 99°$

(c) $\theta = 240°$ (f) $\theta = 225°$

Values of the trigonometric functions of θ and its reference angle θ_R

Let us consider a second quadrant angle θ. Put θ in standard position as shown in the figure 23, and let $P(-x, y)$ be a point on its terminal side. Using the y-axis as an axis of symmetry, reflect P through the y-axis. This will give you another point $P'(x, y)$ which is the image of P on the terminal side of θ_R.

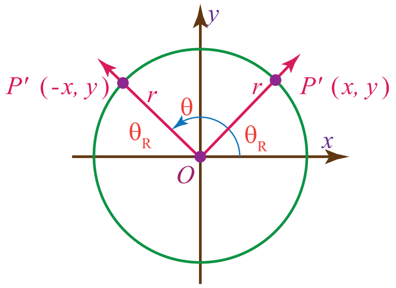


Figure 23.

This implies that $OP = OP'$, that is $OP = OP' = \sqrt{x^2 + y^2} = r$.

Hence, $\sin\theta = \frac{y}{r}$, $\sin\theta_R = \frac{y}{r} \Rightarrow \sin\theta = \sin\theta_R$

$\cos\theta = \frac{-x}{r}$, $\cos\theta_R = \frac{x}{r} \Rightarrow \cos\theta = -\cos\theta_R$.

The values of the trigonometric function of a given angle θ and the values of the corresponding trigonometric functions of the reference angle θ_R are the same in absolute value but they may differ in sign.

Example 18

Express the sine and cosine functions of 160° in terms of its reference angle.

Solution

Remember that an angle with measure 160° is a second quadrant angle .

In quadrant II, only sine is positive.

The reference angle $\theta_R = 180° - 160° = 20°$

Therefore, $\sin 160° = \sin 20°$ and $\cos 160° = -\cos 20°$.

Supplementary angles

Two angles are said to be supplementary, if the sum of their measures is equal to 180°.

Example 19

Pairs of angles with measures of 30° and 150°, 120° and 60°, 45° and 135°, 75° and 105°, 10° and 170° are examples of supplementary angles.

Example 20

Find the values of sin 150° and cos 150°.

Solution

The reference angle $\theta_R = 180° - 150° = 30°$

Therefore, $\sin 150° = \sin 30° = \frac{1}{2}$ and $\cos 150° = -\cos 30° = -\frac{\sqrt{3}}{2}$.

Example 21

Find the values of sin 240° and cos 240°.

Solution

The reference angle $\theta_R = 240° - 180° = 60°$

$\sin 240° = -\sin 60° = -\frac{\sqrt{3}}{2}$ and $\cos 240° = -\cos 60° = -\frac{1}{2}$.

Remember that in quadrant III both sine and cosine are negative.

In general,

If θ is a second quadrant angle, then its reference angle will be (180° – θ). Hence,

$$\sin\theta = \sin(180° - \theta) \text{ and } \cos\theta = -\cos(180° - \theta)$$

If θ is a third quadrant angle, its reference angle will be θ –180°.

Hence, $\sin\theta = -\sin(\theta - 180°)$ and $\cos\theta = -\cos(\theta - 180°)$.

Exercises

1. Express the sine and cosine of each of the following angle measures in terms of their reference angle:
   (a) 105°
   (b) 175°
   (c) 220°
   (d) –260°
   (e) –300°
   (f) 380°

2. Find the values of:
   (a) sin 135° and cos 135°
   (b) cos 143°, if cos 37° = 0.7986
   (c) sin 115°, if sin 65° = 0.9063
   (d) cos 24°, if cos 156° = –0.9135

In this section, you will draw and discuss some properties of the graphs of the two trigonometric functions: sine and cosine.

Graph of the sine function

Example 22

Draw the graph of $y = \sin\theta$, where $0° \leq \theta \leq 360°$.

Solution

To determine the graph of $y = \sin\theta$, we construct a table of values for $y = \sin\theta$, where $0° \leq \theta \leq 360°$ (which is the same as $0 \leq \theta \leq 2\pi$ in radians.)

The tables below show some of the values of $y = \sin\theta$ in the given interval.

θ in deg	0	30	60	90	120	150	180	210	240	270	300	330	360
θ in rad	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2}{3}\pi$	$\frac{5}{6}\pi$	π	$\frac{7}{6}\pi$	$\frac{4}{3}\pi$	$\frac{3}{2}\pi$	$\frac{5}{3}\pi$	$\frac{11}{6}\pi$	2π
***y* = sin θ**	0	0.5	0.87	1	0.87	0.5	0	– 0.5	– 0.87	–1	– 0.87	– 0.5	0

To draw the graph we mark the values of θ on the horizontal axis and the values of y on the vertical axis. Then we plot the points and connect them using a smooth curve.

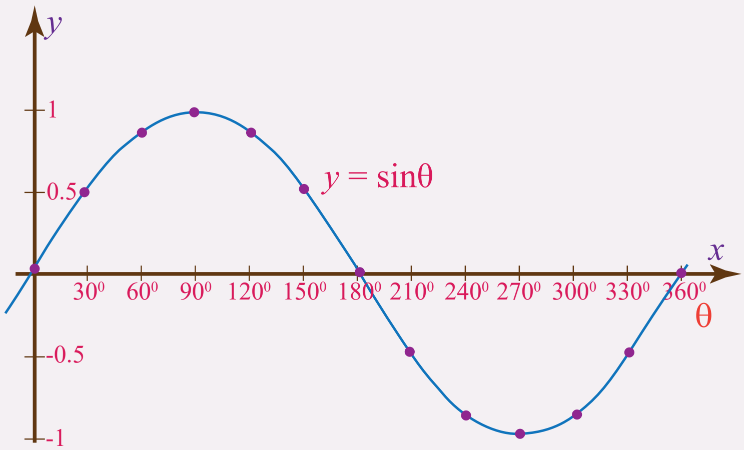


Figure 24.

Domain and range

For any angle θ taken on the unit circle, there is some point $P(x, y)$ on its terminal side. Since $\sin\theta = \frac{y}{1} = y$, the function $y = \sin\theta$ is defined for any angle θ taken on the unit circle.

Therefore, the domain of the sine function is the set of all real numbers. Also, note from the graph that the value of y is never less than –1 or greater than +1.

Note: The domain of the sine function is the set of all real numbers.

The range of the sine function is $\{y \mid -1 \le y \le 1\}$.

Graph of the cosine function

EXAMPLE 23

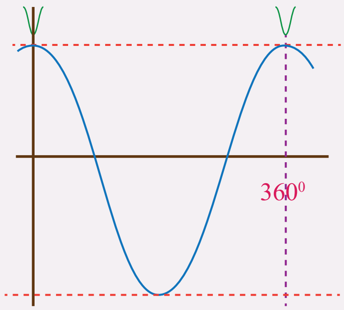
Draw the graph of $y = \cos\theta$, where $0° \le \theta \le 360°$

Solution

To determine the graph of $y = \cos\theta$, we construct a table of values for $y = \cos\theta$, where $0° \le \theta \le 360°$ (which is the same as $0 \le \theta \le \pi$ in radians.)

The tables below show some of the values of $y = \sin\theta$ in the given interval.

θ in deg	0	30	60	90	120	150	180	210	240	270	300	330	360
θ in rad	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2}{3}\pi$	$\frac{5}{6}\pi$	π	$\frac{7}{6}\pi$	$\frac{4}{3}\pi$	$\frac{3}{2}\pi$	$\frac{5}{3}\pi$	$\frac{11}{6}\pi$	2π
$y = \cos\theta$	1	0.87	0.5	0	– 0.5	– 0.87	– 1	–0.87	– 0.5	0	0.5	0.87	1



To draw the graph we mark the values of θ on the horizontal axis and the values of y on the vertical axis. Then we plot the points and connect them using a smooth curve.

Figure 25. Graph of $y = \cos\theta$ for $0° \le \theta \le 360°$

Note: The domain of the cosine function is the set of all real numbers.

The range of the cosine function is $\{y \mid -1 \le y \le 1\}$.

Figure 26 represents the sine and cosine functions drawn on the same co-ordinate system.

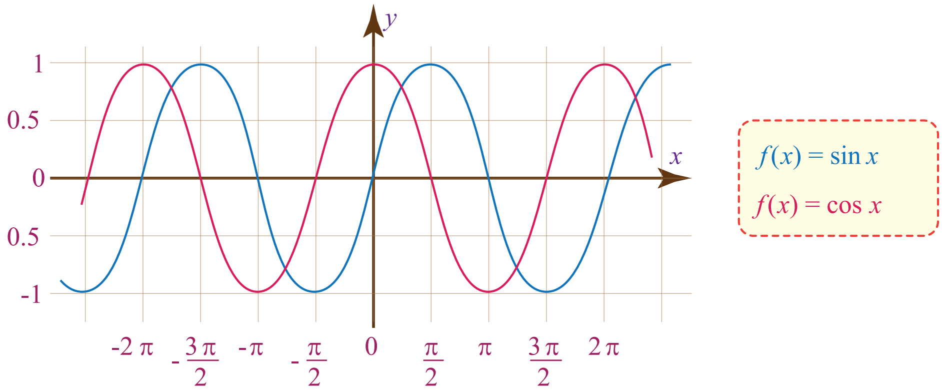


Figure 26.

From this diagram you can see that both sine and cosine curves have the same shape. The curves "follow" each other, always exactly $\frac{\pi}{2}$ radians (90°) apart.

Identification of maximum and minimum values from the graphs

Below is the graph of the function $f(x) = \sin x$ for $0 \leq x \leq 2\pi$.

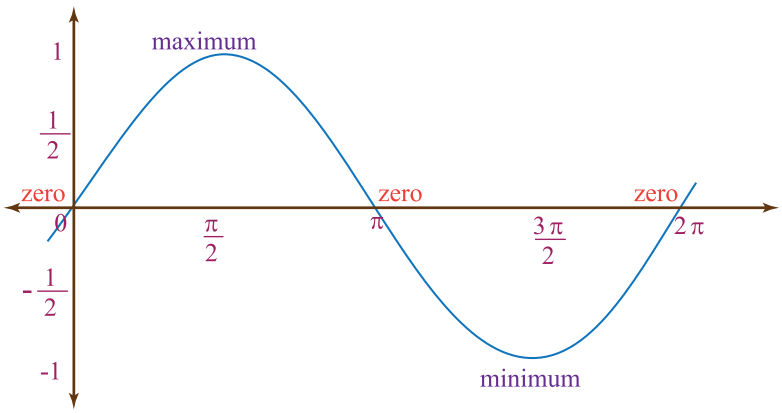


Figure 27. Graph of $f(x) = \sin x$ for $0 \leq x \leq 2\pi$.

The graph of the $f(x) = \sin x$ begins at the zero point, then rises to the maximum value of 1 as we move from 0 to $\frac{\pi}{2}$ rad and it then decreases back to 0 at $x = \pi$

radians. It then decreases and reaches its minimum value −1 at $x = \frac{3\pi}{2}$ and it then goes back up to 0 at 2π radians before starting all over again.

The standard cosine graph behaves in a similar but slightly different way as the sine function. The graph of the $f(x) = \cos x$ begins at the point (0, 1) and the maximum value is 1. Then it falls to 0 as we move from 0 to $\frac{\pi}{2}$ rad and it then decreases back to −1 at $x = \pi$ radians, which is its minimum value. It then increases and reaches its minimum value 0 at $x = \frac{3\pi}{2}$ and it then goes back up to 1 at 2π radians before starting all over again.

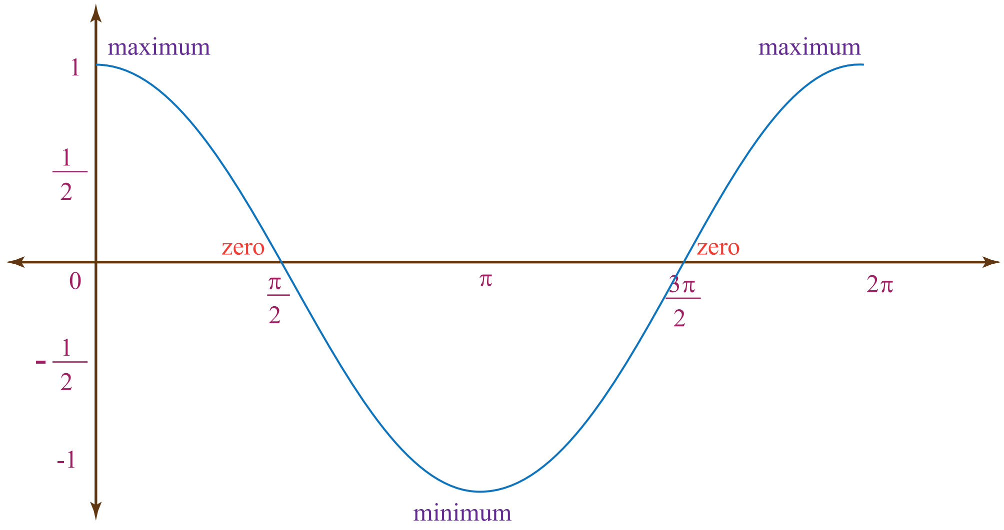


Figure 28. Graph of $f(x) = \cos x$ for $0 \leq x \leq 2\pi$.

Exercises

1. Refer to the graph of $y = \sin\theta$ or the table of values for $y = \sin\theta$ to determine how the sine function behaves as θ increases from 0° to 360° and answer the following:
   (a) As θ increases from 0° to 90°, sin θ increases from _____ to _____ .
   (b) As θ increases from 90° to 180°, sin θ decreases from _____ to _____ .
   (c) As θ increases from 180° to 270°, sin θ decreases from _____ to _____ .
   (d) As θ increases from 270° to 360°, sin θ increases from _____ to _____ .

2. Refer to the graph of $y = \cos\theta$ or the table of values for $y = \cos\theta$ to determine how the cosine function behaves as θ increases from $0°$ to $360°$ and answer the following:
   (a) As θ increases from $0°$ to $90°$, $\cos\theta$ decreases from _____ to _____ .
   (b) As θ increases from $90°$ to $180°$, $\cos\theta$ decreases from _____ to _____ .
   (c) As θ increases from $180°$ to $270°$, $\cos\theta$ increases from _____ to _____ .
   (d) As θ increases from $270°$ to $360°$, $\cos\theta$ increases from _____ to _____ .

Key Terms

- Angle
- Angle in standard position
- Cosine function
- CO–terminal angles
- Degree measure
- Negative angle
- Positive angle
- Quadrantal angle
- Radian measure
- Reference angle
- Sine function
- Supplementary angles
- Trigonometric function
- Trigonometry

Summary

- The rotation of a ray about its vertex from an initial position to a terminal position is called an angle.
- An angle is positive if it is formed by a rotation of a ray in anticlockwise direction and it is negative if it is formed by a rotation of a ray in clockwise direction.
- An angle in the coordinate plane is said to be in standard position, if its vertex is at the origin and its initial side is along the positive x-axis.
- The angle θ subtended at the centre of a circle by an arc equal in length to the radius is 1 radian.
- To convert degrees to radians, multiply the given degree by $\frac{\pi}{180^o}$.
- To convert radians to degrees, multiply the given radian by $\frac{180^o}{\pi}$.
- If θ is an angle in standard position and $P(x , y)$ is a point on the terminal side of θ, other than the origin $O(0 , 0)$, and r is the distance of point P from the origin O, that is, $r = \sqrt{x^2 + y^2}$ then $\sin\theta = \frac{y}{r}$ and $\cos\theta = \frac{x}{y}$.

- Signs of sine, cosine and tangent functions:
  - (i) In the first quadrant both the sine and cosine functions are positive.
  - (ii) In the second quadrant only sine is positive.
  - (iii) In the third quadrant both the sine and cosine functions are negative.
  - (iv) In the fourth quadrant only cosine is positive.
- Functions of negative angles:
  If θ is an angle in standard position, then
  $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$.
- Reference angle θ_R:
  If θ is an angle in standard position whose terminal side does not lie on either coordinate axis, then the reference angle θ_R for θ is the positive acute angle formed by the terminal side of θ and the x-axis.
- The values of the trigonometric function of a given angle θ and the values of the corresponding trigonometric functions of the reference angle θ_R are the same in absolute value but they may differ in sign.
- Supplementary angles:
  Two angles are said to be supplementary, if their sum is equal to 180°. If θ is a second quadrant angle, then its supplement will be $(180° - \theta)$.
  $\sin\theta = \sin(180° - \theta)$ and $\cos\theta = -\cos(180° - \theta)$,
  Co-terminal angles are angles in standard position (angles with the initial side on the positive x-axis) that have a common terminal side.
- Co-terminal angles have the same trigonometric values.
- The domain of the sine function is the set of all real numbers.
- The range of the sine function is $\{y \mid -1 \le y \le 1\}$.
- The domain of the cosine function is the set of all real numbers.
- The range of the cosine function is $\{y \mid -1 \le y \le 1\}$.

Exercises

1. Indicate to which quadrant each of the following angles belong:

(a) 225°	(c) –300°	(e) –700°
(b) 333°	(d) 610°	(f) 900°

(g) –765° (i) 1440°

(h) –1238° (j) 2010°.

2. Convert each of the following to radians:

(a) 40° (d) 330° (g) –220°

(b) 75° (e) –95° (h) –420°

(c) 240° (f) –180° (i) –3060°

3. Convert each of the following angles in radians to degrees:

(a) $\frac{2\pi}{6}$ (d) $\frac{43\pi}{6}$ (g) $\frac{-3\pi}{12}$

(b) $\frac{-2\pi}{3}$ (e) $-\frac{4\pi}{9}$ (h) $\frac{-\pi}{24}$.

(c) $\frac{7\pi}{18}$ (f) 5π

4. Use a unit circle to find the values of sine and cosine of θ when θ is:

(a) 810° (d) – 630° (g) 1080°

(b) – 450° (e) 990° (h) –1170°

(c) 900° (f) – 990°

5. Find the values of sine and cosine functions of θ when θ in radians is:

(a) $\frac{5\pi}{6}$ (c) $\frac{4\pi}{3}$ (f) $\frac{-5\pi}{3}$

(b) $\frac{7\pi}{6}$ (d) $\frac{3\pi}{2}$ (g) $\frac{-7\pi}{4}$

(e) $\frac{5\pi}{3}$ (h) $\frac{-11\pi}{6}$

6. State whether each of the following functional values are positive or negative:

(a) sin 310° (c) cos (–220°) (e) $\sin\left(\frac{-\pi}{6}\right)$

(b) cos 220° (d) sin (–90°)

7. Give a reference angle for each of the following;

(a) 140° (d) 414° (g) 1238°

(b) 260° (e) –190° (h) –1080°

(c) 355° (f) –336°

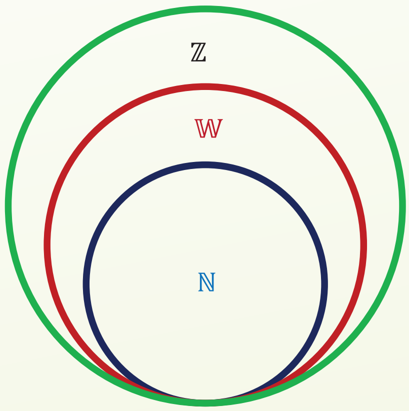
8. Find the value of each of the following:
   (a) sin (–120°) (c) cos (–300°) (e) sin 450°
   (b) cos 600° (d) sin 990° (f) cos (–420°).
9. Evaluate the two trigonometric functions of θ, if θ is in standard position and its terminal side contains the given point $P(x, y)$:
   (a) $P(5, 12)$ (e) $P(15, 8)$
   (b) $P(-7, 24)$ (f) $P(1, -8)$
   (c) $P(5, -6)$ (g) $P(-3, -4)$
   (d) $P(-8, -17)$ (h) $P(0, 1)$

CHAPTER 12

M12CH12

NUMBERS AND NUMERATION

Chapter Contents



Chapter Outcomes

Upon the completion of this chapter learners will able to:

- review real numbers with emphasis on (whole numbers, factors of whole numbers, multiples of whole numbers, prime numbers and prime factorization of whole numbers, integers, ratio of two whole numbers and rational numbers);
- convert from base ten to other bases and verse versa;
- solve problems in modular arithmetic;
- demonstrate identities in Commutative property, Associative property, Distributive property, Binomial expressions and properties of negatives;
- work and solve problems using powers and roots.

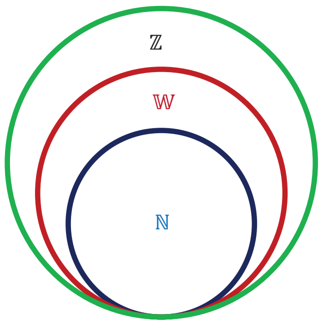
Introduction

Numbers are involved in almost all of our day to day activities and you have been learning about numbers starting from you early grades. The different number systems that you have learned in the previous grades are natural numbers, whole numbers, integers, rational numbers, real numbers and the basic operations on these number systems. In addition, you will learn about Number Bases, Modular Arithmetic and their applications.

Natural numbers, integers, prime and composite numbers

The first set of number that the human being started using were the set of natural numbers and this set is defined as $\mathbb{N}=\{1,2,3,\ldots\}$ later the number 0 was added on the set of natural numbers to form the set of whole numbers and the set of whole numbers is defined as $\mathbb{W}=\{0, 1, 2, 3,\ldots\}$. Whole numbers were not sufficient to represent different situations day to day activities that require of using numbers. For example to express profit and loss, altitudes below sea level, temperatures below 0°C, there was a need to extend the set of whole numbers and include negative numbers like $\cdots, -4, -3, -2, -1$ and the set that was formed by including these numbers with the set of whole number is the set of integers, denoted by. That is, $\mathbb{Z}=\{\cdots, -3, -2, -1, 0, 1, 2, 3, ...\}$

The three set have the following relationship. $\mathbb{N}\subseteq\mathbb{W}\subseteq\mathbb{Z}$ and this relationship can be expressed using a Venn diagram as follows.



The divisibility of some numbers

Given two whole numbers m and n, if there exists another whole number k such that $n=mk$, then we say that m divides n or n is divisible by m.

Example 1

1. $10=2\times5$ and this implies both 2 and 5 divide 10. That is, 10 is divisible by both 2 and 5.
2. 30 is not divisible by 4, because there is no whole number that can be multiplied by 4 to give us 30. For example, $4\times7=28$, $4\times9=36$ and $28<30<36$.

Determine the divisibility of a given by another number worth considering it.

For any natural number n, $n = n \times 1 = n$ and this implies every whole number is divisible by 1.

The different divisibility tests

Divisibility test for 2

A whole number is divisible by 2, if its unit's digit is an even number, that is, the unit's digit of the number is one of the numbers 0, 2, 4, 6, 8.

EXAMPLE 2

Determine if each the following numbers are divisible by 2.

1880, 10401, 10072, 23933, 10004, 231425 and 1905446.

Solution

The unit's digits of 1880, 10072, 10004 and 1905446 are 0, 2, 4 and 6 respectively, which are even. Hence, 1880, 10072, 10004 and 1905446 are divisible by 2.

The unit's digits of 10401, 23933 and 231425 are 1, 3 and 5 respectively, which are not even.

Therefore, 10401, 23933 and 231425 are not divisible by 2.

Divisibility tests for 3 and 9

A whole number is divisible by:

(a) 3 if the sum of its digits is divisible by 3.

(b) 9 if the sum of its digits is divisible by 9.

EXAMPLE 3

1. The sum of the digits of 3000009 is $3+0+0+0+0+0+9=12$ and 12 is divisible by 3. Hence 3000009 is divisible by 3.
2. The sum of the digits of 400001 is $4+0+0+0+0+1=5$ and 5 is not divisible by 3. Hence 400001 is not divisible by 3.
3. The sum of the digits of 4230306 is $4+2+3+0+3+0+6=18$ and 18 is divisible by 9. Hence the 4230306 is divisible by 9.
4. The sum of the digits of 2000000 is $2+0+0+0+0+0+0=2$ and 2 is not divisible by 9. Hence 2000000 is not divisible by 9.

Divisibility test for 4

A whole number is divisible by 4, if the two digits number formed by the last two digits of the given number is divisible by 4.

Example 4

Determine the divisibility of each of the following numbers by 4.

(a) 349156 (b) 324578

Solution

(a) The two digit number formed by the last two digits of the number is 56 and $56 = 4 \times 14$. This implies is 56 divisible by 4.

Hence 349156 is divisible by 4.

(b) The two digit number formed by the last two digits of the number is 78 and 78 is not divisible by 4.

Hence 324578 is not divisible by 4.

Divisibility tests for 5 and 10

(a) A whole number is divisible by 5, if its unit digit is 0 or 5.

(b) A whole number is divisible by 10, if its unit digit is 0.

Example 5

Determine the divisibility of each of the following numbers by 5 and 10.

1105, 100005, 20000225, 23550, 400000 and 10809000.

Solution

The unit's digits of all the numbers 1105, 100005, 20000225, 23550, 400000 and 10809000 are either 5 or 0.

Thus, all of these numbers are divisible by 5.

On the other hand, the unit's digits of 1105, 100005 and 20000225 are all 5. Thus, these numbers are not divisible by 10.

The unit's digits of 23550, 400000 and 10809000 are all 0 and hence all of them are divisible by 10.

Divisibility test for 6

A whole number is divisible by 6, if the number is divisible by both 2 and 3.

Example 6

Determine the divisibility of each of the following numbers by 6.

(a) 100098 (b) 213009.

Solution

(a) Since the unit's digit of 100098 is 8, and 8 is divisible by 2.

Hence 10098 is divisible by 2.

And also, the sum of the digits of 100098 is $1+0+0+0+9+8=18$ and 18 is divisible by 3. Thus 100098 is divisible by 3.

Therefore, 100098 is divisible by 6.

(b) The unit's digit of 213009 is 9 and 9 is not even. This implies, 213009 is not divisible by 2.

Thus, 213009 is not divisible by 6.

Divisibility test for 8

A whole number is divisible by 8 if sum of 4 times the hundreds digit, 2 times the tens digit and the unit digit of the number divisible is by 8 or the number formed by the last three digits of the number is divisible by 8.

Example 7

Determine the divisibility of each of the following numbers by 8.

(a) 107352 (b) 1000022

Solution

(a) $(4\times3)+(2\times5)+2=12+10+2=24$ and is divisible by 8.

Hence 107352 is divisible by 8.

(b) $(4\times0)+(2\times2)+2=6$ and 6 is not divisible by 8.

Thus, 1000022 is not divisible by 8.

Divisors, multiples, prime and composite numbers and prime factorization

If m, n and q are whole numbers and $n=m\times q$ then:

- m and q are called **Factors (Divisors)** of n;
- n is called a **Multiple** of both m and q.

In this case, we write this relation as $n\div m=q$ and $n\div q=m$.

For any number a, $a\times0=0$. That is, zero is a multiple of every number.

DEFINITION

Let m be a natural number and $m \neq 1$. Then

(a) m is said to be a **prime number** if it has exactly two different factors, the number itself and 1;

(b) m is said to be a **composite number** if it has more than two different factors or if it is not a **prime number**.

The number 1 is neither prime nor composite.

EXAMPLE 8

The numbers 2, 3, 7, 11, 13, 17, 19, 23 are all prime numbers and 4, 6, 8, 9, 10, 12 are composite numbers.

DEFINITION

A **prime factor** of a number is a factor of the number that is a prime number and writing a number as a product of all of its prime factors is called **prime factorization** of the number.

Theorem: (Fundamental Theorem of Arithmetic)

Every composite number can be expressed (or factorized) as a product of powers primes and this factorization is unique except order.

The following are steps that we can use to find prime factorization of a number.

1. First divide the number by the smallest prime number that divides the given number.
2. Divide the quotient in Step 1 by the smallest prime that divides it (the quotient).
3. Continue the process until you get a prime quotient.
4. The product of these primes is the prime factorization of the given number.

EXAMPLE 9

Find the prime factorization of each of the following numbers.

(a) 200 (b) 55 (c) 170.

Solution

(a) $200 = 2 \times 100$ (2 is the smallest prime that divides 200).

$100 = 2 \times 50$ (2 is the smallest prime that divides 100).

$50 = 2 \times 25$ (2 is the smallest prime that divides 50).

$25 = 5 \times 5$ (5 is the smallest prime that divides 25) and 5 is a prime number.

Thus, $200 = 2 \times 2 \times 2 \times 5 \times 5 = 2^3 \times 5^2$ is the prime factorization of 200.

(b) $66 = 2 \times 33$ (2 is the smallest prime number that divides 66).

$33 = 3 \times 11$ (3 is the smallest prime number that divides 33) and 11 is a prime number.

Therefore, $66 = 2 \times 3 \times 11$ is the prime factorization of 66.

(c) (2 is the smallest prime number that divides 170).

$85 = 5 \times 17$ (5 is the smallest prime number that divides 85) and 17 is a prime number.

Hence, the prime factorization of 170 is $170 = 2 \times 5 \times 17$.

The set of real numbers

Rational numbers

Definition

For integers a and b and $b \neq 0$, $\frac{a}{b}$ is called fraction. For natural numbers a and b, if $a < b$, then the fraction $\frac{a}{b}$ is called proper fraction, otherwise it is called improper fraction. For natural numbers a and b, the greatest common factor of a and b is 1, if then the faction $\frac{a}{b}$ is in simplest form.

Example 10

The fraction $\frac{30}{40}$ is a proper fraction but it is not in its simplest form, since the greatest common factor of 30 and 40 is 10, not 1.

Definition

The set of numbers of the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$ is called the set of rational numbers, denoted by $\mathbb{Q}$. That is, $\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0 \right\}$.

If a is an integer, then $a = \frac{a}{1}$, which is a rational number. Hence, we have the following relation; $\mathbb{Z} \subseteq \mathbb{Q}$.

Representation of rational numbers by decimals

Fraction is another form of division. For natural numbers a and b, $\frac{a}{b}$ means $a \div b$
When we divide an integer a by another integer b only one of the following two cases will occur.

1. The division process terminates when a zero remainder is obtained. In this case, the decimal obtained is called a terminating decimal.
2. The division process does not terminate, but there is a repeating of some digits. In this case, the decimal obtained is called a repeating decimal.

Example 11

The decimal representation of $\frac{2}{5} = 0.4$ (Terminating Decimal.)

The decimal representation of $\frac{1}{3} = 0.33333...$ (Repeating Decimal.)

Note

In decimal representation of a rational number a repetition of decimals is indicated by putting a bar on the repeating digits. For example, $0.00242424\ldots = 0.00\overline{24}$.
(The digits with the bar repeat themselves in the given order.)

Representation of decimals by fractions

Every number in decimal form, repeating or terminating decimal, can be expressed as a fraction of two integers.

1. If d is a terminating decimal with n-digits after the decimal point, then we write d in fraction form as: $d = \frac{d \times 10^n}{10^n}$.
2. If d is a repeating decimal with k- non-repeating and r-repeating decimals after the decimal point, then we write d in faction form as
$$d = \frac{d \times (10^{k+r} - 10^k)}{(10^{k+r} - 10^k)}$$

Example 12

Express each of the following decimals as fractions.

(a) 14.23 (b) $314.7\overline{02}$

Solution

(a) In 14.23 there are 2 nonrepeating digits after the decimal point. Therefore, a fraction form of 14.23 is $14.23 = \frac{14.23 \times 10^2}{10^2} = \frac{1423}{100}$.

(b) In 34.702, there is 1-nonrepeating digit and 2-repeating digits after the decimal point. Therefore, the fraction form of $34.7\overline{02}$ is:

$$d = \frac{d \times (10^3 - 10^1)}{(10^3 - 10^1)} = \frac{34702.\overline{02} - 347.\overline{02}}{990} = \frac{34355}{990}.$$

Operations on fractions

Given two fractions $\frac{a}{b}$ and $\frac{c}{d}$, where a, b, c, d are integers and $b \neq 0$ and $d \neq 0$:

(i) $\frac{a}{b} + \frac{c}{d} = \frac{(a \times d) + (b \times d)}{b \times d}$ (Sum of fractions)

(ii) $\frac{a}{b} - \frac{c}{d} = \frac{(a \times d) - (b \times d)}{b \times d}$ (Difference of fractions)

(iii) $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$ (Product of fractions)

(iv) $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a \times d}{b \times c}$ (Division of fractions and $c \neq 0$)

Exercises

1. Which one of the following is a prime number?

   (a) 277 (b) 161 (c) 687 (d) 711

2. From the following numbers, which one is not a rational number?

   (a) 4.52323232... (c) 2.0202020...

   (b) 0.01011011101... (d) 0.1010101...

3. What is the fractional form of the decimal number 4.8121212…?

Computer programmers use the idea of number bases and number base in one of the skills which is very useful to them.

The most commonly used number system is the decimal number system, the number system with base 10. The decimal number system uses the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 and all numbers in the decimal number system are formed by combining these ten digits.

Other number bases work in the same manner only with different number of digits. The number system that uses only the digit 0 and 1 is called Binary number system, the number system with base 2.

The number system that uses the digits 0, 1, 2, 3, 4, 5, 6 and 7 is called the Octal number system, the number system with base 8.

Each of the number system uses its digits to form all of the numbers contained within the system. In the decimal number system, the number system with base 10, the position a digit holds within a number can be described with the terms “ones”, “tens”, “hundreds”, “thousands”, etc. and these terms are words that describe powers of 10.

$10^0 = 1$ (ones), $10^1 = 10$ (tens), $10^2 = 100$ (hundreds),

$10^3 = 1000$ (thousands), $10^4 = 10000$ (ten thousands), etc.

We use a similar process to create numbers in other number systems and only difference is that, instead of raising 10 to a power, the new base number is used. For example, when working in base 8, we are working with powers of 8 ($8^0, 8^1, 8^2, 8^3, ...$). Base 2 works with powers of 2, base 3 works with powers of 3, base 4 works with powers of 4 and so on.

Converting a number from base ten to other bases

In order to change from base 10 to a different base, a successive division method is used. The given decimal numeral is divided repeatedly by the appropriate base number, and the remainders, including zero, are noted at each stage. The division process will until there is nothing left to divide. The answer is obtained by reading the remainders upwards, as indicated by the arrows in the following examples.

Example 13

Convert 234_{ten} to a number base 5.

Solution

We divide the number 234 repeatedly by 5.

5	234
5	46 rem 4
5	9 rem 1
5	1 rem 4
	0 rem 1

Then the answer is obtained by reading the reminders upwards as indicated by the arrow.
Thus, $234_{ten} = 1414_{five}$.

Example 14

Convert 65_{ten} to number base 2.

Solution

We divide the number 234 repeatedly by 5.

2	65
2	32 rem 1
2	16 rem 0
2	8 rem 0
2	4 rem 0
2	2 rem 0
2	1 rem 0
2	0 rem 1

Then the answer is obtained by reading the reminders upwards as indicated by the arrow.
Thus, $65_{ten} = 1000001_{two}$.

Exercises

Covert each of the following base ten numerals into the base indicated.

(a) 675 to base 5
(b) 896 to base 4
(c) 3795 to base 2
(d) 3476 to base 5
(e) 2725 to base 2
(f) 1000 to base 8

Converting from other bases to number base ten

Converting from other bases to numbers in base ten is simple as far as you remember each digit in the other base number represent a power of the base number.

Example 15

Convent 1100101_{two}, to the corresponding base ten numeral.

Solution

List the digits in order and count them from the RIGHT to LEFT, starting with zero

Digits	1	1	0	0	1	0	1
Numbering	6	5	4	3	2	1	0

Use this listing to convert each digit to

$$(1\times 2^6)+(1\times 2^5)+(0\times 2^4)+(0\times 2^3)+(1\times 2^2)+(0\times 2^1)+(1\times 2) = 64+32+4+2$$
$$=102.$$

Therefore, 1100101_{two} is equal to 102_{ten}.

Example 16

Convert 457_{eight} to number base 10.

Solution

List the digits in order, and count them from the RIGHT to LEFT, starting with zero:

Digits	4	5	7
Numbering	2	1	0

$$457_{eight} = (4\times 8^2)+(5\times 8^1)+(7\times 8^0) = (4\times 64)+(5\times 8)+(7\times 1) = 303_{ten}.$$

Exercises

Convert each of the following to base 10.

(a) 341_{eight} (b) 3456_{seven} (c) 21203_{four}

Computer programmers use the idea of number bases and number base in one of the skills which is very useful to them.

Activity 1

1. If today is Monday, what day will it be after 19 days?
2. If it is 10 hours now, what will be the time after 29 hours?

From your responses in Activity 1 observe that a week is divided in seven days and a day is divided in 24 hours.

DEFINITION

Two integers a and b are congruent modulo m if their difference is an integer multiple of m, i.e., $b-a=km$ for $m\in\mathbb{Z}$. This equivalence is written as $a\cong b(\text{mod}\,m)$.

EXAMPLE 17

For any integer m, show that every number is congruent to itself modulo m. That is, for any $a,m\in\mathbb{Z}$, $a\cong a(\text{mod}\,m)$.

Solution

For any $a,m\in\mathbb{Z}$, $a-a=$ 0, which is a multiple of m.

Thus, $a\cong a(\text{mod}\,m)$.

EXAMPLE 18

Show that every number is congruent to any other number modulo 1. That is, for any $a,b\in\mathbb{Z}$, $a\cong b(\text{mod}\,1)$.

Solution

Let $a,b\in\mathbb{Z}$. Then $a-b$ is a multiple of 1. Thus, $a\cong b(\text{mod}\,1)$.

EXAMPLE 19

Show that if a and b are any two even integers, then $a\cong b(\text{mod}\,2)$.

Solution

Let $a,b\in\mathbb{Z}$ be two even integers. Then $a-b$ is even and hence it is a multiple of 2. Thus, $a\cong b(\text{mod}\,2)$.

EXAMPLE 20

Determine if each of the following are true.

(a) $37\cong 25(\text{mod}\,3)$ (b) $100\cong 7(\text{mod}\,5)$ (c) $388\cong 126(\text{mod}\,9)$.

Solution

(a) $37-25=12=3\times 4$. This implies, the statement $37\cong 25(\text{mod}\,3)$ is true

(b) $100-7=93$ and 93 is not a multiple of 5. Therefore, the statement $100\cong 7(\text{mod}\,5)$ is false.

(c) $388-126=252=9\times 28$. This implies, $388\cong 126(\text{mod}\,9)$ is true.

Rules of modular arithmetic

ACTIVITY 2

Let a, b, c, d and m be integers such that $a \cong b(\text{mod}\, m)$ and $c \cong d(\text{mod}\, m)$
What can you say about each of the following.
(i) $a+c \cong b+d(\text{mod}\, m)$ (ii) $a-c \cong b-d(\text{mod}\, m)$ (iii) $a \times c \cong b \times d(\text{mod}\, m)$.

Addition

Theorem

Let a, b, c, d and m be integers. If $a \cong b(\text{mod}\, m)$ and $c \cong d(\text{mod}\, m)$, then $a+c \cong b+d(\text{mod}\, m)$.

Proof

Suppose $a \cong b(\text{mod}\, m)$ and $c \cong d(\text{mod}\, m)$. Then $a-b=mq$ and $c-d=mr$ for some $q, r \in \mathbb{Z}$.

This implies $(a-b)+(c-d)=mq+mr$ and $(a-b)+(c-d)=(a+c)-(b+d)$ by the commutative property of addition in $\mathbb{Z}$.

Thus, $(a+c)-(b+d)=m(q+r)$ and hence $a+c \cong b+d(\text{mod}\, m)$.

Theorem

Let a, b, c, d and m be integers. If $a \cong b(\text{mod}\, m)$ and $c \cong d(\text{mod}\, m)$, then $a-c \cong b-d(\text{mod}\, m)$.

Proof

Suppose $a \cong b(\text{mod}\, m)$ and $c \cong d(\text{mod}\, m)$. Then $a-b=mq$ and $c-d=mr$ for some $q, r \in \mathbb{Z}$.

This implies $(a-b)-(c-d)=mq-mr$ and $(a-b)-(c-d)=(a-c)-(b-d)$ by the commutative property of addition in $\mathbb{Z}$.

Thus, $(a-c)-(b-d)=m(q-r)$ and hence $a-c \cong b-d(\text{mod}\, m)$.

EXAMPLE 21

$7003 \cong 3(\text{mod}\, 7)$ and $70004 \cong 4(\text{mod}\, 7)$. Then $7003+70004=77007$ and $3+4=7$.

Thus, $77007 \cong 7(\text{mod}\, 7)$ and $7 \cong 0(\text{mod}\, 7)$.

Therefore, $7003+70004=3+4(\text{mod}\, 7)$.

Example 22

$7004 \cong 4 \pmod 7$ and $709 \cong 2 \pmod 7$. Then $7004 - 709 = 6295$ and $4 - 2 = 2$

As $6295 - 2 = 6293 = 7 \times 899$, then $6295 \cong 2 \pmod 7$.

Multiplication

Theorem

Let a, b, c, d and m be integers. If $a \cong b \pmod m$ and $c \cong d \pmod m$, then $a \times c \cong b \times d \pmod m$.

Proof

Suppose $a \cong b \pmod m$ and $c \equiv d \pmod m$. Then $a - b = mq$ and $c - d = mr$ for some $q, r \in \mathbb{Z}$.

This implies $a = b + mq$ and $c = d + mr$.

Then $ac = (b + mq)(d + mr) = bd + m(qd + br + mqr)$ and $qd + br + mqr \in \mathbb{Z}$.

Thus, $ac \equiv bd \pmod m$.

Example 23

Let a and b be two integers such that $a \equiv 3 \pmod 7$ and $b \equiv 4 \pmod 7$.

Then $ab \equiv 12 \equiv 5 \pmod 7$.

Example 24

Simplify 16^{753} in arithmetic modulo 5.

Solution

We know that $16 \equiv 1 \pmod 5$.

Then $16^{753} \equiv 1^{753} \pmod 5$.

Exercises

1. Find each of the following sums.

   (a) $25 + 17 \pmod 3$

   (b) $90 + 25 \pmod 9$

   (c) $1021 + 345 \pmod 7$

2. Find each of the following difference.
   (a) $25-17(\bmod 3)$
   (b) $90-25(\bmod 9)$
   (c) $1021-345(\bmod 7)$
3. Find each of the following products.
   (a) $25 \times 17 \pmod 3$
   (b) $90 \times 25 \pmod 9$
   (c) $1021 \times 345 \pmod 7$

DEFINITION

For two real numbers a and b;

1. If $a^2 = b$ then a is called a square root of b.
2. The positive square root of a number is called the principal square root of the give number.

EXAMPLE 25

Since $(-3)^2 = 9$ and $3^2 = 9$ both -3 and 3 are square roots of 9. But 3 is the principal square root of 9.

Note:

The sign $\sqrt{2}$ is a radical sign that is used to denote the principal square root.

EXAMPLE 26

$\sqrt{4} = 2$, $\sqrt{36} = 6$ and $\sqrt{\dfrac{25}{49}} = \dfrac{5}{7}$.

DEFINITION

For any two real numbers a and b and a positive integer n;

1. If $a^n = b$ then a is called an n-th root of b
2. The principal n-th root of b, denoted by $\sqrt[n]{b}$, is defined by:

$$\sqrt[n]{b} = \begin{cases} \text{the positive } n\text{-th root of } b \text{ if } b > 0 \\ \text{the negative } n\text{-th root of } b \text{ if } b < 0 \text{ nad if } n \text{ is odd} \\ 0 \text{ if } b = 0. \end{cases}$$

Example 27

(a) Since $(-2)^6 = 64$ and $2^6 = 64$ both -2 and 2 are 6th roots of 64.

(b) $\sqrt[6]{64} = 2$, $\sqrt[3]{27} = 3$, *and* $\sqrt[5]{-32} = -2$.

Note

For any three positive real numbers a, b and c and a positive integer n, if $a^n < b < c^n$ then $a < \sqrt[n]{b} < c$

Example 28

1. Since $3^2 < 10 < 4^2$ then $3 < \sqrt{10} < 4$.
2. Since $2^3 < 20 < 3^3$ then $2 < \sqrt[3]{20} < 3$.

Definition

If b is a real number and n is a positive integer greater than 1, then the power $b^{\left(\frac{1}{n}\right)}$ is defined by $b^{\left(\frac{1}{n}\right)} = \sqrt[n]{b}$.

Here n is called the **index of the radical,** $\sqrt{\ }$ is called the **radical symbol,** a is called the **radicand,** $\sqrt[n]{b}$, is the **radical form** of the n-th root of b and $b^{\left(\frac{1}{n}\right)}$ is the **exponential form** of the n-th root of b.

An expression containing a radical symbol is called a **radical expression**.

Example 29

$$\sqrt[5]{7} = 7^{\frac{1}{5}} \text{ and } \frac{\sqrt{3}}{\sqrt[3]{5}} = \frac{3^{\frac{1}{2}}}{5^{\frac{1}{3}}}.$$

Note

For any two real numbers a and b and a positive integer $n > 1$

1. $a^{\frac{1}{n}} b^{\frac{1}{n}} = \sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab} = (ab)^{\frac{1}{n}}$
2. $\dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}} = \left(\dfrac{a}{b}\right)^{\frac{1}{n}}$ (if $b \neq 0$.)
3. If $a \geq 0$ and m is a positive integer, $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{a}\right)^m$.

Example 30

Simplify each of the following.

(a) $\sqrt[3]{4}\sqrt[3]{2} = \sqrt[3]{4\times 2} = \sqrt[3]{8} = 2$

(b) $\frac{\sqrt{27}}{\sqrt{3}} = \sqrt{\frac{27}{3}} = \sqrt{9} = 3$

(c) $\frac{(128)^{\frac{1}{5}}}{4^{\frac{1}{5}}} = \left(\frac{128}{4}\right)^{\frac{1}{5}} = (32)^{\frac{1}{5}} = \sqrt[5]{32} = 2.$

Scientific notation (Standard form)

The mass of the Earth is approximately 5,972,000,000,000,000,000,000,000 kilograms. The molecular diameter of ammonia is 0.0000000297 centimeters. Very large and very small numbers are prone to errors when writing these numbers or typing them. To prevent these types of errors, these numbers can be written in a form called scientific notation.

Definition

A positive number is said to be in scientific notation (or standard form), if it can be written as a product of the form: $a \times 10^k$ where $1 \leq a < 10$ and k is an integer.

Note

The simplest method for conversion of standard notation to scientific notation is to move the decimal.

1. For very large numbers, move the decimal to the left, until it is after the first numeral. Count the number of places the decimal has moved. This then becomes the exponent on the 10.
2. For very small numbers, move the decimal to the right, until it is after the first non-zero numeral. Count the number of places the decimal has moved. This becomes a negative exponent on the 10.

Example 31

Determine if each of the following are true.

(a) $123,400,000,000,000$

(b) 0.0000000000032

Solution

(a) ince there are 14 digits after the right most digit of the number, the scientific notation of the given number is $123,400,000,000,000 = 1.234\times 10^{14}$

(b) The decimal point has to move 12 places to the right to get a nonzero digit. Therefore, the scientific notation of the number is $0.0000000000032 = 3.2\times 10^{-12}$

Scientific notation to standard form

To convert scientific notation to standard notation, reverse the procedure given above.

1. For very large numbers (positive exponent on the 10), move the decimal to the right the same number of places as given by the exponent.
2. For very small numbers (negative exponent on the 10), move the decimal to the left the same number of places as given by the exponent.

Example 32

Write each of the following numbers in decimal form.

(a) 1.45×10^{9} (b) 2.31×10^{-7}

Solution

(a) Move the decimal point 9 places to the right to obtain 1,450,000,000.

(b) Move the decimal point 5 places to the left to obtain 0.000,000,231.

Rationalizing denominators and simplifying quotients of radical expressions

A simplified radical expression cannot have a radical in the denominator. The procedure for removing a radical from the denominator is called **rationalizing the denominator**. The product property of radicals is used to rationalize a denominator.

Product property of radicals

If a and b are real numbers and n is a positive integer such that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then $\sqrt[n]{a}\sqrt[n]{b}=\sqrt[n]{ab}$, . In particular,

1. if $n=2$ and $a=b$, then $\sqrt{a}\sqrt{a}=\sqrt{a^2}=a$.
2. $\sqrt[n]{a^{n-1}}\sqrt[n]{a}=\sqrt[n]{a^n}=a$.

Example 33

Simplify (rationalize the denominator for) each of the following.

1. $\dfrac{2}{\sqrt{5}}$ 2. $\dfrac{3}{\sqrt{7x}}$ (for $x>0$) 3. $\dfrac{5}{\sqrt[3]{4}}$

Solution

1. $\dfrac{2}{\sqrt{5}}=\left(\dfrac{2}{\sqrt{5}}\right)\times\left(\dfrac{\sqrt{5}}{\sqrt{5}}\right)=\dfrac{2\sqrt{5}}{5}$.

2. $\dfrac{3}{\sqrt{7x}}=\left(\dfrac{3}{\sqrt{7x}}\right)\times\left(\dfrac{\sqrt{7x}}{\sqrt{7x}}\right)=\dfrac{3\sqrt{7x}}{7x}=\dfrac{3\sqrt{7x}}{7x}$.

3. $\dfrac{5}{\sqrt[3]{4}}=\left(\dfrac{5}{\sqrt[3]{4}}\right)\times\left(\dfrac{\sqrt[3]{2}}{\sqrt[3]{2}}\right)=\dfrac{5\sqrt[3]{2}}{\sqrt[3]{4\times2}}=\dfrac{5\sqrt[3]{2}}{\sqrt[3]{8}}=\dfrac{5\sqrt[3]{2}}{2}$.

Rationalize a two-term denominator

Conjugates

The **conjugate** of the two-term expression $a + b$ is $a - b$ and visa versa.

Fact

The product of a square-root expression and it's conjugate is an expression containing no square roots (i.e. a rational expression).

EXAMPLE 34

Simplify

(a) $\dfrac{7}{\sqrt{x}+4}$ (b) $\dfrac{\sqrt{a}+7}{3\sqrt{a}-2}$

Solution

(a) $\dfrac{7}{\sqrt{x}+4}=\left(\dfrac{7}{\sqrt{x}+4}\right)\times\left(\dfrac{\sqrt{x}-4}{\sqrt{x}-4}\right)=\dfrac{7(\sqrt{x}-4)}{x-16}.$

(b) $\dfrac{\sqrt{a}+7}{3\sqrt{a}-2}=\left(\dfrac{\sqrt{a}+7}{3\sqrt{a}-2}\right)\times\left(\dfrac{3\sqrt{a}+2}{3\sqrt{a}+2}\right)=\dfrac{(\sqrt{a}+7)(3\sqrt{a}+2)}{9a-4}$

$$=\frac{3a+23\sqrt{a}+14}{9a-4}.$$

EXERCISES

1. What is the simplified form of the expression $\dfrac{\sqrt[5]{192}}{\sqrt[5]{6}}$?
2. What is the expression that can be used to rationalize the denominator of $\dfrac{7}{3+\sqrt{5}}$?

KEY TERMS

- Base
- Divisibility
- Modular arithmetic
- Powers
- Rational number
- Radicals
- Repeating decimals
- Terminating decimals

Summary

- Given two whole numbers m and n, if there exists another whole number $k$such that $n = mk$, then we say that m divides n or n is divisible by m.
- Divisibility Test for 2: A whole number is divisible by 2, if its unit digit is an even number, that is, the unit's digit of the number is one of the numbers 0, 2, 4, 6, 8.
- Divisibility Tests for 3 and 9: A whole number is divisible by:
  - 3, if the sum of its digits is divisible by 3.
  - 9, if the sum of its digits is divisible by 9.
- Divisibility Test for 4: A whole number is divisible by 4, if the two digits number formed by the last two digits of the given number is divisible by 4.
- Divisibility Tests for 5 and 10
  - A whole number is divisible by 5, if its unit digit is 0 or 5.
  - A whole number is divisible by 10, if its unit digit is 0.
- Divisibility Test for 6: A whole number is divisible by 6, if the number is divisible by both 2 and 3.
- If m, n and q are whole numbers and $n = m \times q$ then:
  - m and q are called **Factors (Divisors)** of n;
  - n is called a **Multiple** of both m and q.
- In this case, the relation is written as $n \div m = q$ and $n \div m = q$.
- For any number a, $n \times 0 = 0$. That is, zero is a multiple of every number.
- Let m be a natural number and $m \neq 1$. Then
  - m is said to be a **prime number** if it has exactly two different factors, the number itself and 1;
  - m is said to be a **composite number** if it has more than two different factors or if it is not a **prime number**.
  - The number 1 is neither prime nor composite.
- A **prime factor** of a number is a factor of the number that is a prime and writing a number as a product of all of its prime factors is called **prime factorization** of the number.

- Every composite number can be expressed (or factorized) as a product of powers primes and this factorization is unique except order.
- For integers *a* and *b* and , is called fraction. For natural numbers and , if , then the fraction is called proper fraction, otherwise it is called improper fraction.
- For natural numbers and , if then the faction, is in simplest form.
- The set of numbers of the form $\frac{a}{b}$, where *a* and *b* are integers and $b \neq 0$ is called the set of rational numbers, denoted by $\mathbb{Q}$ That is, $\mathbb{Q} = \left\{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\right\}$.
- If *a* is an integer, then $a = \frac{a}{1}$, which is a rational number. Hence, we have the following relation; $\mathbb{Z} \subseteq \mathbb{Q}$.
- If *d* is a terminating decimal with *n*-digits after the decimal point, then we write *d* in fraction form as: $d = \frac{d \times 10^n}{10^n}$.
- If *d* is a repeating decimal with *k*- non-repeating and *r*- repeating decimals after the decimal point, then we write *d* in faction form as $d = \frac{d \times (10^{k+r} - 10^k)}{(10^{k+r} - 10^k)}$.
- Given two fractions $\frac{a}{b}$ and $\frac{c}{d}$, where a, b, c, d are integers and $b \neq 0$ and $d \neq 0$:
  - $\frac{a}{b} + \frac{c}{d} = \frac{(a \times d) + (b \times d)}{b \times d}$ (Sum of fractions)
  - $\frac{a}{b} - \frac{c}{d} = \frac{(a \times d) - (b \times d)}{b \times d}$ (Difference of fractions)
  - $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$ (Product of fractions)
  - $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \div \frac{d}{c} = \frac{a \times d}{b \times c}$ (Division of fractions and $c \neq 0$)
- Two integers *a* and *b* are congruent modulo *m* if their difference is by an integer multiple of m, i.e., $b - a = km$ for $m \in \mathbb{Z}$. This equivalence is written as $a \cong b(\bmod m)$.
- Let *a*, *b*, *c*, *d* and *m* be integers. If $a \cong b(\bmod m)$ and $c \cong d(\bmod m)$, then $a + c \cong b + d(\bmod m)$.
- Let *a*, *b*, *c*, *d* and *m* be integers. If $a \cong b(\bmod m)$ and $c \cong d(\bmod m)$, then $a - c \cong b - d(\bmod m)$.
- Let *a*, *b*, *c*, *d* and *m* be integers. If $a \cong b(\bmod m)$ and $c \cong d(\bmod m)$, then $a \times c \cong b \times d(\bmod m)$.

Exercises

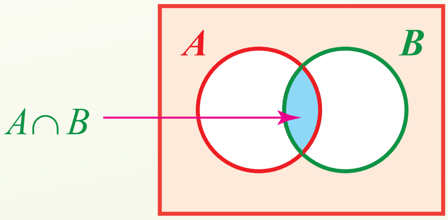
1. After rationalizing the denominator of the expression $\frac{\sqrt{2}}{\sqrt{2}+\sqrt{5}}$, what will be the resulting expression?
2. Determine the simplified form of the expression $\left(\frac{2}{3}\right)^{-2}\left(\frac{2}{\sqrt{3}}\right)^{4} \div \left(\frac{\sqrt{2}}{\sqrt[3]{3}}\right)^{6}$?
3. What is the decimal form of $2.15\times10^{(-3)}$?
4. Find two consecutive integers so that $\sqrt{35}-6$ between these numbers. is located
5. What is the decimal form of $\frac{5^2\times3^7\times10^3}{100\times81\times125}$?
6. What is the simplified form of the expression $\frac{2\sqrt{72}}{3}-\frac{3\sqrt{128}}{4}+5\sqrt{\frac{1}{2}}$?
7. What is the simplified form of the expression $\frac{(x+y)^2-1}{x+y+1}$?
8. When $x\neq-2y$ and $x\neq-y$, what is the simplified form of $\frac{3}{x+2y}-\frac{x-y}{(x+y)(x+2y)}+\frac{2}{x+y}$?
9. For $x>0$ and $x\neq3$, what is the simplified form of the expression $\left(\frac{x^{\left(\frac{5}{2}\right)}+81x^{\left(-\frac{3}{2}\right)}}{x+3}\right)\left(\frac{x^{\left(\frac{3}{2}\right)}}{x-3}\right)$?
10. For $a\neq1$ and $a\neq0$, what is the simplified form of the expression $\left(\frac{a}{1-a}+\frac{a+1}{a}\right)\div\left(\frac{a-1}{a}-\frac{a}{a+1}\right)$?
11. What is the simplified form of $\frac{3\sqrt{24}-2\sqrt{18}}{-\sqrt{2}}$?

M12CH13

CHAPTER 13

SETS

Chapter Contents



Chapter Outcomes

Upon completion of this chapter, learners will:

- define sets and use set notation;
- define and apply subsets;
- discuss universal set, equal sets, equivalent sets and listing the elements of a set;
- discuss Venn diagram and it to illustrate the following operations on sets intersection of sets, disjoint sets, union of sets, and complement of a set;
- discuss properties of set operations;
- solve two sets and three sets problems using Venn diagram;
- review open statements and implication and apply them using sets.

Introduction

In this unit, you will learn about definitions of sets, the different ways to describe sets and their representation through Venn diagrams. Different operations on sets and some applications of sets will also be considered.

Activity 1

Identify all numbers that belong to each of the following collections of numbers, if there are any.

(a) All prime natural numbers less than 10.

(b) All the natural numbers which are less than 100 and divisible by 10.

(c) All the natural numbers which are less than 1.

Each of the collections in Activity 1 are well-defined collections, that is, given a number, one can determine if that number is in the given collection or not. Such collections are called sets.

Definition

A well-defined collection of objects or individuals is called a set.

By well-defined, we mean that, given an object, we are able to determine whether the object is in the set or not, without any ambiguity.

The individual objects in a set are called the elements or the members of the set.

Example 1

Identify each of the following collections as a set or not a set.

(a) The group of all intelligent people in the world.

(b) The collection of all composite natural numbers less than 20.

(c) The group of all grade 12 students in your country.

Solution

(a) The word intelligent is an ambiguous word that, given an individual, different people have different opinions about the individual. Some may say the given individual is intelligent and some may say the individual is not intelligent. Therefore, "The collection of all intelligent people in the world" is not a set.

(b) All the composite natural numbers that less than 20 are 4, 6, 8, 9, 10, 12, 14. 15, 16. 18. That is, this collection of numbers is a well-defined collection, thus it is a set.

(c) Given an individual, it is simple is to determine whether that individual is a grade 12 student or not. This group is a well a defined group and hence it is a set.

Notation

1. Mostly, capital letters, like A, B, C, S, T, U, etc are used to name sets and small letters, like, a, b, c, x, y, z, etc are used to represent elements.
2. The symbol '$\in$' stands for the phrase 'is an element of' (or 'belongs to'). So, $x \in A$ is read as 'x is an element of A' or 'x belongs to A'. We write the statement 'x does not belong to A' as $x \notin A$.
3. A set can be denoted by putting it's elements by using the symbols called braces (or curly brackets) { } and separate them by a comma, whenever it is necessary. For instance, the set of all vowels in the English alphabet is written as {all vowels in the English alphabet} or $\{a, e, i, o, u\}$.

Ways of describing sets

There are different ways to describe sets and the most commonly used ways of describing sets are verbal method, listing or roster method and set builder method.

(i) Verbal method

You can describe a set in words. For example, consider the following sets.

(a) The set of all whole numbers greater than 100.

(b) The set of all integers less than 0.

(c) The set of all grade 12 students in the this country.

All these sets are described using verbal method, that is, in a sentence form.

(ii) The listing method (also called roster or enumeration method)

If the elements of a set can be listed, then you can describe the set by listing its elements. The elements can be listed completely if possible or partially and they follow a certain predefined pattern.

EXAMPLE 2

Describe (express) each of the following sets using the listing method:

(a) The set of the first five English alphabets.

(b) The set of natural numbers less than 100.

(c) The set of non-negative integers.

(d) The set of integers.

Solution

(a) Let A be the set of the first five English alphabets. Then the first five English alphabets are a, b, c, d and e. The we can write A as $A = \{a, b, c, d, e\}$. This listing method is also called complete listing method, because all the elements are completely listed.

(b) Let B be the set of natural numbers less than 100. The natural numbers less than 100 are 1, 2, 3, . . . ,100 and the set is given:

$$B = \{ 1, 2, 3, \ldots, 100\}.$$

The three dots after the element 3 (called an ellipsis which means "and so on") indicate that the elements in the set continue in that manner up to and including the last element 100.

This listing method is called partial listing method, because the elements are partially listed and shown by three dots that they are continuing unto the last element, 100.

(c) Let $\mathbb{W}$ be the set of non-negative integers. Then the non-negative integers are 0, 1, 2, 3 and continuing. Thus, the set is denoted by

$$\mathbb{W} = \{0, 1, 2, 3, \ldots\}.$$

The three dots indicate that the elements continue in the given pattern and there is no last or final element. This listing method is called a partial listing method.

(d) The set of integers is denoted by $\mathbb{Z}$ and is described by

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

The three dots in the left indicate that the elements are continuing to the left as indicated by the first three elements and the three dots in the right indicate that the elements are continuing as indicated by the first three elements. This method is also a partial listing method.

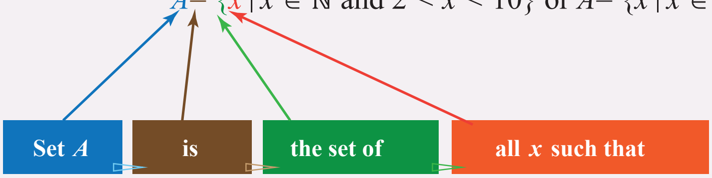
(iii) The set-builder method (also known as method of defining property)

You can define a set by describing the properties rather than listing its elements and this method is called a set builder method. Defining a set by using a set-builder method is also called a set comprehension or set abstraction.

Example 3

Let $A = \{3, 4, 5, 6, 7, 8, 9\}$. Then A can be described using set-builder method.

$$A = \{x \mid x \in \mathbb{N} \text{ and } 2 < x < 10\} \text{ or } A = \{x \mid x \in \mathbb{N} \text{ and } 2 < x < 10\}.$$



Note that "all x such that" may be written as "$x \mid$" using a vertical line "|" or "x:" using a colon "**:**".

Hence we read the above as "set A is the set of all elements x such that x is a natural number between 2 and 10".

Note that in the above set A, the properties that characterize the elements of the set are $x \in \mathbb{N}$ and $2 < x < 10$.

Example 4

Express each of the following sets using set-builder method:

(a) $\mathbb{N} = \{1, 2, 3, \ldots\}$.

(b) $A = \{\text{real numbers between 0 and 1}\}$.

(c) $B = \{\text{integers divisible by 3}\}$.

(d) The real solution set of $|x - 1| = 2$.

Solution

(a) $\mathbb{N} = \{x \mid x \in \mathbb{N}\}$ = the set of natural numbers.

(b) $A = \{x \mid x \in \mathbb{R} \text{ and } 0 < x < 1\}$ = the set of real numbers between 0 and 1.

Note that this set can also be expressed as $A = \{x \in \mathbb{R} \mid 0 < x < 1\}$.

(c) $B = \{x \mid x = 3n, \text{ for some integer } n\}$ or $B = \{3n \mid n \in \mathbb{Z}\}$.

(d) Let S be the solution set of the equation $|x - 1| = 2$. The it can be written as write

$S = \{x \mid x \in \mathbb{R} \text{ and } |x - 1| = 2\}$.

Exercises

1. Which of the following collections are well defined? Justify your answer.
   (a) $\{x \mid x \text{ is an interesting bird}\}$.
   (b) $\{x \mid x \text{ is a good student}\}$.
   (c) The set of natural numbers less than 100.
   (d) $\{y \mid y \text{ is a factor of } 13\}$.
2. Which of the following are true and which are false?
   (a) $2 \in \{-1, 0, 1\}$
   (b) $a \notin \{\{a, c\}\}$
   (c) $6 \in \{\text{factors of } 24\}$
3. Describe each of the following sets using a verbal method:
   (a) $A = \{5, 6, 7, 8, 9\}$
   (b) $M = \{2, 3, 5, 7, 11, 13, \ldots\}$
   (c) $G = \{8, 9, 10, \ldots\}$
   (d) $E = \{1, 3, 5, \ldots, 99\}$

4. Describe each of the following sets using the listing method (if possible):
   (a) The set of prime factors of 300.
   (b) The set of natural numbers that are less than 113 and divisible by 5.
   (c) The set of even natural numbers.
   (d) The set of real numbers between 1 and 3.
5. Describe each of the following sets by
   (i) the listing method. (ii) the set–builder method.
   (a) The set of letters in the word mathematics.
   (b) The set of capital cities in Africa.
   (c) The set of whole numbers between 5 and 13.
   (d) The set of even natural numbers less than 19.
   (e) The set of all odd natural numbers.
6. Describe each of the following sets by
   (i) verbal method. (ii) the set–builder method.
   (a) $\{1, 2, 3, \ldots, 10\}$
   (b) $\{1, 3, 5, 7, \ldots\}$
   (c) $\{5, 10, 15, 20, \ldots\}$
   (d) {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}
   (e) $\{2, 3, 5, 7, 11, \ldots\}$

ACTIVITY 2

What is the relationship between each of the following pairs of sets?

(a) M = {all students in your class whose names begin with a vowel letter};
N = {all students in your class whose names begin with the letter E}.

(b) $A = \{1, 3, 5, 7\}$; $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

(c) $E = \{x \in \mathbb{R} \mid (x-2)(x-3) = 0\}$; $F = \{x \in \mathbb{N} \mid 1 < x < 4\}$.

From your responses in Activity 2, you can have observed that: for some of the pairs.

(i) Every element of one set is contained in the other set.

(ii) Every element of a set is contained in another set and the second set has some elements that are not in the first set.

Definition

A set A is said to be a subset of set B, denoted by $A \subseteq B$, if every element of A is also an element of B.

A set A is said to be a proper subset of a set B, denoted by $A \subset B$, if A is a subset of B and B contains at least one element that is not in A.

Note: If A is not a subset of B, then we denote this by $A \not\subseteq B$.

Example 5

Let $\mathbb{Z} = \{x \mid x \text{ is an integer}\}$; $\mathbb{Q} = \{x \mid x \text{ is a rational number}\}$.

Since every integer is a rational number, then you have the relation $\mathbb{Z} \subseteq \mathbb{Q}$.

Example 6

Let $G = \{0, 1, 2, 3\}$ and $H = \{1, 2, 3, 4, 5\}$. Then

(a) $0 \in G$ but $0 \notin H$ and hence $G \not\subseteq H$;

(b) $5 \in H$, but $5 \notin G$ and hence $H \not\subseteq G$.

Example 7

Let $A = \{0, 1,2,3\}$ and $B = \{-2, -1, 0, 1, 2, 3\}$. Every element of A is an element of B. So, we have the relation $A \subseteq B$, and $-2 \in B$, but $-2 \notin A$. So $A \subset B$.

Note: For any set A

(i) $\varnothing \subseteq A$ (ii) $A \subseteq A$

(iii) A is not a proper subset of itself.

Definition

Let A be any set. The set of all subsets of A is called the power set of A, denoted by $P(A)$. That is, $P(A) = \{S \mid S \subseteq A\}$

Example 8

Let $A = \{0, 1\}$. Then subsets of A are $\varnothing$, $\{0\}$, $\{1\}$ and A itself.

Therefore $P(A) = \{\varnothing, \{0\}, \{1\}, A\}$ and the proper subsets of A are $\varnothing$, $\{0\}$, $\{1\}$.

Activity 3

Complete the following table.

Set	Number of elements of set A	Subsets	Number of Subsets of A	Proper subsets	Number of proper subsets
$A = \varnothing$	0	$\varnothing$	$1 = 2^0$	No proper subset	$0 = 2^0 – 1$
$A = \{0\}$	1	$\varnothing$, $\{0\}$	$2 = 2^1$	$\varnothing$	$1 = 2^1 – 1$
$A = \{0, 1\}$					
$A = \{0, 1, 2\}$		$\varnothing$, $\{0\}$, $\{1\}$, $\{2\}$, $\{0, 1\}$, $\{0, 2\}$, $\{1, 2\}$, $\{0, 1, 2\}$			$7 = 2^3 – 1$

From your responses in Activity 3, you can observe the following generalization about the number of subsets and the number of proper subsets of a given set.

Fact: If a set S has n elements, where n is a non negative integer, then

(i) The number of subsets of S is 2^n and

(ii) The number of proper subsets of S is $2^n – 1$, this is because, the proper subsets of A are all the subsets of A, except A itself.

Example 9

Find the number of subsets and the number of proper subsets of each of the following sets.

(a) $A = \{1, 2, 3, 4\}$ (b) $B = \{a, e, i, o, u\}$

Solution

(a) Set A has 4 elements. Thus, the number of subsets of A is $2^4 = 16$ and the number of proper subsets of A is $2^4 - 1 = 16 - 1 = 15$.

(b) Set B has 5 elements. Thus, the number of subsets of A is $2^5 = 32$ and the number of proper subsets of A is $2^5 - 1 = 32 - 1 = 31$.

Exercises

1. Determine whether each of the following statements is true or false. If it is false, justify your answer.

   (a) $\{1, 4, 3\} \subseteq \{3, 4, 1\}$ (c) $\{4\} \subseteq \{2, \{4\}\}$

   (b) $\{1, 3, 1, 2, 3, 2\} \not\subseteq \{1, 2, 3\}$ (d) $\varnothing \subseteq \{\varnothing, \{4\}\}$

2. Find all the subsets and all the proper subsets of each of the following sets.
   (a) $A = \{a, b, c, d\}$.
   (b) The set of alphabets in the word "book".
   (c) $B = \{2, 4, 6, 8, 10, 12\}$.
   (d) $C = \{0, \{1, 2\}\}$.

Equal and equivalent sets

Activity 4

Consider the following list of pairs of sets.

(a) $A = \{1, 2\}$; $B = \{x \in \mathbb{N} \mid x < 3\}$.

(b) $E = \{-1, 3\}$; $F = \left\{\frac{1}{2}, \frac{1}{3}\right\}$.

(c) $R = \{1, 2, 3\}$; $S = \{a, b, c\}$.

(d) $G = \{x \in \mathrm{N} \mid x \text{ is a factor of } 6\}$; $H = \{x \in \mathrm{N} \mid 6 \text{ is a multiple of } x\}$.

(e) $X = \{1, 1, 3, 2, 3, 1\}$; $Y = \{1, 2, 3\}$.

Identify the pairs of sets:

1. That have exactly the same elements;
2. That have the same number of elements, but need not be the same elements.

From your responses in Activity 4, observe that sets that have exactly the same elements are called equal sets and the pairs of sets that have the same number of elements are called equivalent sets.

A. Equality of sets

Definition

Given two sets A and B, if every element of A is also an element of B and if every element of B is also an element of A, then the sets A and B are said to be equal, written as $A = B$.

Thus, $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$.

Thus, two sets are equal if every element of one set is an element of the other set.

Example 10

Consider the following sets.

$$E = \{x \in \mathbb{R} \mid (x-2)(x-3) = 0\} \text{ and } F = \{x \in \mathbb{N} \mid 1 < x < 4\}.$$

Show that $E = F$.

Solution

By completely listing the elements of each set, we have $E = \{2, 3\}$ and $F = \{2, 3\}$.

We see that E and F have exactly the same elements. So they are equal, that is $E = F$.

Observe that $E \subseteq F$ and $F \subseteq E$.

Example 11

Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 4, 2, 3\}$.

Then $A = B$, since these sets contain exactly the same elements.

Note: If A and B are not equal, we write $A \neq B$.

Example 12

Let $C = \{-1, 3, 1\}$ and $D = \{-1, 0, 1, 2\}$.

$C \neq D$, because $2 \in D$, but $2 \notin C$.

B. Equivalence of sets

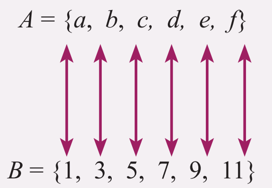
Definition

Two sets A and B are said to be equivalent, written as $A \leftrightarrow B$ (or $A \sim B$), if there is a one-to-one correspondence between the elements of the two sets.

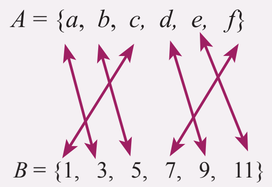
Observe that two finite sets A and B are equivalent, if and only if the number of elements in A and the number of elements in set B are equal.

Example 13

Consider the sets $A = \{a, b, c, d, e, f\}$ and $B = \{1, 3, 5, 7, 9, 11\}$. Even though these two sets are not equal, they have the same number of elements. So, for each member of set B you can find a unique member in set A that corresponds with the given element in B.



The double arrow shows how each element of a set is matched with an element of another set. This matching could be done in different ways, for example:



There is a one–to–one correspondence between A and B and hence A and B are equivalent sets, written as $A \leftrightarrow B$.

Example 14

Let $A = \{a, e, \pi\}$ and $B = \{1, 2, 3\}$.

A and B have three elements each and hence A and B are equivalent sets.

That is, $A \leftrightarrow B$.

Example 15

Consider the two sets $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{W} = \{0, 1, 2, 3, \ldots\}$.

Then the correspondence $n \leftrightarrow n - 1$ for any natural number n is a one-to-one correspondence between the two sets $\mathbb{N}$ and $\mathbb{W}$.

Thus, the two sets are equivalent, that is, $\mathbb{N} \leftrightarrow \mathbb{W}$.

Note

Equal sets are always equivalent since each element can be matched with itself, but equivalent sets are not necessarily equal. For example,

$$\{1, 2\} \leftrightarrow \{a, b\} \text{ but } \{1, 2\} \neq \{a, b\}.$$

Exercises

Determine each of the following pairs represents equal sets or equivalent sets.

1. $\{a, b, e, i, o, u\}$ and $\{2, 4, 6, 8, 10\}$.
2. $\{\varnothing\}$ and $\varnothing$.
3. $\{x \in \mathbb{N} \mid x < 10\}$ and $\{1,2, 3, 5, 7, 11, 13, 17, 19\}$.
4. $\{1, \{2, 4\}\}$ and $\{a, b, c\}$.
5. $\{x \in \mathbb{N} \mid x < x\}$ and $\{x \in \mathbb{N} \mid x < 1\}$.
6. The set of natural numbers and the set of integers.

Activity 5

How many elements does each of the following sets contain?

(a) $A = \{x \mid x \text{ is a real number whose square is negative one}\}$

(b) $B = \{x \mid x \in \mathbb{N} \text{ and } 2 < x < 11\}$

(c) $C = \{x \mid x \in \{1, 2, 3\}\}$

(d) $D = \{x \mid x \text{ is an integer}\}$

(e) $E = \{2, 4, 6, \ldots\}$

Observe from Activity 5, observe that a set may have no element, a finite number of elements or it has elements that not finite.

A. Empty Set

Definition

A set that contains no element is called an empty set, or null set or a void set. Empty set is denoted by either $\varnothing$ or $\{ \}$.

Example 16

(a) If $A = \{x \mid x$ is a real number and $x^2 = -1\}$, as there is no real number whose square is -1, $A = \varnothing$.

(b) If $B = \{x \in \mathbb{R} \mid x \neq x\}$, then for any real number x, $x = x$. So $B = \varnothing$.

(c) Let M be the set of months of a year with 32 days. There is no month of a year with 32 days, as each month of a year has at most 31 days. Therefore, $M = \varnothing$.

B. Finite and Infinite Sets

Definition

A set S is called finite, if S is empty set or it contains n elements, for some positive integer n.

A set S is called infinite, if it is not finite.

Notation: If a set S is finite, then we denote the number of elements of S by $n(S)$. If $S = \varnothing$, then $n(S) = 0$.

Example 17

Identify each of the following sets as finite and infinite.

(a) $A = \{-1, 0, 1\}$.

(b) $B = \{ x \in \mathbb{R} \mid x^2 = -1\}$.

(c) $C = \{x \in \mathbb{N} \mid x$ is a multiple of $3\}$.

(d) $D = \{x \in \mathbb{N} \mid x$ is a factor of $108\}$.

(e) $E = \{2, 4, 6, \ldots\}$.

(f) $F = \{x \mid x$ is a real number and $0 < x < 1\}$.

Solution

(a) A has only three elements, -1, 0 and 1. Therefore, A is a finite set and $n(A) = 3$.

(b) $B = \varnothing$. Then B is finite and $n(B) = 0$.

(c) $C = \{3, 6, 9, \ldots\}$ and the multiples of 3 are not finite. So C is an infinite set.

(d) $D = \{1, 2, 3, 4, 6, 9, 12, 18, 27, 36, 54, 108\}$ and D has 12 elements. Therefore, $n(D) = 12$.

(e) E contains the set of even natural numbers, which are not finite. Thus, E is an infinite.

(f) There are infinitely many real numbers between 0 and 1. Therefore, F is an infinite set.

C. Universal Set

DEFINITION

A set that contains all the elements under consideration in a particular discussion is called a **universal set**. A universal set is usually denoted by U.

The main purpose of the "universal set" is to focus your attention on the objects of interest and its need will become apparent in our discussions on set operations. However, there may be more than one universal set associated with a given set.

EXAMPLE 18

Consider two sets, $A = \{x, y, z\}$ and $B = \{1, 2, 3, x, y\}$, then $U = \{1, 2, 3, x, y, z\}$ can be considered as a universal set associated with both sets A and B.

EXAMPLE 19

Consider the following group in a secondary school.

G = {all Grade 12 students}.

I = {all students interested in a school play}.

R = {all class representatives of each class in the school}.

S = {all students in the school}.

Each set G, I and R is a subset of S. Then S is a universal set in this particular discussion.

To illustrate various relationships that can arise between sets, we often use pictorial representations called Venn diagrams. These diagrams consist of rectangles or closed curves, usually circles and the elements of the sets are written in their respective circles.

For example, the relationship '$A \subset B$' can be illustrated by the following Venn diagram.

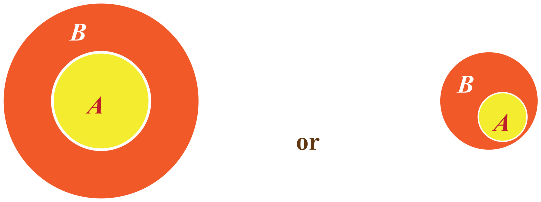


Figure 1.

Example 20

Represent the following pairs of sets using Venn diagrams:

(a) $A = \{a, b, c, d\}$; $B = \{a, d\}$.

(b) $C = \{2, 4, 6, 8,\ldots\}$; $D = \{1, 3, 5, 7,\ldots\}$.

(c) $E = \{2^n \mid n \in \mathbb{N}\}$; $F = \{2n \mid n \in \mathbb{N}\}$.

(d) $A = \{1, 3, 5, 7, 9\}$; $B = \{2, 3, 5, 8\}$; $C = \{1, 5, 7\}$.

Solution

(a)

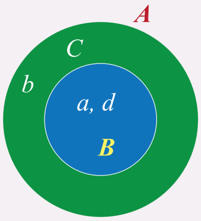


Figure 2.

(c)

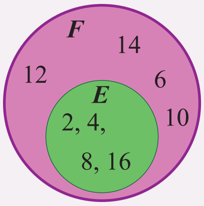


Figure 4.

(b)

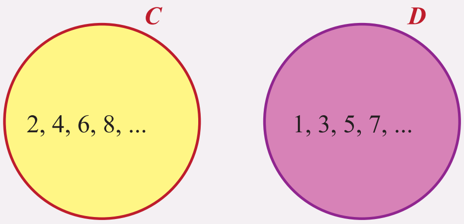


Figure 3.

(d)

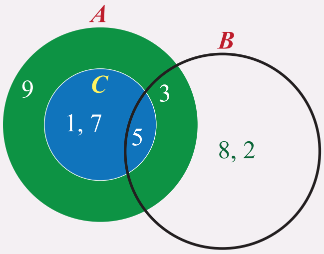


Figure 5.

Exercises

1. Draw Venn diagrams to illustrate the relationships between the following pairs of sets:

   (a) $A = \{1, 9, 2, 7, 4\}$; $L = \{4, 9, 8, 2\}$.

(b) B = {the vowels in the English alphabet}
M = {the first five letters of the English alphabet}.

(c) $C = \{1, 2, 3, 4, 5\}$; $M = \{6, 9, 10, 8, 7\}$.

(d) $F = \{3, 7, 11, 5, 9\}$; O = {all odd numbers between 2 and 12}

2. For each of the following, draw a Venn diagram to illustrate the relationship between the sets:

(a) U = {all animals}; C = {all cows}; G ={all goats}

(b) U = {all people}; M = {all males}; B = {all boys}

3. What is the relationship between the following pairs of sets?

(a) $\mathbb{W} = \{ 0, 1, 2,\ldots\}$ and $\mathbb{N} = \{ 1, 2, 3,\ldots\}$.

(b) $\mathbb{W} = \{0, 1, 2,\ldots\}$ and $\mathbb{Z} = \{\ldots,-3, -2, -1, 0, 1, 2,\ldots\}$.

(c) $\mathbb{N} = \{ 1, 2, 3,\ldots\}$ and $\mathbb{Z} = \{\ldots,-3, -2, -1, 0, 1, 2,\ldots\}$.

(d) $\mathbb{Z} = \{\ldots,-3, -2, -1, 0, 1, 2,\ldots\}$ and $\mathbb{Q} = \left\{ \frac{a}{b} : a,b \in \mathbb{Z},\ b \neq 0 \right\}$.

Express the relationship between each pair using a diagram.

Express the relationship of all the sets, $\mathbb{W}$, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ using one diagram.

There are different operations on sets and some of these operations are intersection, union, complement, relative complement and symmetric difference of sets.

Activity 6

Consider the following two sets. $G = \{2, 4, 6, 8, 10, 12\}$ and $H = \{1, 2, 3, 4, 5\}$.

(a) Determine the set that contain elements from either of the two sets..

(b) Determine the set that contain elements from both sets.

From your responses in Activity 6, the set that contains elements from either of the two sets is called the union of the two sets and the set of elements that are in both sets is called the intersection of the two sets.

A. Union of sets

Definition

The union of two sets A and B, denoted by $A \cup B$ and read "A union B", is the set of all elements that are members of set A or set B or both of the sets.

That is, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

Example 21

Let $A = \{a, b, c, d, e\}$ and $B = \{c, d, e, f, g\}$. Then $A \cup B = \{a, b, c, d, e, f, g\}$.

Example 22

(a) $\{a, b\} \cup \{c, d, e\} = \{a, b, c, d, e\}$

(b) $\{1, 2, 3, 4, 5\} \cup \varnothing = \{1, 2, 3, 4, 5\}$

Example 23

The *red* shaded region of the diagram in the figure below represents $A \cup B$.

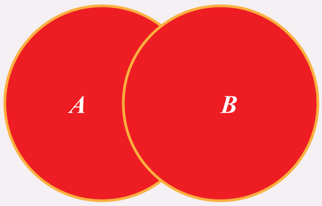


Figure 6.

Properties of the union of sets

Activity 7

Let $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$, $C = \{3, 4, 5, 6\}$ and $D = \{2, 4\}$ and the universal set $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

1. Find
   (a) $A \cup B$ (b) $B \cup A$ (c) $A \cup D$
   (a) What is the relationship between $A \cup$B and $B \cup A$?
   (b) What is the relationship between A and D?
   (c) What is the relationship between $A \cup D$ and A?
2. Find
   (a) $A \cup B$ (c) $B \cup C$
   (b) $(A \cup B) \cup C$ (d) $A \cup (B \cup C)$
   What is the relationship between $(A \cup B) \cup C$ and $A \cup (B \cup C)$?
3. Find $A \cup \varnothing$, what is the relationship between $A \cup \varnothing$ and A?
4. Find $A \cup B$, what is the relationship between $A \cup B$ and U?

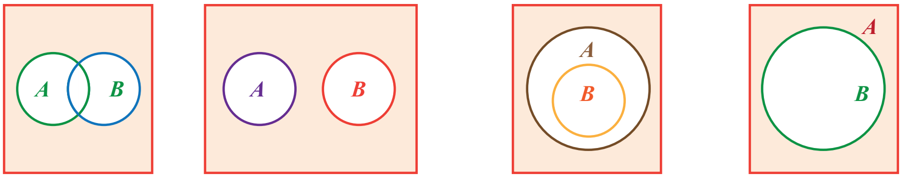
From your responses in Activity 7, you must have observed the following properties of union of sets.

For any sets A, B and C and a universal set U:

1. $A \cup B = B \cup A$ (Union of sets is Commutative)
2. $(A \cup B) \cup C = A \cup (B \cup C)$ (Union of sets is Associative)
3. $A \cup \varnothing = A$ ($\varnothing$ is Identity for Union)
4. $A \subseteq B$, then $A \cup B = B$.
5. $A \cup U = U$.

Exercises

1. Given $A = \{1, 2, \{3\}\}$, $B = \{2, 3\}$ and $C = \{\{3\}, 4\}$, find:
   (a) $A \cup B$
   (b) $B \cup C$
   (c) $A \cup C$
   (d) $A \cup (B \cup C)$
   (e) $(A \cup B) \cup C$
2. Determine each of the following statements as true or false:
   (a) If $x \in A$ and $x \notin B$, then $x \notin (A \cup B)$.
   (b) If $x \in (A \cup B)$ and $x \notin A$, then $x \in B$.
   (c) If $x \notin A$ and $x \notin B$, then $x \notin (A \cup B)$.
   (d) For any set A, $A \cup A = A$.
   (e) For any set A, $A \cup \varnothing = A$.
   (f) If $A \subseteq B$, then $A \cup B = B$.
   (g) For any two sets A and B, $A \subseteq (A \cup B)$ and $B \subseteq (A \cup B)$.
   (h) For any three sets A, B and C, if $A \subseteq B$, and $B \subset C$, then $A \cup B = C$.
   (i) For any three sets A, B and C, if $A \cup B = C$, then $B \subset C$.
   (j) If $A \cup B = \varnothing$, then $A = \varnothing$ and $B = \varnothing$.
3. Using copies of the Venn diagrams below, shade $A \cup B$.



(a) (b) (c) (d)

Figure 7.

B. Intersection of sets

Definition

The intersection of two sets A and B, denoted by $A \cap B$ and read as "A intersection B", is the set of all elements in both sets A and B. That is, $A \cap B = \{ x \mid x \in A \text{ and } x \in B\}$.

Example 24

Let $A = \{a, b, c, d, e\}$ and $B = \{a, e, i, o, u\}$. Then find $A \cap B$.

Solution

a and e are the only elements that are common for both A and B.

That is, $A \cap B = \{a, e\}$.

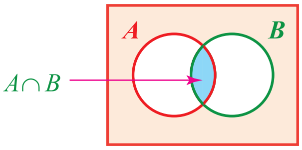
Example 25

Let $C = \{2, 4, 6, ...\}$ (multiples of 2) and $D = \{3, 6, 9, ...\}$ (multiples of 3). Then find $C \cap D$.

Solution

The common elements in both sets C and D are the multiples of 6.

Then $C \cap D = \{6, 12, 18, ...\}$, that is, multiples of 6.



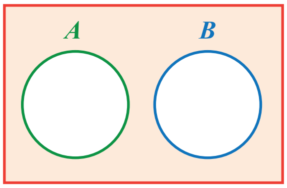
Using the Venn diagram, $A \cap B$ is represented by the shaded region with ***light blue***.

Figure 8.

Definition

Two sets are said to be disjoint if they have no common element.

Sets A and B are disjoint, if and only if $A \cap B = \varnothing$.



In the Venn diagram, the sets A and B are disjoint.

Here $A \cap B = \varnothing$

Figure 9.

Example 26

Let $A = \{1, 2, 3\}$ and $B = \{5, 6, 7, 8\}$. Then find $A \cap B$.

Solution

There is no element that is common for both A and B.

That is, $A \cap B = \varnothing$ and hence A and B are disjoint sets.

Example 27

Let $\mathbb{E}$ be the set of even natural numbers and $\mathbb{O}$ be the set of odd natural numbers. Find $\mathbb{E} \cap \mathbb{O}$.

Solution

There is no natural number that is both even and odd. That is, $\mathbb{E} \cap \mathbb{O} = \varnothing$ and hence $\mathbb{E}$ and $\mathbb{O}$ are disjoint sets.

Properties of intersection of sets

Activity 8

Let $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the universal set and let
$A = \{0, 2, 3, 5, 7\}$, $B = \{0, 2, 4, 6, 8\}$ and
$C = \{x \in \mathbb{N} \mid x \text{ is a factor of } 6\}$

1. Find
   (a) $A \cap B$ (b) $B \cap A$
   What is the relationship between $A \cap B$ and $B \cap A$?
2. Find
   (a) $A \cap B$ (b) $(A \cap B) \cap C$ (c) $B \cap C$ (d) $A \cap (B \cap C)$
   What is the relationship between $(A \cap B) \cap C$ and $A \cap (B \cap C)$?
3. Find $A \cap U$. What is the relationship between $A \cap U$ and A?

From your responses in Activity 8, observe the following properties of intersection. For any sets A, B and C and the universal set U.

1. $A \cap B = B \cap A$ (Intersection of sets is Commutative)
2. $(A \cap B) \cap C = A \cap (B \cap C)$ (Intersection of sets is Associative)
3. $A \cap U = A$. (U is the Identity for Intersection)
4. If $A \subseteq B$, then $A \cap B = A$.

Properties of intersection and union of sets

Activity 9

Let $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{1, 2, 3, a, b, c\}$ and $C = \{0, 2, 4, 6, 8\}$. Then determine each of the following sets.

(a) $A \cup (B \cap C)$ (c) $A \cap (B \cup C)$
(b) $(A \cup B) \cap (A \cup C)$ (d) $(A \cap B) \cup (A \cap C)$

From your responses in Activity 9, observe that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Distributive properties

For any sets three A, B and C

1. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. (Union is distributive over the intersection of sets).
2. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Intersection is distributive over the union of sets).

Exercises

1. Given $A = \{a, b, \{c\}\}$, $B = \{b, c\}$ and $C = \{\{c\}, d\}$, find:
   - (a) $A \cap B$
   - (b) $A \cap C$
   - (c) $B \cap C$
   - (d) $A \cap (B \cap C)$
2. State whether each of the following statements is true or false:
   - (a) If $x \in A$ and $x \notin B$, then $x \in (A \cap B)$.
   - (b) If $x \in (A \cap B)$, then $x \in A$ and $x \in B$.
   - (c) If $x \notin A$ and $x \in B$, then $x \in (A \cap B)$.
   - (d) For any set A, $A \cap A = A$.
   - (e) If $A \subseteq B$, then $A \cap B = A$.
   - (f) For any two sets A and B, $A \cap B \subseteq A$ and $A \cap B \subseteq B$.
   - (g) If $A \cap B = \varnothing$, then $A = \varnothing$ or $B = \varnothing$.
   - (h) If $(A \cup B) \subseteq A$, then $B \subseteq A$.
   - (i) If $A \subseteq B$, then $A \cap B = B$.
   - (j) If $A \subseteq B$, then $A \cap B = \varnothing$.
   - (k) If $A \subseteq B$, then $B' \subseteq A'$.
3. In each of the following Venn diagrams, shade each of the following sets.
   - (a) $(A \cup B) \cap C$.
   - (b) $(A \cap B) \cup C$.
   - (c) $(A \cup B) \cup C$.
   - (d) $(A \cap B) \cap C$.

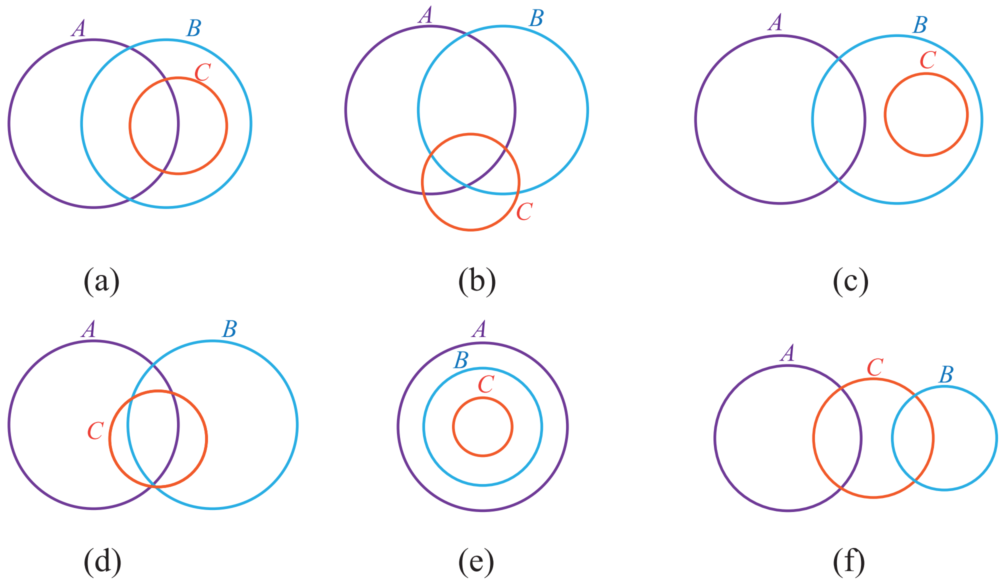


Figure 10.

C. Difference and symmetric difference of sets

(i) The relative complement (or difference) of two sets

Definition

The relative complement of a set B with respect to a set A (or the difference between A and B), denoted by $A - B$, read as "A difference B", is the set of all elements in A that are not in B.

That is, $A - B = \{x | x \in A \text{ and } x \notin B\}$.

Note: $A - B$ is sometimes denoted by $A \backslash B$. (read as "A less B")

$A - B$ and $A \backslash B$ are used interchangeably.

Using a Venn diagram, $A \backslash B$ can be represented by shading the region in A which is not part of B.

$A \backslash B$ is shaded in *light green*.

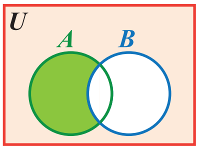


Figure 11.

Example 28

If $A = \{x, y, z, w\}$ and $B = \{a, b, x, y\}$, then find:

(a) the complement of B relative to A

(b) $B \backslash A$

(c) $B \backslash B$

Solution

(a) Note that finding "the complement of B relative to A" is the same as finding "the relative complement of B with respect to A". That is $A \backslash B$.

So, $A \backslash B = \{z, w\}$.

(b) $B \backslash A = \{a, b\}$.

(c) $B \backslash B = \varnothing$.

Activity 10

Let $A = \{0, 2, 3, 5, 7\}$, $B = \{0, 2, 4, 6, 8\}$ and $C = \{1, 2, 3, 6\}$. Find:

(a) $A \backslash B$ (b) $B \backslash A$ (c) $(A \backslash B) \backslash C$ (d) $A \backslash (B \backslash C)$

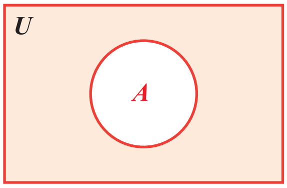
From the results of the Activity 10, observe that the relative complement of sets is neither commutative nor associative.

(ii) The complement of a set

Definition

Let A be a subset of a universal set U. The complement (or absolute complement) of A, denoted by A' or A^C, is defined to be the set of all elements of U that are not in A.

i.e., $A' = \{x | x \in U \text{ and } x \notin A\}$.



Using a Venn diagram, we can represent A' by the shaded region as shown in Figure 12.

Note that for any set A and universal set U,

$$A' = U \backslash A$$

Figure 12.

Example 29

In copies of the Venn diagram on the right, shade

(a) $A \backslash B$

(b) $(A \cap B)'$

(c) $A \cap B'$

(d) $A' \cup B'$

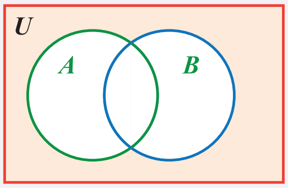


Figure 13.

Solution

(a) Since $A \backslash B$ is the set of all elements in A that are not in B, we shade the region in A that is not part of B (*shaded in green in* Figure 14).

$A - B$ is shaded

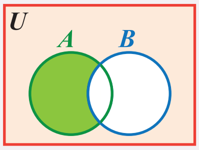
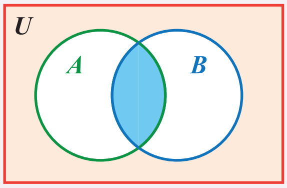


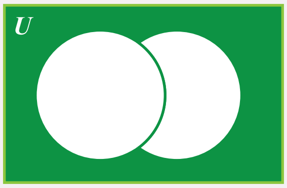
Figure 14.

(b) First we shade the region $A \cap B$; then $(A \cap B)'$ is the region outside $A \cap B$.

Figure 15. i.



ii.

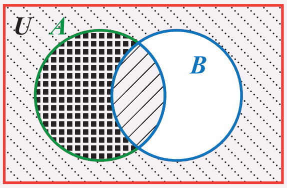


$A \cap B$ is the shaded (*blue*) region. $(A \cap B)$' is the *green* shaded region.

(c) First we shade A with strokes that slant upward to the right (////) and shade B' with strokes that slant downward to the right (\\\\).

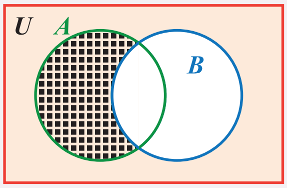
Then $A \cap B'$ is the cross-hatched region.

Figure 16. i.



A and B' are shaded

ii.



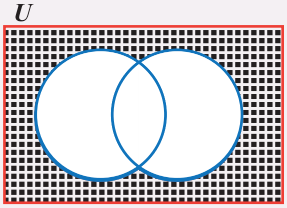
$A \cap B'$ is shaded

Note that the region of $A \backslash B$ is the same as the region of $A \cap B'$.

(d) First we shade A', the region outside A, with strokes that slant upward to the right (////) and then shade B' with strokes that slant downward to the right (\\\\).

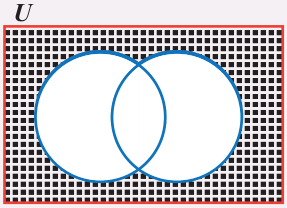
Then $A' \cup B'$ is the total shaded region.

Figure 17. i.



A' or B' are shaded

ii.



$A' \cup B'$ is shaded

Note that the region of $(A \cap B)'$ is the same as the region $A' \cup B'$.

Note: When we draw two overlapping circles within a universal set, four regions are formed. Every element of the universal set U is in exactly one of the following regions.

(i) in A and not in B ($A \backslash B$)

(ii) in B and not in A ($B \backslash A$)

(iii) in both A and B ($A \cap B$)

(iv) in neither A nor B ($(A \cup B)'$)

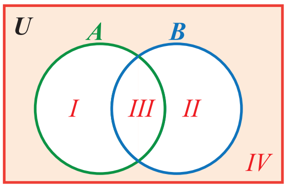


Figure 18.

For any two sets A and B, the following properties hold true:

1. $A \backslash B = A \cap B'$

2. $(A \backslash B) \cup B = A \cup B$

Activity 11

Let $U = \{1, 2, 3, \dots, 10\}$, $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8, 10\}$

(a) $(A \cup B)'$ (b) $A' \cap B'$ (c) $(A \cap B)'$ (d) $A' \cup B'$

Your responses in Activity 11 lead you to the following important laws called De Morgan's laws.

Theorem: De Morgan's law

For any two sets A and B

1. $(A \cap B)' = A' \cup B'$
2. $(A \cup B)' = A' \cap B'$

Exercises

1. Given $A = \{a, b, c\}$ and $B = \{b, c, d, e\}$ find:
   (a) The relative complement of A with respect to B.
   (b) The complement of B relative to A.
   (c) The complement of A relative to B.
2. In each of the Venn diagrams given below, shade $A \setminus B$.

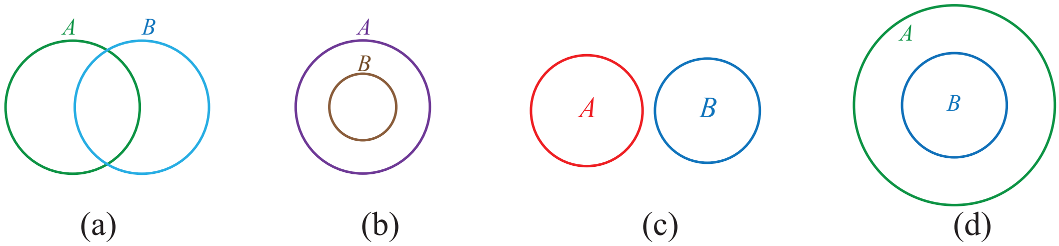


Figure 19.

3. Determine whether each of the following statements is true or false:
   (a) If $x \in A$ and $x \notin B$ then $x \in (B \setminus A)$
   (b) If $x \in (A \setminus B)$ then $x \in A$
   (c) $B \setminus A \subseteq B$, for any two sets A and B
   (d) $(A \setminus B) \cap (A \cap B) \cap (B \setminus A) = \varnothing$, for any two sets A and B
   (e) If $A \setminus B = \varnothing$ then $A = \varnothing$ and $B = \varnothing$
   (f) If $A \subseteq B$ then $A \setminus B = \varnothing$
   (g) If $A \cap B = \varnothing$ then $(A \setminus B) = A$
   (h) $(A \setminus B) \cup B = A \cup B$, for any two sets A and B
   (i) $A \cap A' = \varnothing$
4. Let $U = \{1, 2, 3, \ldots, 8, 9\}$ be the universal set and $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 4, 5, 6\}$. List the elements of each of the following:
   (a) A' (d) $(A \setminus B)'$ (g) $(A')'$
   (b) B' (e) $A' \cap B'$ (h) $B \setminus C$
   (c) $(A \cup C)'$ (f) $(A \cup B)'$ (i) $B \cap C'$

(iii) The symmetric difference between two sets

Activity 12

Let $A = \{a, b, d\}$ and $B = \{b, d, e\}$. Then find:

(a) $A \cap B$ (c) $A \setminus B$ (e) $(A \cup B) \setminus (A \cap B)$

(b) $A \cup B$ (d) $B \setminus A$ (f) $(A \setminus B) \cup (B \setminus A)$

Compare the results of (e) and (f).

The result of the above *A*ctivity leads us the following definition.

Definition

Let A and B be any two sets. The symmetric difference between A and B, denoted by $A \Delta B$, is the set of all elements in $A \cup B$ that are not in $A \cap B$.

That is $A \Delta B = \{x | x \in (A \cup B) \text{ and } x \notin (A \cap B)\}$

or $A \Delta B = (A \cup B) \setminus (A \cap B)$.

Using a Venn diagram, $A \Delta B$ is illustrated by shading the region in $A \cup B$ that is not part of $A \cap B$ as shown.

$A \Delta B$ is the shaded dark brown region.

From Activity 12 and the above Venn diagram, you can observe that

$$A \Delta B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A).$$

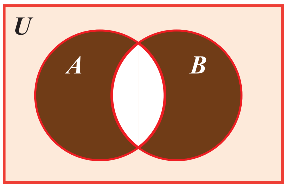


Figure 20.

Note: If $A \cap B = \varnothing$ then $A \Delta B = A \cup B$.

Example 30

Let $A = \{-1, 0, 1\}$ and $B = \{1, 2\}$. Find $A \Delta B$.

Solution

$A \cup B = \{-1, 0, 1, 2\}$; $A \cap B = \{1\}$.

Therefore, $A \Delta B = (A \cup B) \setminus (A \cap B) = \{-1, 0, 2\}$.

Example 31

Let $A = \{a, b, c\}$ and $B = \{d, e\}$. Find $A \Delta B$.

Solution

$A \cup B = \{a, b, c, d, e\}$; $A \cap B = \varnothing$.

Therefore, $A \Delta B = (A \cup B) \setminus \varnothing = \{a, b, c, d, e\} = A \cup B$.

EXERCISES

1. If $A \cap B = \{1, 0, -1\}$ and $A \cap C = \{0, -1, 2, 3\}$, then find $A \cap (B \cup C)$.
2. Simplify each of the following by using Venn diagram or any other property.
   (a) $A \cap (A \cup B)$ (c) $A \cap (A' \cup B)$
   (b) $P' \cap (P \cup Q)$ (d) $P \cup (P \cap Q)$

C. Cartesian Product of Sets

Ordered pair

An *ordered pair* is an element (x, y) formed by taking x from one set and y from another set. In (x, y), we say that x is the first element or coordinate and y is the second element or coordinate.

Such a pair is ordered in the sense that (x, y) and (y, x) are not equal unless $x = y$.

Equality of ordered pairs

Two ordered pairs (a, b) and (c, d) are said to be equal, written as $(a, b) = (c, d)$, if and only if $a = c$ and $b = d$.

EXAMPLE 32

The hourly temperature of a certain place is shown in the following table.

Time (in Hours)	9	10	11	12	13	14	15
Temp (Degree Celsius)	21	22	25	29	28	22	26

At x hours the temperature was y Degree Celsius.

That is, as a set of ordered pair (x, y), the information can be given as:

$\{(9, 21), (10, 22), (11, 25), (12, 29), (13, 28), (14, 22), (15, 26)\}$.

DEFINITION

Given two non-empty sets A and B, the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$ is called the Cartesian product of A and B, denoted by $A \times B$ (read "A cross B").

That is, $A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$.

Note that the sets A and B in the definition can be the same or different.

EXAMPLE 33

If $A = \{1, 2, 3\}$ and $B = \{4, 5\}$, then

$A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$.

Example 34

Let $A = \{a, b\}$, then form $A \times A$.

Solution

$A \times A = \{(a, a), (a, b), (b, a), (b, b)\}$.

Example 35

Let $A = \{-1, 0\}$ and $B = \{-1, 0, 1\}$.

Find $A \times B$ and illustrate it by means of a diagram.

Solution

$A \times B = \{(-1, -1), (-1, 0), (-1, 1),$
$(0, -1), (0, 0), (0, 1)\}$

The diagram is as shown in Figure 21.

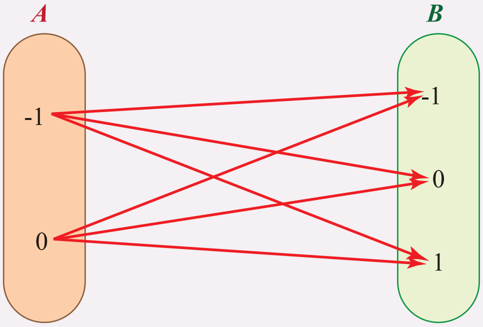


Figure 21.

Note: If A and B are finite sets, $n(A \times B) = n(A) \times n(B)$.

Activity 13

1. Let $A = \{2, 3\}$ and $B = \{0, 1, 2\}$. Find:
   (a) $A \times B$ (b) $B \times A$ (c) $n(A \times B)$
2. Let $A = \{a, b\}$ $B = \{c, d, e\}$ and $C = \{f, e, c\}$. Find:
   (a) $A \times (B \cap C)$ (c) $(A \times B) \cap (A \times C)$
   (b) $A \times (B \cup C)$ (d) $(A \times B) \cup (A \times C)$

From the result of the Activity, you can conclude that:

For any sets A, B and C

(i) $A \times B \neq B \times A$, for $A \neq B$ (*Cartesian product of sets is not commutative).*

(ii) $n(A \times B) = n(A) \times n(B) = n(B \times A)$.

(iii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$. (*Cartesian product is distributive over intersection).*

(iv) $A \times (B \cup C) = (A \times B) \cup (A \times C)$. (*Cartesian product is distributive over union).*

Exercises

1. Given $A = \{2\}$ $B = \{1, 5\}$ $C = \{-1, 1\}$ find:
   (a) $A \times B$
   (b) $B \times A$
   (c) $B \times C$
   (d) $A \times (B \cap C)$
   (e) $(A \cup C) \times B$
   (f) $(A \times B) \cup (A \times C)$
   (g) $B \times B$
2. If $B \times C = \{(1, 1), (1, 2), (1, 3), (4, 1), (4, 2), (4, 3)\}$, find:
   (a) B
   (b) C
   (c) $C \times B$
3. If $n(A \times B) = 18$ and $n(A) = 3$ then find $n(B)$.
4. Let $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the universal set and $A = \{0, 2, 4, 6, 8, 9\}$, $B = \{1, 3, 6, 8\}$ and $C = \{0, 2, 3, 4, 5\}$. Find:
   (a) $A' \times C'$
   (b) $B \times A'$
   (c) $B \times (A' \setminus C)$
5. If $(2x + 3, 7) = (7, 3y + 1)$, find the values of x and y.

For two sets A and B, the number of elements that are either in set A or set B, denoted by $n(A \cup B)$, may not necessarily be $n(A) + n(B)$ as it can be seen in the Figure 22.

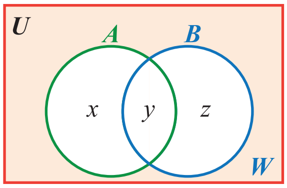


Figure 22.

In this figure, suppose the number of elements in the closed regions of the Venn diagram are denoted by x, y, z and w.

$n(A) = x + y$ and $n(B) = y + z$.

So, $n(A) + n(B) = x + y + y + z$.

$n(A \cup B) = x + y + z = n(A) + n(B) - y$

That is, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

Number of elements in $A \cup B$.

For any two finite sets A and B;

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

(ii) If $A \cap B = \varnothing$, then $n(A \cup B) = n(A) + n(B)$.

(iii) $n(A \setminus B) = n(A) - n(A \cap B)$

Example 36

Among 1500 students in a St. Theresa Convent school, 13 students failed in English examination, 12 students failed in Mathematics Examination and 7 students failed in both English and Mathematics Examinations.

(i) How many students failed in either English or in Mathematics examinations?

(ii) How many students passed both in English and in Mathematics examinations?

Solution

Let E be the set of students who failed in English, M be the set of students who failed in mathematics and U be the set of all students in the school.

Then, $n(E) = 13$, $n(M) = 12$, $n(E \cap M) = 7$ and $n(U) = 1500$.

(i) $n(E \cup M) = n(E) + n(M) - n(E \cap M) = 13 + 12 - 7 = 18$.

(ii) The set of all students who passed in both subjects is $U \setminus (E \cup M)$.

$n(U \setminus (E \cup M)) = n(U) - n(E \cup M) = 1500 - 18 = 1482$.

Exercises

1. For $A = \{2, 3, \ldots 6\}$ and $B = \{6, 7, \ldots 10\}$ show that:
   (a) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
   (b) $n(A \times B) = n(A) \times n(B)$
   (c) $n(A \times A) = n(A) \times n(A)$
2. If $n(C \cap D) = 8$ and $n(C \setminus D) = 6$ then find $n(C)$.
3. Using a Venn diagram, or a formula, answer each of the following:
   (a) Given $n(Q \setminus P) = 4$, $n(P \setminus Q) = 5$ and $n(P) = 7$ find $n(Q)$.
   (b) If $n(R' \cap S') + n(R' \cap S) = 3$, $n(R \cap S) = 4$ and $n(S' \cap R) = 7$, find $n(U)$.
4. Indicate whether the statements below are true or false for all finite sets A and B. If a statement is false give a counter example.
   (a) $n(A \cup B) = n(A) + n(B)$
   (b) $n(A \cap B) = n(A) - n(B)$
   (c) If $n(A) = n(B)$ then $A = B$
   (d) If $A = B$ then $n(A) = n(B)$
   (e) $n(A \times B) = n(A) \cdot n(B)$
   (f) $n(A) + n(B) = n(A \cup B) - n(A \cap B)$
   (g) $n(A' \cup B') = n((A \cup B)')$

(h) $n(A \cap B) = n(A \cup B) - n(A \cap B') - n(A' \cap B)$

(i) $n(A) + n(A') = n(U)$

5. Suppose A and B are sets such that $n(A) = 10$, $n(B) = 23$ and $n(A \cap B) = 4$, then find:
   (a) $n(A \cup B)$
   (b) $n(A \setminus B)$
   (c) $n(A \Delta B)$
   (d) $n(B \setminus A)$
6. If $A = \{x \mid x$ is a non-negative integer and $x^3 = x\}$, then how many proper subsets does A have?
7. Of 100 students, 65 are members of a mathematics club and 40 are members of a physics club. If 10 are members of neither club, then how many students are members of:
   (a) Both clubs?
   (b) Only the mathematics club?
   (c) Only the physics club?
8. The following Venn diagram shows two sets A and B. If $n(A) = 13$, $n(B) = 8$, then find:
   (a) $n(A \cup B)$
   (b) $n(U)$
   (c) $n(B \setminus A)$
   (d) $n(A \cap B')$

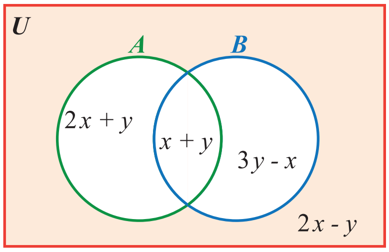


Figure 23.

Key Terms

- Cartesian product
- Complement
- Disjoint sets
- Element
- Empty set
- Finite set
- Infinite set
- Intersection of sets
- Power set
- Proper subset
- Relative complement
- Set
- Subset
- Symmetric difference
- Union of sets
- Universal set

Summary

- A set is a well-defined collection of objects. The objects of a set are called its elements (or members).
- Sets are described in the following ways:
  - (a) Verbal method
  - (b) Listing method
    - (i) Partial listing method
    - (ii) Complete listing method
  - (c) Set-builder method
- The universal set is a set that contains all elements under consideration in a discussion.
- The complement of a set A is the set of all elements that are found in the universal set but not in A.
- A set S is called finite if and only if it is the empty set or has exactly n elements, where n is a natural number. Otherwise, it is called infinite.
- A set A is a subset of B if and only if each element of A is in set B.
- Given a set A,
  - (i) $P(A)$, the power set of a set A, is the set of all subsets of A.
  - (ii) If $n(A) = n$, then the number of subsets of A is 2^n.
- Two sets A and B are said to be equal if and only if $A \subseteq B$ and $B \subseteq A$.
- Two sets A and B are said to be equivalent if and only if there is a one-to-one correspondence between their elements.
- Given two sets A and B:
  - (i) A set A is a proper subset of set B, denoted by $A \subset B$, if and only if $A \subseteq B$ and $B \not\subseteq A$.
  - (ii) A has n elements, then the number of subsets of A is 2^n.
  - (iii) If $n(A) = n$, then the number of proper subsets of A is $2^n - 1$.
- Operations on sets; for any sets A and B,
  - (i) $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.
  - (ii) $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.
  - (iii) $A - B$ (or $A \setminus B$) $= \{x \mid x \in A \text{ and } x \notin B\}$.

(iv) $A \Delta B = \{x \mid x \in (A \cup B) \text{ and } x \notin (A \cap B)\}$.

(v) $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$.

- Properties of union, intersection, symmetric difference and Cartesian product: For all sets A, B and C:

(i) Commutative properties

(a) $A \cup B = B \cup A$

(b) $A \cap B = B \cap A$

(c) $A \Delta B = B \Delta A$

(iv) Associative properties

(a) $A \cup (B \cup C) = (A \cup B) \cup C$

(b) $A \Delta (B \Delta C) = (A \Delta B) \Delta C$

(c) $A \cap (B \cap C) = (A \cap B) \cap C$

(iv) Identity properties

(a) $A \cup \varnothing = A$

(b) $A \cap U = A$ (U is a universal set)

(iii) Distributive properties

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(c) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(d) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(v) De Morgan's Law

(a) $(A \cup B)' = A' \cap B'$

(b) $(A \cap B)' = A' \cup B'$

(iii) For any set A

(a) $A \cup A' = U$

(b) $(A')' = A$

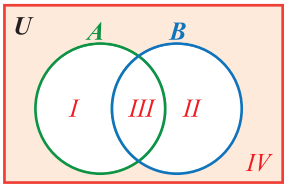
(c) $A \cap A' = \varnothing$

(d) $A \times \varnothing = \varnothing$

EXERCISES

1. Which of the following are sets?
   (a) The collection of all tall students in your class.
   (b) The collection of all natural numbers divisible by 3.
   (c) The collection of all students in your school.
   (d) The collection of all intelligent students in Grade 12.
   (e) The collection of all subsets of the set $\{1, 2, 3, 4, 5\}$.
2. Rewrite the following statements, using the correct notation:
   (a) B is a set whose elements are x, y, z and w.
   (b) 3 is not an element of set B.
   (c) D is the set of all rational numbers between $\sqrt{\ }$ and $\sqrt{5}$.
   (d) H is the set of all positive multiples of 3.
3. Which of the following pairs of sets are equivalent?
   (a) $\{1, 2, 3, 4, 5\}$ and $\{m, n, o, p, q\}$
   (b) $\{x \mid x$ is a letter in the word mathematics$\}$and $\{y \in \mathbb{N} \mid 1 \leq y \leq 11\}$
   (c) $\{a, b, c, d, e, f, \ldots m\}$ and $\{1, 2, 3, 4, 5, \ldots 13\}$
4. Which of the following represent equal sets?
   $A = \{a, b, c, d\}$ $\quad\quad$ $B = \{x, y, z, w\}$
   $C = \{x \mid x$ is one of the first four letters in the English alphabet$\}$
   $D = \varnothing$ $\quad$ $E = \{0\}$ $\quad$ $F = \{x \mid x \neq x\}$ $\quad$ $G = \{x \in \mathbb{Z} \mid -1 < x < 1\}$
5. If $U = \{a, b, c, d, e, f, g, h\}$, $A = \{b, d, f, h\}$ and $B = \{a, b, e, f, g, h\}$, find the following:
   (a) A'
   (b) B'
   (c) $A \cap B$
   (d) $(A \cap B)'$
   (e) $A' \cap B'$
6. In the Venn diagram given below, write the region labelled by I, II, III and IV in terms of A and B.

Figure 24.



7. For each of Questions (a), (b) and (c), copy the following Venn diagram and shade the regions that represent:
   (a) $A \cap (B \cap C)$.
   (b) $A \backslash (B \cap C)$.
   (c) $A \cup (B \backslash C)$.

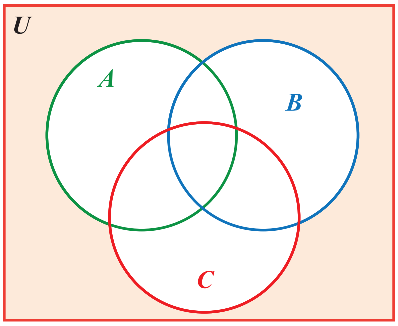


Figure 25.

8. Let $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2, 3, 4, 5\}$, $B = \{0, 2, 1, 6, 8\}$ and $C = \{3, 6, 9\}$. Then find:
   (a) A'
   (b) $B \backslash A$
   (c) $A \cap C'$
   (d) $C \times (A \cap B)$
   (e) $(B \backslash A) \times C$
9. Suppose B is a proper subset of C,
   (a) If $n(C) = 8$, what is the maximum number of elements in B?
   (b) What is the least possible number of elements in B?
10. If $n(U) = 16$, $n(A) = 7$ and $n(B) = 12$, find:
   (a) $n(A')$
   (b) $n(B')$
   (c) Greatest possible $n(A \cap B)$
   (d) Least possible $n(A \cup B)$
11. In a class of 31 students, 22 students study physics, 20 students study chemistry and 5 students study neither. Calculate the number of students who study both subjects.
12. Suppose A and B are sets such that $A \cup B$ has 20 elements, $A \cap B$ has 7 elements, and the number of elements in B is twice that of A. What is the number of elements in:
   (a) A?
   (b) B?
13. State whether each of the following is finite or infinite:
   (a) $\{x \mid x$ is an integer less than $5\}$
   (b) $\{x \mid x$ is a rational number between 0 and $1\}$
   (c) $\{x \mid x$ is the number of points on a 1 cm-long line segment$\}$
   (d) The set of all trees found in Africa.
   (e) The set of wheat seeds in 1,000 quintals.
   (f) The set of students in this class who are 10 years old.

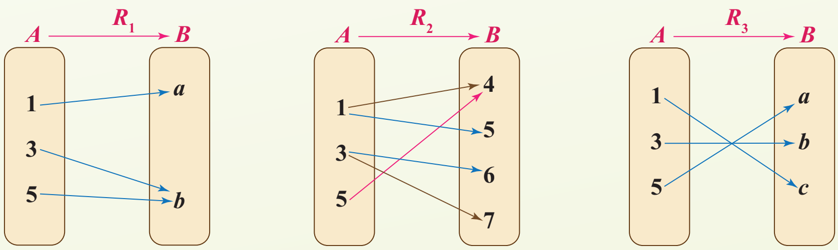
14. How many letters in the English alphabet precede the letter v? (Think of a shortcut method).
15. Of 100 staff members of a school, 48 drink coffee, 25 drink both tea and coffee and everyone drinks either coffee or tea. How many staff members drink tea?
16. Given that set A has 15 elements and set B has 12 elements, determine each of the following:
    (a) The maximum possible number of elements in $A \cup B$.
    (b) The minimum possible number of elements in $A \cup B$.
    (c) The maximum possible number of elements in $A \cap B$.
    (d) The minimum possible number of elements in $A \cap B$.

M12CH14

CHAPTER 14

RELATIONS AND FUNCTIONS

Chapter Contents



Chapter Outcomes

Upon completion of this Chapter, learners will:

- define and discuss relations and functions;
- numerical mappings;
- solve problems on relations, functions and mappings;
- calculate ratio and proportion;
- define variation and solve problems on variation.

Introduction

In our daily life, we come across many patterns that characterize relations such as brother and sister, mother and child, teacher and student, etc. Similarly, in mathematics also, we come across different relations such as number a is less than number b, set A is subset of set B, and so on. In all these cases, we find that a relation involves pairs of objects in some specific order. In this unit, you will learn how to link pairs of objects from two sets and then introduce relations between the two objects in the pair. You also learn here about special relations which are called functions.

In our daily life we usually talk about relations between various things. For example, we say someone is the father of another person, 5 is greater than 3, Paris is the capital city of France, etc.

Activity 1

1. Let $A = \{1, 2, 4, 6, 7\}$ and $B = \{5, 12, 7, 9, 8, 3\}$
   List all ordered pairs (x, y) which satisfy each of the following sentences where $x \in A$, $y \in B$.
   (a) x is greater than y
   (b) y is a multiple of x
   (c) The sum of x and y is odd
   (d) x is half of y
2. Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
   List all ordered pairs (x, y) which satisfy each of the following sentences, where $x, y \in A$.
   (a) y is a multiple of x
   (b) x is the square of y
   (c) x is less than y
   (d) x is a prime factor of y
3. Let U = $\{x$: x is a student in your class$\}$
   (i) In each of the following, list all ordered pairs (x, y) which satisfy the given sentence where $x, y \in U$.
      (a) x is taller than y
      (b) x is younger than y
   (ii) Discuss other ways that you can relate the students in your class.

As you have noticed from the above Activity, each sentence involves what is intuitively understood to be a relationship. Expressions of the type "is greater than", "is multiple of", "is a factor of", "is taller than", etc. which express the relation are referred to as relating phrases.

From Activity 1 you might have observed the following:

(i) In considering relations between objects, order is often important.

(ii) A relation establishes a pairing between objects.

Therefore, from a mathematical stand point, the meaning of a relation is more precisely defined as follows.

DEFINITION

Let A and B be non-empty sets. A relation R from A to B is any subset of $A \times B$.
In other words, R is a relation from A to B if and only if $R \subseteq (A \times B)$.

EXAMPLE 1

Let $A = \{ 1, 2, 3, 4\}$ and $B = \{ 1, 3, 5\}$

(i) $R_1 = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$ is a relation from A to B because $R_1 \subseteq (A \times B)$.

Notice that we can represent R_1 in the set builder method as

$R_1 = \{ (x, y) \mid x \in A, y \in B, x < y\}$

(ii) $R_2 = \{(1, 1), (2, 1), (3, 1), (3, 3), (4, 1), (4, 3)\}$ is a relation from A to B because $R_2 \subseteq (A \times B)$.

In the set builder method, R_2 is represented by $R_2 = \{ (x, y) \mid x \in A, y \in B, x \geq y\}$.

EXAMPLE 2

Let $A = \{1, 2, 3\}$ then observe that

$R_1 = \{(1, 2), (1, 3), (2, 3)\}$, $R_2 = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$
and $R_3 = \{(x, y) \mid x, y \in A, x + y \text{ is odd}\}$ are relations on A.

EXERCISES

1. For each of the following relations, determine the relating phrase:

   (a) $R = \{(x, y): x \text{ is taller than } y\}$

   (b) $R = \{(x, y): y \text{ is the square root of } x\}$

   (c) $R = \{(x, y): y = 2x\}$

2. Let $A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$

   (a) $R = \{(2, 2), (4, 4), (6, 6)\}$ is a relation on A. Express the relation using set builder method.

(b) Is $R = \{(2, 1), (2, 3), (2, 5), (1, 2), (3, 4), (5, 6)\}$ a relation from A to B?

(c) If R is a relation from A to B given by $R = \{(x, y): y = x - 1\}$, then list the elements of R.

3. If $R = \{(x, y): y = 2x + 1\}$ is a relation on A, where $A = \{1, 2, 3, 4, 5, 6\}$, then list the elements of R.

4. Write some ordered pairs that belong to the relation given by
$R = \{(x, y): y < 2x;\ x \in \mathbb{Z} \text{ and } y \in \mathbb{Z}\}$.

Domain and range of a relation

ACTIVITY 2

Let $A = \{1, 2, 4, 6, 7\}$ and $B = \{5, 12, 7, 9, 8, 3\}$

Let R_1 and R_2 be relations given by:

$$R_1 = \{(x, y) \mid x \in A, y \in B, x > y\} \text{ and } R_2 = \left\{(x, y) : x \in A,\ y \in B,\ x = \frac{1}{2}y\right\}$$

Represent each of the following sets using complete listing method.

(a) $D = \{x : (x, y) \in R_1\}$

(b) $D = \{x : (x, y) \in R_2\}$

(c) $R = \{y : (x, y) \in R_1\}$

(d) $R = \{y : (x, y) \in R_2\}$

From your responses in Activity 2, observe that, in each case, the set represented by D contains all the first coordinates of the given relation and the set represented by R contains the second coordinates of the give relation.

In the above discussion the set of all the first coordinates of the ordered pairs of a relation R is called the domain of R and the set of all second coordinates of the ordered pairs of R is called the range of R.

We give the definition of domain and range formally as follows.

DEFINITION

Let R be a relation from a set A to a set B. Then

1. the set of all the first coordinates of R is called the domain of R, denoted by Dom(R). That is, $\text{Dom}(R) = \{x : (x, y) \text{ belongs to } R \text{ for some } y\}$

2. the set of all the second coordinates of R is called the range of R, denoted by Ran(R). That is, $\text{Ran}(R) = \{y : (x, y) \text{ belongs to } R \text{ for some } x\}$

Example 3

Given the relation $R = \{(1, 3), (2, 5), (7, 1), (4, 3)\}$, find the domain and range of the relation R.

Solution

Since the domain contains the first coordinates, $\text{Dom}(R) = \{1, 2, 7, 4\}$ and the range contains the second coordinates, $\text{Ran}(R) = \{3, 5, 1\}$.

Example 4

Given $A = \{1, 2, 4, 6, 7\}$ and $B = \{5, 12, 7, 9, 8, 3\}$

Find the domain and range of the relation $R = \{(x, y): x \in A, y \in B, x > y\}$

Solution

If we describe R by complete listing method, we will find

$$R = \{(4, 3), (6, 3), (7, 3), (6, 5), (7, 5)\}.$$

This shows that the domain of $R = \{4, 6, 7\}$ and the range of $R = \{3, 5\}$.

Exercises

1. For the relation given by the set of ordered pairs $\{(5, 3), (-2, 4), (5, 2), (-2, 3)\}$ determine the domain and the range of the relation.
2. Let $A = \{1, 2, 3, 4\}$ and $R = \{(x, y): y = x + 1; x, y \in A\}$ List the ordered pairs that satisfy the relation and determine the domain and the range of R.
3. Find the domain and the range of each of the following relations:
   (a) $R = \left\{(x, y): y = \sqrt{x-1}\right\}$ (b) $R = \left\{(x, y): y = x^2 + 1\right\}$
   (c) $R = \{(x, y): y$ is a mathematics teacher in section $qx\}$
4. Let $A = \{x: 1 \leq x < 10\}$ and $B = \{2, 4, 6, 8\}$. If R is a relation from A to B given by $R = \{(x, y): x + y = 12\}$, then find the domain and the range of R.

In this section, you shall learn about particular types of relations which are called functions, the domain and range of a function, and combinations of functions.

Activity 3

Consider the following relations

$R_1 = \{(1, 2), (3, 4), (2, 5), (5, 6), (4, 7)\}$ $R_3 = \{(1, 2), (1, 4), (2, 5), (2, 6), (4, 7)\}$

$R_2 = \{(1, 2), (3, 2), (2, 5), (6, 5), (4, 7)\}$

(a) What differences do you see between these relations?

(b) How are the first elements of the coordinates paired with the second elements of the coordinates?

From your responses in Activity 3, observe that, there are relations in which every first coordinate is paired with exactly one second coordinate. Such relations are called functions.

Definition

A function is a relation such that no two ordered pairs have the same *first*-coordinates and different *second*-coordinates.

Example 5

Consider the relation $R = \{(1, 2), (7, 8), (4, 3), (7, 6)\}$.

Since 7 is paired with both 8 and 6, the relation R is not a function.

Example 6

Let $R = \{(1, 2), (7, 8), (4, 3)\}$. This relation is a function because no *first*-coordinate is paired (mapped) with more than one element of the *second*-coordinate.

Example 7

Consider the following arrow diagrams.

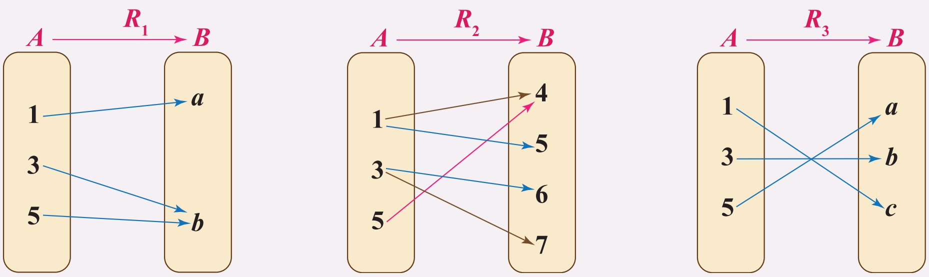


Figure 1.

Which of these relations are functions?

Solution

R_1 is a function, because no first-coordinate is paired (mapped) with more than one element of the second-coordinate.

R_2 is not a function because 1 and 3 are both mapped onto two numbers.

R_3 is a function, because no first-coordinate is paired (mapped) with more than one element of the second-coordinate

Example 8

The relation $R = \{(x, y): y \text{ is the father of } x\}$ is a function because no child has more than one father.

Example 9

Consider the relation $R = \{(x, y): x \text{ is the mother of } y\}$.

This relation is not a function, because a woman can have more than one child.

Domain and range of a function

In the previous section you learnt about the domain and range of a relation. As a function is a special type of a relation, the domain and range of a function are determined in exactly the same way.

Definition

Let R be a function. Then

1. The set of all the first coordinates of R is called the domain of R, denoted by Dom(R). That is, $\text{Dom}(R) = \{x : (x, y) \text{ belongs to } R \text{ for some } y\}$
2. The set of all the second coordinates of R is called the range of R, denoted by Ran(R). That is, $\text{Ran}(R) = \{y : (x, y) \text{ belongs to } R \text{ for some } x\}$

Example 10

For each of the following functions, determine the domain and the range.

(a) $R = \{(2, -1), (4, 3), (0, 1)\}$ (b) $S = \{(2, -1), (4, 3), (0, -1), (3, 4)\}$

Solution

(a) $\text{Dom}(R) = \{0, 2, 4\}$ and $\text{Ran}(R) = \{-1, 1, 3\}$

(b) $\text{Dom}(S) = \{0, 2, 3, 4\}$ and $\text{Ran}(S) = \{-1, 3, 4\}$

You will now consider some functions that are defined by a formula.

Example 11

Is the relation $R = \{(x, y): x \text{ and } y \text{ are real numbers and } x = y^2\}$ a function?

Solution

This is not a function because numbers for x are paired with more than one number in y. For example, $(9, -3)$ and $(9, 3)$ satisfy the relation with 9 being mapped to both -3 and 3.

Example 12

Is $R = \{(x, y) : x$ and y real numbers and $y = |x|\}$ a function?

Solution

Since for every number there is unique absolute value, each number x is mapped to one and only one number y, so the relation $R = \{(x, y) : y = |x|\}$ is a function.

Notation: If x is an element in the domain of a function f, then the element in the range that is associated with x is denoted by $f(x)$ and is called the image of x under the function f. This means $f = \{(x, y) : y = f(x)\}$

The notation $f(x)$ is called function notation. Read $f(x)$ as "f of x".

Note: f, g and h are the most common letters used to designate a function. But, any letter of the alphabet can be used.

A function from A to B can sometimes be denoted as $f: A \rightarrow B$, where the domain of f is A and the range of f is a subset of B, in which case we say B contains the images of the elements of A under the function f.

Example 13

Consider the function $R = \{(x, y)$: x and y are real numbers $y = |x|\}$. Here the rule $y = |x|$ can be written as $f(x) = |x|$. As a result of which, $f(0) = |0| = 0$, $f(-2) = |-2| = 2$ and $f(3) = |3| = 3$.

Example 14

If $R = \{(x, y)$: x and y are real numbers and y is twice $x\}$, then we can denote this function by $f(x) = 2x$.

Example 15

Consider $f(x) = 2x + 2$.

Since $f(x) = 2x + 2$ is defined for all $x \in \mathbb{R}$, the domain of the function is the set of all real numbers. The range is also $\mathbb{R}$ since every real number y has a real number x such that $y = f(x) = 2x + 2$.

Example 16

Let $f(x) = \sqrt{x+3}$

Since the expression in the radical must be non-negative, $x + 3 \geq 0$.

This implies $x \geq -3$. So the domain is the set $D = \{x\colon x \geq -3\}$.

Since the value of $\sqrt{x+3}$ is always non-negative, the range is the set

$R = \{y\colon y \geq 0\}$.

Example 17

Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 7, 9\}$

If $f: A \to B$ is the function given by $f(x) = 2x + 1$, then find the domain and the range of f.

Solution

Since $f(1) = 3 \in B$, $f(2) = 5 \in B$, $f(3) = 7 \in B$ and $f(4) = 9 \in B$, the domain of f is $D = \{1, 2, 3, 4\}$ and the range of f is $R = \{3, 5, 7, 9\}$.

Remark: If $f: A \to B$ is a function, then, for any $x \in A$ the image of x under f, $f(x)$ is called the functional value of f at x. For example, if $f(x) = x - 3$, then the functional value of f at $x = 5$ is $f(5) = 5 - 3 = 2$. Finding the functional value of f at x is also called evaluating the function at x.

Example 18

Take $f(x) = \sqrt{x+3}$ and evaluate:

(a) $f(3)$

(b) $f(6)$

Solution

(a) $f(-3) = \sqrt{-3+3} = \sqrt{0} = 0$

(b) $f(x) = \sqrt{6+3} = \sqrt{9} = 3$

Example 19

For the function $f(x) = 1 - x^2$

(a) Find the domain and the range

(b) Evaluate $f(2)$ and $f(-1)$

Solution

(a) The domain of the function is $D = \{x: x \in \mathbb{R}\}$, since it is defined for all real numbers.
The range is $R = \{y : y \leq 1\}$

(b) $f(2) = 1 - (2)^2 = 1 - 4 = -3$ and $f(-1) = 1 - (-1)^2 = 1 - 1 = 0$.

Exercises

1. Determine whether each of the following relations is a function or not, and give reasons for those that are not functions.

   (a) $R = \{(-1, 2), (1, 3), (-1, 3)\}$

   (b) $R = \{(1, 1), (1, 3), (-1, 3), (2, 1)\}$

   (c) $R = \{(x, y): y$ is the area of triangle $x\}$

(d) $R = \{(x, y): x$ is the area of triangle $y\}$

(e) $R = \{(x, y): y$ is a multiple of $x\}$

2. Find the domain and the range of each of the following functions:

(a) $f(x) = 3$

(b) $f(x) = 1 - 3x$

(c) $f(x) = \sqrt{x-4}$

(d) $f(x) = |x| - 1$

(e) $f(x) = \dfrac{1}{2-x}$

(f) $f(x) = \sqrt{2-x}$

(g) $f(x) = \sqrt{x-3}$

3. If $f(x) = 2x + \sqrt{x+5}$, evaluate each of the following:

(a) $f(-4)$

(b) $f(5)$

If functions are defined using numbers, we can add, subtract, multiply and divide functions with functions.

1. Sum of Functions

Let f and g be two functions whose values are real numbers. The sum of f and g, denoted by $f + g$, is a function which is defined $(f + g)(x) = f(x) + g(x)$ for each x in the domain of both f and g.

The domain of $f + g$ is the intersection of the domain of f and the domain g.

Example 20

If $f(x) = 2 - x$ and $g(x) = 3x + 2$ then the sum of these functions is given by

$(f + g)(x) = (2 - x) + (3x + 2) = 2x + 4$, which is also a function.

The domain of $f = \mathbb{R}$ and the domain of $g = \mathbb{R}$.

The function $(f + g)(x) = 2x + 4$ has also domain $= \mathbb{R}$.

Example 21

Let $f(x) = 2x$ and $g(x) = \sqrt{2x}$. Determine

(a) The sum $f + g$

(b) The domain of $(f + g)$

Solution

(a) $(f + g)(x) = f(x) + g(x) = 2x + \sqrt{2x}$.

(b) Domain of $f + g = \{x : x \geq 0\}$.

2. Difference of Functions

Let f and g be two functions whose values are real numbers. The difference of f and g, denoted by $f - g$ is a function which is defined $(f - g)(x) = f(x) - g(x)$.for each x in the domain of both f and g.

The domain of $f - g$ is the intersection of the domain of f and the domain g.

Example 22

If $f(x) = 3x + 2$ and $g(x) = x - 4$, then the difference of these functions is
$(f - g)(x) = f(x) - g(x) = (3x + 2) - (x - 4) = 2x + 6$ and the domain of $f - g = \mathbb{R}$.

Example 23

Let $f(x) = 2x$ and $g(x) = \sqrt{1-x}$. Determine:

(a) The difference $f - g$

(b) The domain of $f - g$

Solution

(a) $(f - g)(x) = f(x) - g(x) = 2x - \sqrt{1-x}$.

(b) Domain of $f - g = \{x : x \leq 1\}$.

3. Product of functions

Let f and g be two functions whose values are real numbers. The product of f and g, denoted by $f.\,g$ is a function which is defined $(f.\,g)(x) = f(x) . g(x)$, for each x in the domain of both f and g.

The domain of $f.\,g$ is the intersection of the domain of f and the domain g.

Example 24

If $f(x) = 2x$ and $g(x) = 3 - x$, then the product of these functions
$(fg)(x) = f(x)\,g(x) = (2x)\,(3 - x) = 6x - 2x^2$ and the domain of $fg = \mathbb{R}$.

Note: The domain of the sum, difference and product of functions f and g is the intersection of the domain of f and the domain of g.

4. Quotient of Functions

Let f and g be two functions whose values are real numbers and $g \neq 0$. The quotient of f and g, denoted by $\frac{f}{g}$, is the function defined $\left\{\frac{f}{g}\right\}(x) = \frac{f(x)}{g(x)}$.

The domain of $\frac{f}{g}$ is the intersection of the domain of f and the domain g without the numbers that make g zero.

That is $\text{Dom}\left(\frac{f}{g}\right) = (\text{Dom}(f) \cap \text{Dom}(g))\backslash\{x \in \text{Dom}(g): g(x) = 0\}$.

EXAMPLE 25

If $f(x) = 3$ and $g(x) = 2 + x$ then the quotient of these functions $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{3}{2+x}$ and the domain of $\left(\frac{f}{g}\right) = \mathbb{R}\backslash\{-2\}$.

EXAMPLE 26

Let $f(x) = \frac{x}{x-2}$ and $g(x) = \frac{x-3}{2x}$.

1. Find

(a) $f+g$ (b) $f-g$ (c) fg (d) $\frac{f}{g}$

2. Determine the domain of each of the functions in (1) above.

Solution

1.

(a) $(f+g)(x) = f(x) + g(x) = \frac{x}{x-2} + \frac{x-3}{2x} = \frac{3x^2-5x+6}{2x(x-2)}$

(b) $(f-g)(x) = f(x) - g(x) = \frac{x}{x-2} - \frac{x-3}{2x} = \frac{x^2+5x-6}{2x(x-2)}$

(c) $(fg)(x) = f(x)\,g(x) = \left(\frac{x}{x-2}\right)\left(\frac{x-3}{2x}\right) = \frac{x(x-3)}{2x(x-2)} = \frac{x-3}{2(x-2)}$

(d) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\frac{x}{x-2}}{\frac{x-3}{2x}} = \left(\frac{x}{x-2}\right)\left(\frac{2x}{x-3}\right) = \frac{2x^2}{x^2-5x+6}$

2. Domain of $f+g$ = Domain of $f-g$ = Domain of fg

$= \mathbb{R}\backslash\{0, 2\}$ or $(-\infty, 0) \cup (0, 2) \cup (2, \infty)$

Domain of $\left(\frac{f}{g}\right) = \mathbb{R}\backslash\{0, 2, 3\}$ or $(-\infty, 0) \cap (0, 2) \cap (2, 3) \cap (3, \infty)$.

EXAMPLE 27

Let $f(x) = 8 - 3x$ and $g(x) = -x - 5$. Determine:

(a) $2f + g$

(b) $3g - 2f$

(c) $(3f)g$

(d) $\dfrac{4g}{3f}$

Solution

(a) $2f(x) + g(x) = 2(8 - 3x) + (-x - 5) = 11 - 7x.$

(b) $3g(x) - 2f(x) = 3(-x - 5) - 2(8 - 3x) = -3x - 15 - 16 + 6x = 3x - 31.$

(c) $(3f(x))g(x) = 3(8 - 3x)(-x - 5) = 9x^2 + 21x - 120.$

(d) $\dfrac{4g(x)}{3f(x)} = \dfrac{4(-x-5)}{3(8-3x)} = \dfrac{-4x-20}{24-9x}.$

Through the above examples, you have seen how to determine the combination of functions. Now, you shall discuss how to evaluate functional values of combined functions for given values in the domains in the examples that follow.

EXAMPLE 28

Let $f(x) = 2 - 3x$ and $g(x) = x - 3$. Evaluate $\left(\dfrac{f}{g}\right)(4)$ and $(f + g)(4)$

Solution

$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} = \dfrac{2-3x}{x-3}$. So $\left(\dfrac{f}{g}\right)(4) = \dfrac{2-3(4)}{4-3} = -10.$

$(f + g)(x) = f(x) + g(x) = -2x - 1$. So $(f + g)(4) = -2(4) - 1 = -9.$

EXAMPLE 29

Let $f(x) = x - 1$ and $g(x) = 3x$. Determine:

(a) $(2f+3g)(1)$

(b) $\dfrac{f}{2g}(3)$

Solution

(a) $(2f + 3g)(1) = 2f(1) + 3g(1) = 2(1 - 1) + 3(3(1)) = 9.$

(b) $\left(\dfrac{f}{2g}\right)(3) = \dfrac{f(3)}{2g(3)} = \dfrac{3-1}{2(3)(3)} = \dfrac{2}{18} = \dfrac{1}{9}.$

EXERCISES

1. If $f = \{(1, 2), (-3, 2), (2, 5)\}$ and $g = \{(2, 4), (1, 5), (3, 2)\}$. Find:
   (a) $f + g$ and $f - g$
   (b) the domains of $(f + g)$ and $f - g$.
2. Let $f = \{(2, 3), (4, 9), (3, -8)\}$ and $g = \{(1, 2), (2, 5), (3, 10), (4, 17)\}$. Determine:
   (a) $-2f$ (b) fg (c) $(fg)(2)$ (d) g^2
3. Write down the domain of each function in question number 2.
4. Let $f(x) = \dfrac{2}{x-1}$ and $g(x) = \dfrac{2x-2}{3x+3}$. Find:
   (a) $f + g$ (b) fg
   (c) domain of $(f + g)$ and fg
5. Let $f(x) = 3x - 3$ and $g(x) = \dfrac{2}{x-1}$. Evaluate:
   (a) $(2fg)(2)$ (b) $\left(\dfrac{f}{g} - 2f\right)(3)$ (c) $(f - g)(4)$

Ratio

ACTIVITY 4

Suppose there are 4 blue balls, 5 white balls and 3 red balls. Write a simple comparison for each of the following.

(a) The number of blue balls to the number of red balls.
(b) The number of red balls to the number of white balls.
(c) The number of blue balls to the number of white balls.

In your responses in Activity 4, you have been comparing two quantities.

A ratio is a comparison of two quantities and ratio can be expressed in the form of a fraction.

For example, the ratio of 3 to 4 can be written in three ways:

Using the word to like 3 to 4
Using a colon like 3:4
Using a fraction like $\dfrac{3}{4}$

These are all read 3 to 4.

Example 30

In a basket, there are 15 mangos and 10 apples. Write the following rations using all the three different forms.

(a) Mangos to apples
(b) Apples to mangos
(c) Mangos to all the fruits
(d) Apples to all the fruits

Solution

	Word	Colon	Fraction
(a) Mangos to apples	15 to 10	15:10	$\frac{15}{10}$
(b) Apples to mangos	10 to 15	15:10	$\frac{10}{15}$
(c) Mangos to all the fruits	15 to 25	15:25	$\frac{15}{25}$
(d) Apples to all the fruits	10 to 25	10:25	$\frac{10}{25}$

Note

1. A ratio $\frac{a}{b}$ is in its simplest form if the greatest common factor of a and b is 1.
2. For a ratio $\frac{a}{b}$ and a nonzero number c, you have the following

$$\frac{a}{b} = \frac{a \times c}{b \times c} \text{ and } \frac{a}{b} = \frac{a \div c}{b \div c}$$

Example 31

Find three ratios equal to the ratio $\frac{15}{12}$.

Solution

$$\frac{15}{12} = \frac{15 \div 3}{12 \div 3} = \frac{5}{4} \left(\frac{5}{4} \text{is the ratio in the simplest form}\right)$$

$$\frac{15}{12} = \frac{15 \times 2}{12 \times 2} = \frac{30}{24}$$

$$\frac{15}{12} = \frac{10 \times 3}{12 \times 3} = \frac{45}{36}$$

Example 32

Arsenal won 13 games and lost 7. Find each ratio.

(a) Games won to games lost

(b) Games won to games played

(c) Games played to games lost

Solution

(a) The ratio of games won to games lost is 13:7.

(b) The number of games played is 3 + 7 = 20. Thus the ratio of games won to games played is 13:20.

(c) The ratio of games played to games lost is 20:7.

Example 33

Reduce the ratio 9:15 to its simplest form.

Solution

*B*oth values 9 and 15 are divisible by 3.

That is 9 ÷ 3 = 3 and 15 ÷ 3 = 5.

Thus, 9:15 = 3:5 and 3:5 is the simplest form of the given ratio.

Proportion

Activity 5

Which of the following pairs of halfrons are equivalent fraction?

(a) $\frac{1}{2}$ and $\frac{2}{3}$ (b) $\frac{2}{3}$ and $\frac{4}{6}$ (c) $\frac{5}{4}$ and $\frac{15}{12}$

From your responses to Activity 5, observe that two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equal, written as $\frac{a}{b}=\frac{c}{d}$ if and only if $a\times d=b\times c$. Comparison of two ratios is called a proportion.

A **proportion** is a statement that two ratios are equal or proportions are two equal ratios.

A proportion shows that the numbers in two different ratios compare to each other in the same way. The proportion $\frac{a}{b}=\frac{c}{d}$ is read as *a* is to *b* as *c* is to *d*. This proportion can also be written as $a:b=c:d$.

In proportion $a:b=c:d$. b and c are called the **means** or the terms in the middle of the proportion and a and d are called the **extremes** or the terms at the beginning and end of the proportion.

EXAMPLE 34

Determine if the statement $\frac{12}{18}=\frac{8}{12}$ is a proportion.

Solution

Determine if the cross products are equal. We have

The product of the means = $18\times 8=144$

Product of the extremes = $12\times 12=144$

Since the product of the means and the product of the extremes are equal, the statement is a proportion.

EXAMPLE 35

If 10 copies of a book cost 850 LRD, what is the cost of 30 books?

Solution

If 10 copies cost 850 LRD, then 1 copy of the book costs $\frac{850}{10}=85$ LRD and 30 copies of the book cost $85\text{ LRD}\times 30=2550$ LRD.

EXERCISES

1. A man drives 120 km in 2 hours and a woman drives 22.5 km in 30 minutes. What is the ratio of the man's speed to that of the woman speed?
2. The ratio of the age of a boy to the age of her father is 2:5. If the sum of their age is 70 years, how old is the father?
3. At a given meeting, there were 30 males and 20 females. Write each ratio as a fraction in simplest form:
   (a) Males to females
   (b) Female to all participants of the meeting
   (c) All participants of the meeting to males
4. Determine if the quantities in each pair of ratios are proportional.
   (a) 5 adults to 10 children and 6 adults to 12 children
   (b) 12 cm by 8 cm and 16 cm by 12 cm

5. Determine whether each pair of ratios forms a proportion or not.

   (a) $\frac{3}{2}$ and $\frac{6}{4}$ (b) $\frac{1}{30}$ and $\frac{40}{120}$ (c) $\frac{15}{40}$ and $\frac{5}{4}$

ACTIVITY 6

Consider the following expressions.

(a) $xy = 10$ (b) $y = \frac{2}{x}$ (c) $y = 2x$

For each of the above expressions,

1. What will happen to x if the values y increase?
2. What will happen to x if the values y decrease?
3. What will happen to y if the values x increase?
4. What will happen to y if the values x decrease?

From your responses in Activity 6, observe that for some of the expressions, as one of the variables increases, the other variable increases and vice versa. On the other hand, in some of the expressions, as one of the variables increases, the other decreases.

Given two variables related by an equation, if one changes the other also changes, the relation is called a variation.

There are two types of variations: direct variation and indirect variation.

Direct Variation

If the two variables change in the same sense; i.e. if one increases, so does the other, then it is called a direct variation.

The statements: "y varies directly as x", "y is directly proportional to x" and "y is proportional to x" expresses a direct variation and it is translated mathematically as $y = kx$, where k is the constant of variation.

EXAMPLE 36

If y varies directly as x and $y = 30$ when $x = 6$, find the variation constant and the equation of variation.

Solution

First we write the statement "*y varies directly as x*" by $y = kx$. Solve for k by substituting the given values in the equation.

That is, $k=\frac{y}{x}=\frac{30}{6}=5.$

Then the equation of variation is $y=5x$.

Example 37

If x varies directly as y and $x = 45$ when $y = 5$, what is the value of y when $x = 90$?

Solution

First we write the statement "*x varies directly as y*" by $y = kx$.

Then $k=\frac{y}{x}=\frac{5}{45}=\frac{1}{9}.$

This implies, the equation of variation is $y=\frac{1}{9}x$ and when $x=90$, $y=\frac{1}{9}\times 90=10.$

Inverse variation

A given variation is an **Inverse or Indirect Variation** if one going up causes the other to go down or vice versa.

Inverse variation occurs whenever a situation produces pairs of numbers whose product is constant. For two quantities x and y, an increase in x causes a decrease in y or vice versa. We can say that y varies inversely as x or $y=\frac{k}{x}$ if and only if $xy=k$.

Example 38

Find the equation and solve for k if y varies inversely as x and $y = 6$ when $x = 10$.

Solution

If y varies inversely as x, then $xy = k$.

Thus, $y = 6$ when $x = 10$ implies $k = 6 \times 10 = 60$.

Example 39

If y varies inversely as x and $y = 10$ when $x = 2$, find y when $x = 10$.

Solution

If y varies inversely as x, then $xy = k$.

Thus, $y = 10$ when $x = 2$ implies $k = 2 \times 10 = 20$.

This implies $y=\frac{k}{x}=\frac{20}{10}=2.$

Exercises

1. Write an equation for the following statements:
   (a) The cost C of fish varies directly as its weight w in kilograms.
   (b) An employee's salary S varies directly as the number of days d he has worked.
   (c) The area A of a square varies directly as the square of its side s.
   (d) The cost of electricity C varies directly as the number of kilowatt hour.
2. Find the constant of variation and write the equation representing the relationship between the quantities in each of the following.
   (a) y varies directly as x and $y = 12$ when $x = 5$.
   (b) y varies inversely as x and $y = 18$ when $x = 3$.
   (c) y varies directly as x and $y = 10$ when $x = 2$.
   (d) y varies inversely as x and $y = 12$ when $x = 2$.
3. Express each of the following statements as a mathematical equation.
   (a) The rate r at which a person types a certain manuscript varies direclty with hours h.
   (b) The length l of a rectangular field varies inversely as its width w.
   (c) The acceleration a of a car is inversely proportional to its mass m.

Key Terms

- Direct Variation
- Domain
- Function
- Indirect Variation
- Proportion
- Ratio
- Range
- Relation
- Variation

Summary

- In a relation, two things are related to each other by a relating phrase.
- Mathematically, a relation is a set of ordered pairs. If A and B are two non-empty sets, then the relation from A to B is a subset of $A \times B$ that satisfies the relating phrase.

- If A and B are any sets and $R \subseteq (A \times B)$, we call R a binary relation from A to B or a binary relation between A and B. A relation $R \subseteq (A \times A)$ is called a relation in or on A.
- The set $\{x: (x, y) \in R \text{ for some } y\}$ is called the domain of the relation R. The set $\{y: (x, y) \in R \text{ for some } x\}$ is called the range of the relation R.
- A function is a special type of a relation in which each x-coordinate is paired with exactly one unique y-coordinate.
- A function from A to B can sometimes be denoted as $f: A \rightarrow B$, where the domain of f is A and the range of f is a subset of B, in which case B contains the images of the elements of A by the function f.
- Let f and g be functions. We define the sum $f + g$, the difference $f - g$, the product fg, and the quotient $\frac{f}{g}$ as:

$f+g : (f+g)(x) = f(x) + g(x)$ $\qquad$ $fg : (fg)(x) = f(x)\, g(x)$

$f-g : (f-g)(x) = f(x) - g(x)$ $\qquad$ $\frac{f}{g} : \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)};\, g(x) \neq 0$

- Two or more quantities can be compared using ratios and proportions.
- Two ratios form a proportion if the product of the means and the extremes are equal.
- Two quantities are in a variation if one is a constant multiple of the other.
- Two quantities are in direct variation if one increases (respectively decreases) the other also increases (respectively decreases).
- Two quantities are invers variation if one increases (respectively decreases) the other decreases (respectively increases).

Exercises

1. For the relation $\{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$ find the domain and the range.
2. If the domain of the relation $R = \{(x, y): y = x + 3\}$ is $A = \{1, 2, 3, 4\}$ then list all the ordered pairs that are members of the relation and find the range.
3. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{a, b, c\}$
   (a) Find $A \times B$.
   (b) Determine relations as subsets of $A \times B$ such that:
   (i) $R_1 = \{(x, y): x \text{ is odd}\}$ $\qquad$ (ii) $R_2 = \{(x, y): 1 \le x \le 3\}$

4. Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 5\}$
   (a) If R is a relation from A to B then, is it true that R is also a relation from B to A? Explain your answer.
   (b) If $R \subseteq (A \times B)$ such that $R = \{(2, 4), (2, 2), (4, 4), (4, 2)\}$, then is R also a relation from B to A?
   (c) What can we conclude from B?
5. Let $R = \{(x, y): x \text{ is taller than } y\}$.
   (a) Does (x, x) belong to the relation? Explain.
   (b) Is it true that if (x, y) belongs to R, then (y, x) also belongs to R?
   (c) If (x, y) and (y, z) belong to R, then is it true that (x, z) belongs to R?
6. Let $R = \{(x, y) : y = x\}$. Show that each of the statements in Question 5 is true.
7. Find the domain and the range of each of the following relations:
   (a) $R = \{(x, y): y = 2x\}$
   (b) $R = \{(x, y): y = |x|\}$
   (c) $R = \{(x, y): x, y \in \{1, 2, 3, 4, 5\} \text{ and } y = 2x - 1\}$
   (d) $R = \{(x, y): \ y = \sqrt{x^2 - 4}\ \}$
8. Sketch the graph of each of the following relations and determine the domain and the range:
   (a) $R = \{(x, y): y \geq -2x+3\}$
   (b) $R = \{(x, y): y = 2x + 1\}$
   (c) $R = \{(x, y): y < -x + 3\}$
   (d) $R = \{(x, y): y \geq |x|\}$
   (e) $R = \{(x, y): y \leq x \text{ and } y \geq 1 - x\}$
   (f) $R = \{(x, y): y \leq |x| \text{ and } y \geq 0\}$
   (g) $R = \{(x, y): y = x+1 \text{ and } y = 1 - x\}$
   (h) $R = \{(x, y): y \leq x+1, y \geq 1 - x \text{ and } x \geq 0\}$
   (i) $R = \{(x, y): y > x - 2, y \geq -x - 2 \text{ and } y \leq 4\}$
9. Let $f(x) = 2x + 1$ and $g(x) = -3x - 4$
   (i) Determine:
      (a) $f + g$
      (b) $f - g$
      (c) fg
      (d) $\dfrac{f}{g}$
   (ii) Evaluate:
      (a) $(2f + 3g)(1)$
      (b) $(3fg)(3)$
      (c) $\dfrac{3f}{2g}(4)$

(iii) Find the domain of $\frac{f}{g}$:

10. Let $f(x) = \frac{x+4}{2x}$ and $g(x) = \frac{2x+4}{x+1}$.

(i) Determine:

(a) fg

(b) $\frac{g}{f}$

(c) $2f - \frac{f}{g}$

(ii) Find the domains of

(a) fg

(b) $\frac{g}{f}$

(c) $2f - \frac{f}{g}$

(iii) Evaluate

(a) $(f-g)(1)$

(b) $\frac{g}{f}(2)$

(c) $\left(2f - \frac{f}{g}\right)(3)$

11. A mobile phone technician uses the linear function $c(t) = 2t + 15$ to determine the cost of repair, where t is the time in hours and $c(t)$ the cost in LRD. How much will you pay if it takes him 3 hours to repair your mobile?

12. A real estate sells houses for 10,000 LRD flat price plus 100 LRD per one square metre.

(a) Find the function that represents the cost of the house that has an area of x m^2.

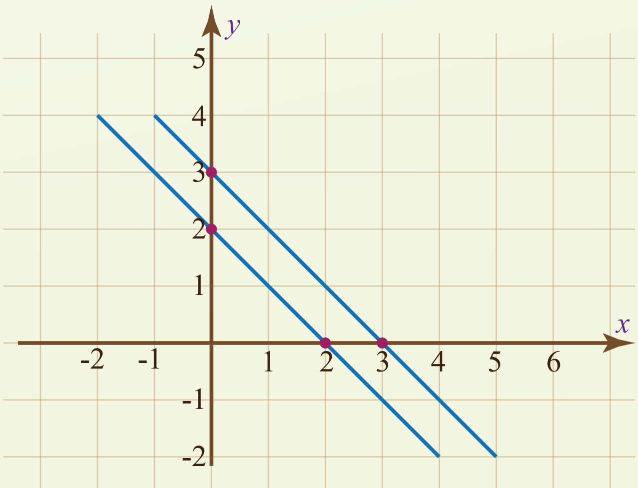
(b) Calculate the cost of the house that has an area of 80 m^2.

M12CH15

CHAPTER 15

ALGEBRAIC PROCESSES

Chapter Contents



Chapter Outcomes

Upon completion of this chapter, learners will:

- simplify algebraic expressions;
- factorize algebraic expressions;
- solve algebraic fractions;
- substitution;
- solve equations and inequalities;
- quadratic equations;
- solve Simultaneous linear equations;
- solve Simultaneous linear inequalities;
- formulate and Solve word problem.

Activity 1

1. Evaluate $3((154 - 26) \div 2^4) + 12 \times 3$. How many operations are involved in the above problem?
2. Evaluate $\dfrac{\left((x-y)^2 + 3x - \dfrac{20}{y}\right)}{4}$ for $\boldsymbol{x} = \mathbf{8}$, $\boldsymbol{y} = \mathbf{5}$. How many operations are involved here?

In mathematics in order to avoid ambiguity of calculation as to which operation should be performed first we have to use a standard rule in precedence of operations:

The order of operations is as follows:

1. Evaluate the parenthesis, if there are any, and if they require evaluation. If more than two parenthesis are there begin from the inner most one.
2. Evaluate the powers or the exponents.
3. Multiply or divide-it doesn't matter. Proceed from left to right.
4. Add or subtract. Proceed from left to right.

Example 1

Calculate $32 - 4[(3 + 6)^2 \div 3] + 2$

Solution

$$\begin{aligned} 32 - 4[(3 + 6)^2 \div 3] + 2 &= 32 - 4\,[(9)^2 \div 3] + 2 \\ &= 32 - 4[81 \div 3] + 2 \\ &= 32 - 4[27] + 2 \\ &= 32 - 108 + 2 \\ &= -76 + 2 = -74. \end{aligned}$$

Example 2

Evaluate the expression $(a + c)^2 - bd$ given: $a = 5$, $b = 0.25$, $c = 3$, $d = 8$

Solution

$(-5 + 3)^2 - (0.25)8 = (-2)^2 - 2 = 4 - 2 = 2.$

Rule 1

For any two variables (values) x and y, the following rules hold true.

1. Commutative property of addition. $x + y = y + x$
2. Commutative property of multiplication. $xy = yx$

The commutative properties mean you can add terms or values in any order; and also we can multiply terms or values in any order. This helps us to collect like terms to one side regardless of their order.

Use the following rules in the simplification of algebraic expressions

For any two variables x and y:

1. $x = 1x = 1x.$
2. $-x = -1x = -1x.$
3. $x - x = x + (-x) = 0$ for any number x.
4. $x + 0 = 0 + x = x$ for any number x.
5. $0x = x0 = 0$ for any number x.
6. $\frac{x}{x} = 1$ unless x is equal to 0. The expression is undefined when $x = 0$.
7. xy is meant $x \times y$ and the operation between x and y is multiplication.
8. when we multiply same variable repeatedly like $x \times x$ we write $x \times x = x^2$ rather than xx and $x \times x \times x = x^3$ rather than xxx.
9. $a(x + y)$ is meant $a \times (x + y)$ i.e. the operation between a and $x + y$ is multiplication.

Multiplication of a number by a monomial

When we multiply a monomial by a number, the number multiplies the numerical coefficient of the monomial.

Example 3

$-8 \times y = -8y$, $-1 \times y = -y$, $4 \times -y = -4y$, $4 \times (-3xy) = -12xy$

Activity 2

1. Compare the values of $xy + xz$ and $x(y + z)$
   (a) when $x = 5, y = 6, z = 10$ (b) when $x = -25, y = 16, z = 8$
2. Compare the values of $xy - xy$ and $x(y - z)$
   (a) when $x = 12, y = 8, z = 3$ (b) when $x = 11, y = -8, z = -4$

From your responses in Activity 2, you must have observed that the following properties of addition and multiplication.

Rule 2: For any three variables x, y and z the following rules hold.

1. Distributive property of multiplication over addition:
   $x(y + z) = xy + xz.$
2. Distributive property of multiplication over subtraction:
   $x(y - z)= xy - xz.$

The distributive property of multiplication over addition helps as to factor out common factors from the sum or difference of two algebraic expressions. This also helps us to sum up like terms.

For example,

$$4x + 5x = 4 \times x + 5 \times x = (4 + 5) \times x = 9 \times x = 9x.$$

$$3xy^2 + 6x^2y = 3xy \times y + 3xy \times x = 3xy(y + x).$$

Addition of like terms is based on distributive property of multiplication over addition.

Example 4

$x + 5x =1 \times x +5 \times x = (1 + 5) \times x = 6 \times x = 6x.$

Addition of numbers is just done as usual.

Example 5

$x - 4x = 1 \times x - 4 \times x = (1 - 4) \times x = -3 \times x = -3x$

In addition of like terms, we do not display to show this distributive properties.

Note that $x - 4x = x + -4x = 1 \times x + (-4) \times x$

$x - 4x = x + -4x = 1 \times x + (-4) \times x = (1 + (-4)\ x = -3x$ or

$x - 4x = 1 \times x - 4 \times x = (1 - 4) \times x = -3x.$

Activity 3

1. If x, y and z represent numbers, evaluate $(x+y)+z$ and $x+(y+z)$ for different values of the three variables and compare your results.
2. If x, y and z represent numbers, evaluate $(x\times y)\times z$ and $x\times(y\times z)$ for different values of the three variables and compare your results.

From your responses if Activity 3, you must have observed that the following properties of addition and multiplication.

Definition

For any three variables x, y and z, the following rules hold.

1. Associative property of addition

   $x + (y + z) = (x + y) + z.$

2. Associative property of multiplication

   $x(yz) = xyz.$

Terms can be grouped using associative property of addition or associative property of multiplication. Note that grouping is not rearranging. In grouping terms stay where they are but parenthesis is introduced to show which operation is to be performed first.

Example 6

Consider $x + 3x + 5x$

Use either of the groupings:

$(x + 3x) + 5x$ or $x + (3x + 5x)$

As a result, $x + 3x + 5x = (x + 3x) + 5x = 4x + 5x = 9x$ or

$x + 3x + 5x = x + (3x + 5x) = x + 8x = 9x.$

Example 7

Simplify $x^2 + 3x^2 + 5xy - 2xy.$

Solution

Collect like terms to one side so:

$x^2 + 3x^2 + 5xy - 2xy = (x^2 + 3x^2) + (5xy - 2xy) = 4x^2 + 3xy.$

Example 8

$3x \times 12x \times 3y = (3x \times 12x) \times 3y = 36x^2 \times 3y = (36 \times 3) \times (x^2 \times y) = 108 \times x^2y = 108x^2y.$

Exercises

1. By using Distributive property of multiplication over addition (subtraction), find the following sums.

   (a) $x + 5x$

   (b) $3x - x$

   (c) $-x - 3x$

   (d) $x - 10x$

   (e) $x + 5x + 7x$

2. Factor out the common factors of each of the following the algebraic expression
   (a) $xy + x^2y^2$
   (b) $4x + 8xy$
   (c) $3x^2y + 6xy$
   (d) $3xy + 6x^2y^3$
   (e) $5x - 10x^2$
3. Simplify the following algebraic expressions. Use Associative property of addition.
   (a) $x + (3x + y)$
   (b) $5y^2 + (y^2 - 2xy)$
   (c) $(x + y) - 3y$
   (d) $(2x + 3) + 12$
4. Simplify the following algebraic expressions
   (a) $3[4x - (2x - 5)]$
   (b) $pq^2 + 4p^2q^2 + 3pq^2$
   (c) $-[(5 - 2p) - (3p + 10)]$
   (d) $3a - 2c + 4 + 6c - 2a$
5. Simplify each of the following algebraic expression.
   (a) $m^2 - 2\{n - 4 - [5 - 3(m^2 - 2n)] + 7n\}$
   (b) $4r - \{(s - 2r) - [4s - (r - s)]\}$
   (c) $2x - [4 + 5x - 3(8 - 2x)]$

Multiplication of Monomial by Binomial

You remember that a monomial is an algebraic expression involving only one term and that a binomial is an algebraic expression with two terms.

Activity 4

1. What is the value of $10 \times (12 + 13)$? How many different ways of calculation can you use? What mathematical rule have you used in one of the methods?
2. Identify the monomials and binomials of each of the product and work out the product. Observe the differences in each problem
   (a) $213 \times (12 + 127)$
   (b) $15 \times (x + 1)$
   (c) $2 \times (x + y)$
   (d) $(3 + 14)\ 5x$
   (e) $5y \times (1 - 7x)$
   (f) $2(x - 1)$
   (g) $4y \times (2x + 3y)$
   (h) $2x \times (5y - 7x)$

Activity 5

1. Find all possible factors of:
   (a) x^2y
   (b) xy^2
   (c) List those factors common to x^2y and xy^2.
   (d) Of all the common factors of x^2y and xy^2 which you listed out in (c) identify the one that contains all other common factors of x^2y and xy^2 as its factor.
2. Repeat question (1) for x^2yz, x^2y^2z and x^2yz^2.

In your responses in Activity 5, you have been finding common factors for two expression and common factors are important in simplifying algebraic expressions. Given two terms in algebraic expression the common factors are those expressions which divide both of them.

Example 9

(a) $3uv$ and $6u$ have common factors 1, 3, u and $3u$.

(b) $2xy$ and $4xyz$ have common factors 1, 2, x, y, $2x$, $2y$, $2xy$ and xy.

The highest common factor is, as was the case with numbers, the largest factor that divides two expressions .So the highest common factor of $3uv$ and $6u$ is (from Example 9(a)) is $3u$; the highest common factor of $2xy$ and $4xyz$ (from example 9(b)) is $2xy$.

Definition

The common factor of two algebraic expression is any expression which is the factor of both algebraic expressions.

Example 10

Consider the algebraic expressions x^2y^3 and x^3y^2

The factors of $x^2y^3$are: 1, x, x^2, y, xy, x^2y, y^2, xy^2, xy^3, x^2y^2, y^3, x^2y^3.

The factors of x^3y^2 are : 1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, x^3y^2.

The common factors for the two algebraic expressions are: 1, x, x^2, y, y^2, xy, x^2y, xy^2, x^2y^2.

The highest common factor of two algebraic expressions is the algebraic expression that is the common factor and contains other common factors as its factor. The highest common factor is written in short as GCD.

Example 11

Show x^2y^2 is the GCD of x^2y^3 and x^3y^2.

Solution

As shown above, x^2y^2 is the common factor of x^2y^3 and x^3y^2. And all common factors of x^2y^3 and x^3y^2, namely

$1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, x^3y^2$ are factors of x^2y^2. Hence x^2y^2 is the GCD of x^2y^3 and x^3y^2.

Example 12

Find the GCD of $18a$ and $45b$.

Solution

$18a = 2 \times 3 \times 3 \times a$

$45b = 3 \times 3 \times 5 \times b$

Therefore, GCD($18a$ and $45b$) $= 3 \times 3 = 9$.

Shorter Method of Finding the GCD of Two or More Algebraic Expressions

(i) *Identify all the variables involved in the algebraic expressions.*

(ii) *For each variable in* (i), *find the highest power which is the factor of all the given algebraic expressions.*

(iii) *Find the GCD of the numerical coefficients of the given algebraic expressions.*

(iv) *Form the algebraic expression which is the product of the powers in (*ii) *and the GCD of the numerical coefficient in* (iii).

Example 13

Find the GCD $2xy^2$, $6x^2y^2$ and $4x^3yz$.

Solution

The variables involved in the three algebraic expressions are x, y and z. The highest power of x, which is common factor of $2xy^2$, $6x^2y^2$ and $4x^3yz$, is x.

The highest power of y, which is common factor of $2xy^2$, $6x^2y^2$ and $4x^3yz$, is y.

The highest power of z, which is common factor of $2xy^2$, $6x^2y^2$ and $4x^3yz$, is 1.
The numerical coefficients are 2, 6 and 4 and the GCD of 2, 4 and 6 is 2.
Therefore, the GCD of $2xy^2$, $6x^2y^2$ and $4x^3yz = x \times y \times 1 \times 2 = 2xy$

Example 14

Factorize each of the following expression.

(a) $x^2 + 4x$ (b) $3x^3y^2 - (x^2y)^3$

Solution

(a) $x^2 + 4x = x\,(x + 4)$

(b) $3x^3y^2 - 6x^2y^3 = (3x^2y^2)x - (3x^2y^2)\,2y = 3x^2y\,(x - 2y)$

Example 15

9xy and $15xz$ have greatest common divisions $3x$.

Example 16

$6a$ and $5b$ have greatest common division 1.

Exercises

Find the greatest common divisions of each of the following pairs of expressions.

(a) $6x$, $18y$
(b) $12mm$, $8m$
(c) $3uv$, $4uv$
(d) $18mp$, $9mn$
(e) $27xyz$, $45xz$

Factorizing Quadratic Trinomials

Example 17

$9x+24y=3(3x+8y)$.

Example 18

$9x^2 + 3x + 15x^3 = 3(3x^2 + x + 5x^3)$

But the terms in the brackets have x as a common factor:

$9x^2+3x+15x^3=3x(3x+1+5x^2)$

This is where have to stop our factorization process since the terms in the brackets have no further common factors.

Example 19

$2ab^2 + ab^2c + 3ab = ab(2b + bc + 3)$.

Example 20

$-2xy^2 - 4x^2y = -2xy(y + 2x)$.

Example 21

Simplify $5(x+2)+y(x+2)=(5+y)(x+2)$.

We note that $(x+2)$ is a s a common factor and factoring it out we have:

$5(x+2)+y(x+2)=(x+2)(5+y)$.

Example 22

$7(y+1)-x(y+1)=(y+1)(7-x)$.

Exercises

1. Factorize the following expression.
   (a) $7x+ 4$
   (b) $20x-4$
   (c) $18xy-3yz$
   (d) $12mn+18mp$
   (e) $16m^2-4m$
   (f) $3x^2+6x-18$
   (g) $-6x-24$
   (h) $-2xy-8x$
   (i) $24mn-16m^2n$
   (j) $-x^2y-y^2x$
   (k) $12m^2n+24m^2n^2$
   (l) $12m^2n+24m^2n^2$ 2
2. Factorize the following expressions:
   (a) $4(x+3)+m(x+3)$
   (b) $x(x-1)+5(x-1)$
   (c) $y(y+4)-6(y+4)$
   (d) $x^2(x+7)+x(x+7)$
   (e) $3x(x-4)-7(x-4)$

One use of factorization of algebraic expression is to simplify algebraic fractions. Using the same method as with ordinary fractions we can cancel out common factors in algebraic fractions to make a simpler equivalent fraction.

Example 23

$$\frac{x}{2x} = \frac{1 \times \cancel{x}}{2 \times \cancel{x}} = \frac{1}{2}.$$

By canceling the common x in the numerator and the denominator.

Example 24

$\frac{5x^2y}{15xy} = \frac{5xy \times x}{5xy \times 3} = \frac{x}{3}$ (By canceling the common factor $5xy$)

Example 25

$$\frac{4a + 2ab}{2a} = \frac{2a(2+b)}{2a} = 2 + b.$$

Noting that $2a$ is a factor in common for the two terms in the sum and then canceling.

Example 26

$$\frac{7x^2}{5y} \times \frac{15yz}{x} = \frac{x \times 7x}{5y} \times \frac{5y \times 3z}{x}$$

$$= \frac{\cancel{x} \times 7x \times \cancel{5} y \times 3z}{\cancel{5} y \times \cancel{x}} = 7x \times 3z = 21xz.$$

Note that, when either the numerator or the denominator is completely cancelled they become 1, not 0.

Example 27

$$\frac{x}{3} \div \frac{2x^2}{3} = \frac{x}{3} \times \frac{3}{2x^2} = \frac{1}{2x}.$$

Example 28

$$\frac{6x+18}{20} \div \frac{3x+9}{15} = \frac{6(x+3)}{20} \times \frac{15}{3(x+3)}$$

$$= \frac{6 \times 15}{20 \times 3} = \frac{3 \times 2 \times 5 \times 3}{5 \times 2 \times 2 \times 3} = \frac{3}{2}.$$

Exercises

1. Simplify each of the following expressions the following.

(a) $\frac{3x}{15}$

(b) $\frac{2x+10}{4}$

(c) $\frac{x^2-4x}{x+4}$

(d) $\frac{3x^2-9x}{2x-6}$

(e) $\frac{5x+20}{x+4}$

(f) $\frac{9x+27}{9x+18}$

(g) $\frac{6ab+2a}{2b}$

(h) $\frac{16m^2n-8mn}{12m-6}$

2. Simplify each of the following.

(a) $\frac{3x+9}{14}\times\frac{7x+21}{x+3}$

(b) $\frac{3mp+4p}{8p}\times\frac{12p^2}{3m+4}$

(c) $\frac{x^2-5x}{2x+10}\times\frac{3x+15}{4x}$

(d) $\frac{16}{2mp+4m}\times\frac{6m^2+8m}{12}$

(e) $\frac{24x-8}{12}\div\frac{9x-3}{6}$

(f) $\frac{x^2+2x}{5}\div\frac{2x+4}{20}$

(g) $\frac{p^2+pq}{7p}\div\frac{8p+8q}{21q}$

(h) $\frac{5xy-15y}{4x-12}\div\frac{6y^2}{x+y}$

An equation in one variable, say x, that can be written in the form $ax+b=0$, where a and b are specified numbers such that $a \neq 0$, is called **linear equation in one variable**.

Example 29

$3x-2=0$, $-3x+4=0$, $\frac{1}{2}x-9=0$ are examples of a linear equation.

The following basic **rules of equation are important in solving linear equations**.

Rule 1: *If equal values are added to both sides of equals, the results are equal.*

That is, if $a=b$, then $a+c=b+c$

Rule 2: If equals are subtracted from equals, the results are equal. i.e., if $a=b$, then $a-c=b-c$.

Rule 3: If equals are multiplied by equals, the results are equal. i.e., if $a=b$, then $ac=bc$.

Rule 4: If equals are divided by equals, the results are equal. i.e., if $a=b$, then $\frac{a}{c}=\frac{b}{c}$, $c\neq 0$

For example, for $4x=8$ we divide both sides of the equation by 4 to obtain $\frac{4x}{4}=\frac{8}{4}$ or $x=2$. (We divide by 4 to reverse the multiplication of x by 4).

Example 30

Find the solution set of the equation $3x-5=10$.

Solution

$3x-5+5=10+5$ (We add 5 to both sides of the given equation)

$3x=15$. (Next, divide both sides of this equation by 3 to separate x.)

$\frac{3x}{3}=\frac{15}{3}$ or $x=5$.

Therefore, the solution set (S.S) is $\{5\}$.

***Check*:** $(3\times5)-5=15-5=10$ (true).

Example 31

Find the solution set of the equation $\frac{x+3}{2}=4$.

Solution

$2\left(\frac{x+3}{2}\right)=2\times4$ (Multiply both sides of the given equation by 2). This gives us:

$x+3=8$ Next, subtract 3 from both sides of this equation

$x+3-3=8-3$ or $x=5$.

Therefore, S.S $=\{5\}$.

***Check*:** $\frac{5\ \ 3}{2}=\frac{8}{2}=$ (true).

Notice that the equations given in Example 30 and Example 31 above have the same solution set. Such equations are called equivalent as defined below.

Two equations are said to be **equivalent** if they have exactly the same solution set.

This fact and the four rules of equation permit us to transform some relatively complex equations to simpler equivalent linear equations. We illustrate this by the following examples.

EXAMPLE 32

Find the solution set of $7x-4=3(5-x)+1$.

Solution

In such case, apply basic rules of equation to bring the variables to the left side and all constant numbers to the right side of the equation as follows:

$7x-4=3(5-x)+1$ (The given equation)

$7x-4=15-3x+1$ (Removing parenthesis by distribution; i.e., $3(5-x)=15-3x$)

$7x-4=-3x+16$ (Combining like terms; i.e., $15+1=16$)

$7x-4+3x=16$ (Adding $3x$ to both sides to remove $-3x$ from the right side)

$7x+3x=16+4$ (Adding 4 to both sides to remove -4 from the left side)

$10x=20$ (Combining like terms)

$\frac{10x}{10}=\frac{20}{10}=2$. That is, $x=2$.

Therefore, the Solution Set = $\{2\}$.

The solution set of an equation can be the set of all real numbers if the equation is satisfied by every rational number; as in the following example.

EXAMPLE 33

Solve $3x-2(x-3)+1=x+7$

Solution

$3x-2(x-3)+1=x+7$ (The given equation)

$3x-2x+6+1=x+7$ (Removing parentheses by distribution; i.e., $-2(x-3)$ $=-2x+6$)

$x+7=x+7$ (Combining like terms; i.e., $3x - 2x = x$, and $6+1=7$)

$7 = 7$ (Subtracting x from both sides)

This is always true whatever the value of x is. This means the given equation is satisfied if you take any number for x as you wish.

Thus, Solution Set = $\mathbb{R}$, set of all real numbers.

There are also some equations which cannot be satisfied by any number. For example, $1 + x = x$ can never become true because there is no number which when 1 is added to it the result is the number itself. The solution set of such equation is **empty set,** written as {} or ∅. For such equation, the usual solution procedure leads to a false statement (false equality). For example, an attempt to solve $1 + x = x$ leads to the following:

$1 + x - x = x - x$ (Subtracting x from both sides of the equation)

$1 = 0$ which is false

Example 34

Find the solution set of $5 + 3(1 - x) = 2(1 - 5x) + 7x$.

Solution

$5 + 3(1 - x) = 2(1 - 5x) + 7x$ (The given equation)

$5 + 3 - 3x = 2 - 10x + 7x$ (Removing parentheses by distribution)

$8 - 3x = 2 - 3x$ (Combining like terms; i.e., $5 + 3 = 8$ and $-10x + 7x = -3x$)

$8 - 3x + 3x = 2 - 3x + 3x$ (Adding $3x$ to both sides)

$8 = 2$ which is false.

This means the solution set of the given equation is empty, ∅.

Exercises

1. Solve each of the following equations.

(a) $2x - 1 = 9$

(b) $3x = 12$

(c) $3x - 2 = 10$

(d) $2x - 8 = 0$

(e) $2x - 3 = 5$

(f) $x - 4 = 0$

(g) $-3x + 4 = 0$

(h) $6x + 8 = 0$

2. Which one of the following pairs of linear equations are NOT equivalent in the set of integers?
   (a)
   (b)
   (c) $\frac{1}{2}x=3$; $2x-10=2$
   (d) $2-5x=7$, $x+1=0$
3. What is the solution set of equation $3(2x+1)-2(3x-4)=11$ in the set of rational number $\mathbb{Q}$?
4. Which one of the following equations has NO solution in the set of integers $\mathbb{Z}$?
   (a) $3x+2=5$
   (b) $3-4x=1$
   (c) $4x+1=2x-3$
   (d) $\frac{1}{2}x-1=x$
5. If x is a rational number, what is the solution set of equation
   $5(x+3)-2(x+5)=14$?
6. What is the solution set of the equation $\frac{2}{3}x+\frac{5}{6}=\frac{1}{2}\left(x-\frac{1}{3}\right)$?
7. If x is a positive rational number, what is the solution set of equation $3(5x+2)=2(5x-2)$?
8. Musa is x years old. Her father is 4 times as old as Musa. Her mother is 7 years younger than her father. If their three ages are add up to 101 years, how old is Musa?
9. If 5 times a number is decreased by 24, the result is 3 times the number. What is the number?

Linear Inequalities

The form of a linear inequality is similar to a linear equation except that the equality symbol (=) is replaced by one of the inequality symbols ($<$, $\leq$, $>$, $\geq$).

DEFINITION

An inequality in one variable, say x, that can be written in the form

$ax+b<0$ or $ax+b\leq 0$ or $ax+b>0$ or $ax+b\geq 0$

where a and b are specified numbers such that $a \neq 0$, is called a **linear inequality**.

A solution set for an inequality is the set of all numbers that satisfies the inequality (makes the statement of the inequality true).

Notation: $\{x: x < c\}$ or $\{x \mid x < c\}$ read as " the set of x **such that** x is less than c. This notation represents the set (collection) of **all numbers** each of which is **less than** c.

$\{x: x \leq c\}$, $\{x: x > c\}$ and $\{x: x \geq c\}$ are interpreted similarly.

The solution set of an inequality may contain infinitely many solutions. For instance, the inequality $x < 3$ is satisfied by all real numbers that are less than 3. Hence, we may write the solution set of $x < 3$ as $\{x: x < 3\}$; that is, the set of all real number less than 3.

Two inequalities are said to be **equivalent** if they have exactly the same solution set. Linear inequalities can be solved in the manner almost similar to solving linear equations.

That is, we solve a given inequality by transforming it to a simpler equivalent linear inequality using the following **basic rules of inequality** (Each of the following rules holds if $<$ is replaced by any of $\leq, >$ or $\geq$.)

Rule 1: When equal values are added to (or subtracted from) both sides of a given inequality, the result is the same (equivalent) inequality. That is,

If $a<b$, then $a+c<b+c$;

also, $a-c<b-c$

For example, for $x - 3 < 0$ we may add 3 to both sides of the inequality to obtain $x-3+3<0+3$ or $x < 3$.

Rule 2: When both sides of a given inequality are multiplied (or divided) by the same **positive** number, the result is the same (equivalent) inequality; i.e., for a positive number c

If $a<b$, then $ac<bc$. Similarly, if $a<b$, then $\frac{a}{c}<\frac{b}{c}$.

(This rule applies also if we take any other inequalities instead of $<$)

For example, for $\frac{x}{2}\leq 5$ we multiply both sides of the inequality by 2 (which is positive) to obtain $2\left(\frac{x}{2}\right)\leq 2\times 5$ or $x \leq 10$.

The above rules are the same as the rules for equation. However, note that in **Rule 2** above the multiplier (or divisor) c is required to be **positive**. However, if we multiply (or divide) both sides of a given inequality by a **negative** number, the result is true only if we **reverse the inequality**; i.e., change $<$ to $>$, $>$ to $<$, $\leq$ to $\geq$ or $\geq$ to $\leq$. For example, if both sides of the following inequality is multiplied by -1, then the result is the inequality to its right side.

$2<5 \Rightarrow -2>-5$ (Read the symbol '$\Rightarrow$' as 'implies')

$-5<0 \Rightarrow -(-5)>-0$ or $5>0$

$4>2 \Rightarrow -4<-2$

$-2>-3 \Rightarrow -(-2)<-(-3)$ or $2<3$.

And so on. In general, we have the following rule.

Rule 3: Multiplication (or division) of both sides of a given inequality by the same **negative** number **reverses** the inequality. That is,

If $a<b$ and $c<0$, then $ac>bc$. Similarly, if $a<b$ and $c<0$, then $\frac{a}{c}>\frac{b}{c}$.

If $a>b$ and $c<0$, then $ac<bc$. Similarly, if $a>b$ and $c<0$, then $\frac{a}{c}<\frac{b}{c}$.

(Likewise, multiplication (or division) of both sides of $a\leq b$ or $a\geq b$ by a negative number reverses the inequality.)

Example 35

Find the solution set of the inequality $3x-5\geq 10$.

Solution

$3x-5+5\geq 10+5$ (Adding 5 to both sides of the given inequality).

$3x\geq 15$. (Next, divide both sides of this inequality by 3, which is positive).

$\frac{3x}{3}\geq\frac{15}{3}$ or $x\geq 5$.

Therefore, S.S $=\{x\colon x\geq 5\}$.

The solution of an inequality is sometimes required to be only in a given **domain** (set). In this case, you have to restrict the solution set of the inequality to the numbers that lie only within the given domain. For instance, if the question is to find the solution set of

$x<3$ in the set of **natural numbers,** then you have to look only for the natural numbers that are less than 3. These are 1 and 2 only. Hence, the set of solution of $x<3$ in the set of **natural numbers** is $\{1, 2\}$.

However, the solution set of $x < 3$ in the set of **whole numbers** is $\{0,1, 2\}$.

Recall the following sets:

- The set of natural numbers, $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$
- The set of whole numbers, $\mathbb{W} = \{0, 1, 2, 3, 4, \ldots\}$
- The set of integers, $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- The set of rational numbers, $\mathbb{Q} = \left\{\frac{a}{b} \mid a, b \in \mathbb{Z},\ b \neq 0\right\} = \left\{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{N}\right\}$

Example 36

Find the solution set of $x-4(x+1) \geq -13-(x-2)$ in the set of whole numbers, $\mathbb{W}$.

Solution

$x-4(x+1) \geq -13-(x-2)$ (The given inequality)

$x-4x-4 \geq -13-x+2$ (Removing parentheses by distribution)

$-3x-4 \geq -11-x$ (Combining like terms; i.e., $x-4x=-3x$ and $-13+2=-11$)

$-3x+x \geq -11+4$ (Adding x to both sides to take it to left; and add 4 to both sides)

$-2x \geq -7$ (Next, division of both sides of this by -2 reverses the inequality)

$\frac{-2x}{-2} \leq \frac{-7}{-2}$ or $x \leq \frac{7}{2}$; i.e., $x \leq 3.5$

Thus, the solution of the given inequality in $\mathbb{W}$ is every whole number less than or equal to 3.5. These are 0, 1, 2, and 3. Therefore, S.S= $\{0, 1, 2, 3\}$.

Some inequalities may have no solution in a specified domain as in the following example.

Example 37

Find the solution set of $7x - 1 \leq 3x + 2$ in the set of **natural numbers**, $\mathbb{N}$.

Solution

$7x - 1 \leq 3x + 2$ (The given inequality)

$7x - 3x \leq 2 + 1$ (Adding $3x$ to both sides to take it to left; and also add 1 to both sides)

$4x \leq 3$ (Next, divide both sides of this by 4 to separate x)

$\frac{4x}{4} \leq \frac{3}{4}$ or $x \leq \frac{3}{4}$.

So, the given inequality is satisfied only by numbers less than or equal to $\frac{3}{4}$. But there is no natural number which satisfies this inequality because **none** of the element of {1, 2, 3, … } is less than or equal to $\frac{3}{4}$. Therefore, the solution set of the given inequality in ℕ is ∅, empty.

Exercises

1. If x is a whole number, what is the solution set of the inequality $\frac{2}{7}(14x-21) \leq 6$?
2. What is the solution set of the inequality $\frac{2}{3}x - \frac{3}{4} \leq \frac{3}{2} - \frac{4}{3}x$ in the set of positive rational numbers?
3. What is the truth set of the inequality $3x \leq \frac{1}{2}(x+15)$ in the set of positive integers?
4. What is the solution set of the inequality $6 - \frac{5}{2}x > 1$ in the set of rational numbers?

An equation of the form $ax+by+c=0$, where a, b and c are constants and a and b are not both zero is called a linear equation in two variables x and y.

A solution to the linear equation $ax+by+c=0$ is an ordered pair (x_0, y_0) of numbers such that $ax_0 + by_0 + c = 0$.

Example 38

Consider the linear equation $2x-3y+10=0$. Then:

(–2, 2) is a solution of the linear equation, since 2 (–2) – 3 (2) + 10 = 0.

On the other hand, (1,1) is not a solution of the given equation, since $2(1)-3(1)+10=9\neq 0$.

DEFINITION

A set of two or more linear equations is called a system of linear equations. A system of two linear equations in two variables x and y is a set of equations that can be written in the form:

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

where a_1, a_2, b_1, b_2, c_1 and c_2 are constants.

A solution to a system of linear equations $\begin{cases} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{cases}$

is an ordered pair (x_0, y_0) of real numbers such that $\begin{cases} a_1x_0 + b_1y_0 = c_1 \\ a_2x_0 + b_2y_0 = c_2. \end{cases}$

EXAMPLE 39

Given the system of linear equations $\begin{cases} 2x+y=8 \\ x-5y=-7, \end{cases}$

which one of the following ordered pairs is a solution to the given system?

(a) (1, 6) (b) (13, 4) (c) (3, 2) (d) (0,4)

Solution

(a) $2(1)+6=2+6=8$ and $1-5(6)=1-30=-29\neq-7$ This implies (1, 6) is a solution of $2x+y=8$, but not a solution of $x-5y=-7$. Therefore, (1, 6) is not a solution of the given system.

(b) $2(13)+4=26+4=30\neq8$ and $13-5(4)=13-20=-7$ This implies, (13, 4) is not a solution of $2x+y=8$, but it is a solution of $x-5y=-7$.
Then (13, 4) is not a solution of the given system.

(c) $2(3)+2=6+2=8$ and $3-5(2)=3-10=-7$
Hence (3, 6) is a solution of the system.

There are different methods of solving systems of linear equations. Some of these methods are: Elimination, Graphical and Substitution. These methods are discussed below.

1. Elimination Method

Steps in Solving System of Linear Equations by Elimination Method

Given a system of two linear equations in two variables:

Step 1: Make the coefficients of one of the two variables equal but opposite in sign in the two equations. This can be done by multiplying with an appropriate number.

Step 2: Add the two equations to eliminate one of the two variables and solve for the remaining variable.

Step 3: Substitute the value in **Step 2** in one of the two equations and solve for the remaining variable.

To illustrate these steps let us consider the following examples.

Example 40

Solve each of the following system of linear equations by Elimination Method.

(a) $\begin{cases} 2x-3y=-6 \\ x+y=4 \end{cases}$ (b) $\begin{cases} x-5y=4 \\ -3x+15y=1 \end{cases}$ (c) $\begin{cases} 5x+y=1 \\ 10x+2y=2 \end{cases}$

Solution

(a) Multiply the second equation by –2and add the two equations

$$+\begin{cases} 2x-5y=-6 \\ -2x-2y=-8 \end{cases}$$
$$\overline{\quad 0-7y=-14 \quad}$$

This implies $y=2$.

Then substitute 2 in the first equation: $2x-5(2)=-6$.

This implies $2x-10=-6 \Rightarrow 2x=-6+10=4 \Rightarrow x=\frac{4}{2}=2.$

Therefore, the solution set of the system is $\{(2, 2)\}$.

(b) Multiply the first equation by 3 and add the two equations.

$$+\begin{cases} 3x-15y=12 \\ -3x+15y=1 \end{cases}$$
$$\overline{\quad 0=13 \quad}$$

But this is always **False.** Hence the system has no solution or the system has empty solution.

(c) Multiply the first equation by -2 and add the two equations.

$$+\begin{cases} -10x-2y=-2 \\ 10x+2y=2 \end{cases}$$
$$0=0$$

This is always **True.** Therefore, the set of all ordered pairs (*x, y*) of numbers such that $y=1-5x$ are solutions, that is, $\{(x,y) \mid y=1-5x \text{ and } x\in\mathbb{R}\}$.

Hence the system has infinitely many solutions.

Note

Given a system of linear equations: $\begin{cases} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{cases}$,

where $a_2b_2c_2\neq 0$ (i.e. $a_2\neq 0,\ b_2\neq 0,\ c_2\neq 0$), we have the following points about solutions of the system of equations.

1. If $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$, then the given system has only one solution.
2. If $\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$, then the given system has no solution
3. If $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$, then the system has infinitely many solutions. That is, an ordered pair that satisfies one of the two equations also satisfies the other.

Example 41

(a) Consider the system $\begin{cases} 2x-3y=-6 \\ x+y=4 \end{cases}$. Since $\frac{2}{1}\neq\frac{-3}{1}$, the system has only one solution.

(b) Considere the system $\begin{cases} x-5y=4 \\ -3x+15y=1 \end{cases}$. We have $\frac{1}{-3}=\frac{-5}{15}\neq\frac{4}{1}$. Then the system has no solution or empty solution.

(c) Consider the system $\begin{cases} 5x=y=1 \\ 10x=2y=20 \end{cases}$ $\frac{5}{10}=\frac{1}{2}=\frac{1}{2}$. Hence the system has infinitely many solutions.

2. Graphical Method

Steps in Solving system of Linear Equations by Graphical Method

Given a system of linear equations:

$$\begin{cases} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2, \end{cases}$$

The graph of the two linear equations $a_1x+b_1y=c_1$ and $a_2x+b_2y=c_2$ are lines.

Step 1: First draw the graph of the two lines given by the two equations on the same coordinate system.

Step 2: There are only three possibilities for the two lines.

(a) The two lines intersect at a point. In this case, the point of intersection of the two lines is a solution to the given system.

(b) The two lines do not intersect (that is, the two lines are parallel). In this case, the system has no solution.

(c) The two lines coincide one onto another. In this case, a solution to one of the two equations is a solution to the other. In this case the system has infinitely many solutions.

Example 42

Solve each of the following system of linear equations by graphical method.

(a) $\begin{cases} x+y=2 \\ 2x+2y=6 \end{cases}$

(b) $\begin{cases} 2x-y=0 \\ x-2y=0 \end{cases}$

(c) $\begin{cases} x-3y=1 \\ 2x-6y=2 \end{cases}$

Two draw a line; take two different points on the line and draw the line that passes through these two points. Two points can be obtained by taking arbitrary values for one of the two variables, mostly for x, and solve for the other.

Solution

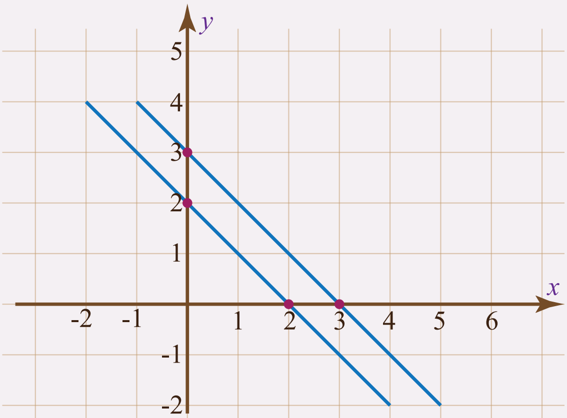
(a) First draw the graphs of the two lines in the same coordinate plane as shown below.

For the equation $x+y=2$.

x	0	2
y	2	0

For the equation $2x+2y=6$.

x	0	3
y	3	0



From the graph we can see that the two lines are parallel, that is, they do not intersect. Hence the system has no solution.

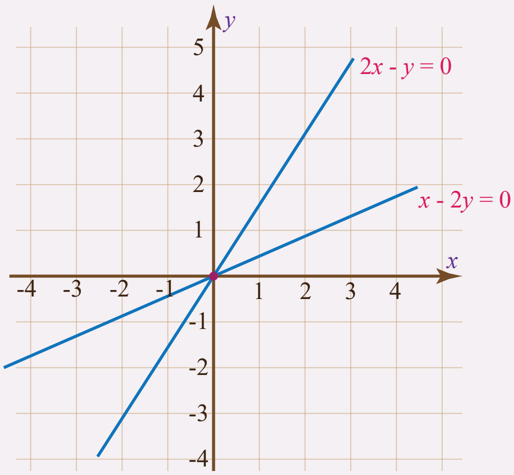
(b) Draw the graphs of the two lines in the same coordinate plane as shown below.

For the equation $2x-y=0$.

x	0	1
y	0	2

For the equation $x-2y=0$.

x	0	2
y	0	1



From the graph we can see that the two lines intersect only at (0,0). Hence, the system has only one solution(0,0).

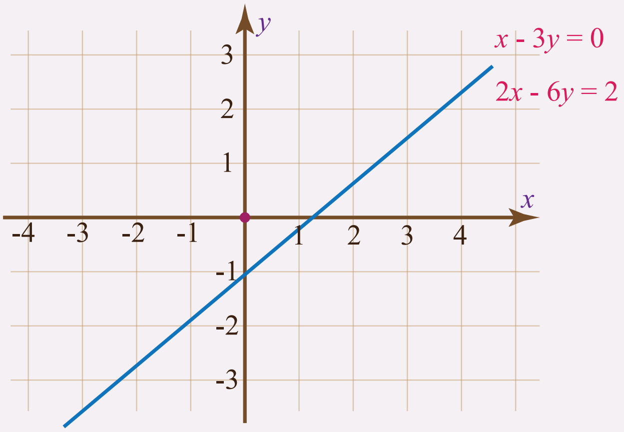
(c) Again draw the graph of each of the two lines in the same coordinate plane as shown below.

For the equation $x-3y=1$.

x	0	1
y	$-\frac{1}{3}$	0

For the equation $2x-6y=2$.

x	0	1
y	$-\frac{1}{3}$	0



From the graph we can see that the two lines coincide. Hence, the system has infinitely many solutions. That is, a solution of one is also a solution to the other.

3. Substitution Method

Steps in Solving Systems of Linear Equations by Substitution Method

Given a system $\begin{cases} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{cases}$

Step 1: For one of the two equations, write one of the two variables in terms of the other.

Step 2: Substitute your result in **Step 1** in the other equation and solve for the second variable.

Step 3: Substitute the value in **Step 2** in one of the two equations and solve for the remaining variable.

These values will give you the solutions of the given equation.

Example 43

Solve each of the following system of linear equations by substitution method.

(a) $\begin{cases} 3x-4y=7 \\ x+2y=9 \end{cases}$ (b) $\begin{cases} x+y= \\ 2x+2y=7 \end{cases}$ (c) $\begin{cases} x-2y=4 \\ -2x+4y=-8 \end{cases}$

Solution

(a) Consider the second equation $x+2y=9$ Solving for x in terms of y will give us: $x=-2y+9$ and substitute this in the first equation, $3(-2y+9)+2y=7$ and solve for $y.-6y+27+2y=7 \Rightarrow -4y=7-27=-20 \Rightarrow y=\frac{-20}{-4}=5.$

Then substitute $y=5$ in the second equation and solve for

$$x+2(5)=9 \Rightarrow x=9-10=-1.$$

Therefore, $(5,-1)$ is the solution of the given system.

(b) From the first equation solve for x in terms of y $x=5-y$ and substitute this in the second equation. $2(5-y)+2y=7 \Rightarrow 10-2y+2y=7 \Rightarrow 10=7.$

This is always False. Hence the system has no solution or the solution set of the given system is empty set.

(c) From the first equation solve for x in terms of y $x = 4 + 2y$ and substitute this in the second equation. $-2(4+2y)+4y=-8 \Rightarrow -8-4y+4y=-8 \Rightarrow -8=-8.$

This is always True. Hence the system has infinitely many solutions.

Exercises

Determine the solution set (if any) of each of the following systems of linear equations.

(a) $\begin{cases} -2x+y=4 \\ 2x+y=4 \end{cases}$

(b) $\begin{cases} \frac{1}{2}x-2y=5 \\ x+4y=7 \end{cases}$

(c) $\begin{cases} 3x-0.5y=6 \\ -2x+y=4+2y \end{cases}$

(d) $\begin{cases} \frac{2}{3}x+2y=5 \\ 2x+6y=2 \end{cases}$

(e) $\begin{cases} \frac{3}{2}x-2y=1 \\ 3x-4y=2 \end{cases}$

In this section you will learn three different techniques in solving quadratic equations. There are Factorization, Completing the Square Method and Using the Quadratic Formula.

DEFINITION

An equation that can be form reduced to the form:

$ax^2 + bx + c = 0,$

where *a, b* and *c* are constants and $a \neq 0$ is called quadratic equation.

Note

A given expression is a perfect square, if it is the square of a given expression.

For two real numbers :

(i) $(a+b)^2 = a^2 + 2ab + b^2 =$ (Perfect Square)

(ii) $(a-b)(a+b) = a^2 - b^2$ (Difference of Two Squares).

(iii) $a^3 + b^3 = (a+b)(a^2 - ab + b^3)$ (Sum of two cubes)

(iv) $a^3 - b^3 = (a-b)(a^2 + ab + b^3)$ (Difference of two cubes)

EXAMPLE 44

(a) $x^2 - 16 = (x-4)(x+4)$ (Difference of Two Squares).

(b) $x^2 + 6x + 9 = (x+3)^2$ (Perfect Square).

(c) $x^3 - 27 = (x-3)(x^2 + 3x + 9)$ (Difference of two cubes)

(d) $x^3 + 27 = (x+3)(x^2 - 3x + 9)$ (Sum of two cubes).

Steps in Solving Quadratic Equations Using Factorization Method

Step1: First write the given equation in the form $ax^2 + bx + c = 0$.

Step2: Factorize $ax^2 + bx + c = 0$ as $(dx+c)+(fx+g) = ax^2 + bx + c$.

Step3: Then solve $(dx+e)(fx+g) = 0$.

That is: $dx + e = 0$ or $fx + g = 0$.

In solving quadratic equations using factorization method the following are very important.

1. For any two real numbers :
2. $ab = 0$ implies $a = 0$ or $b = 0$. (This is called zero product rule.)
3. To factorize the expression $ax^2 + bx + c$, find two numbers p and q such that $q + p = b$ and $pq = ac$.

EXAMPLE 45

Solve the quadratic equation $x^2 + 7x + 12 = 0$ by Factorization Method.

Solution

First factorize $x^2 + 7x + 12$.

Find such that $p + q = 7$ and $pq = 12$ .Then $p = 3$ and $q = 4$ and

$$\begin{aligned} x^2 + 7x + 12 &= x^2 + (3 + 4)x + 12 \\ &= x^2 + 3x + 4x + 12 \\ &= x(x + 3) + 4(x + 3) \\ &= (x + 3)(x + 4) \end{aligned}$$

Now let us solve $(x + 3)(x + 4) = 0$. Then $x + 3 = 0$ or $x + 4 = 0$.

This implies $x = -3$ or $x = -4$ and hence the solution set is $\{-3, -4\}$.

For the quadratic equation $ax^2 + bx + c = 0$ sometimes it is difficult to find two numbers such that $q + p = b$ and $pq = ac$. Because of these we use another method of solving quadratic equations.

Steps in Solving a Quadratic Equations by Completing the Square Method

Step1: First reduce the given quadratic equation to the form

$ax^2 + bx + c = 0 \ (a \neq 0)$.

Step2: Multiply both sides of the given equation by $\frac{1}{a}$ so that the equation becomes:

$$x^2+\frac{b}{a}x+\frac{c}{a}=0.$$

Step3: Add $\frac{-c}{a}$ on both sides of the equation

$$x^2+\frac{b}{a}x=\frac{-c}{a}.$$

Step4: Add $\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}$ on both sides of $x^2+\frac{b}{a}x=-\frac{c}{a}$.

That is : $x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2}$ and $x^2+\frac{b}{a}x+\frac{b^2}{4a^2}$ is a perfect square.

This implies $\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$.

Then $x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}=\pm\frac{\sqrt{b^2-4ac}}{2a}$ and hence $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Example 46

Solve the quadratic equation: $2x^2+6x+2=0$ by Completing the Square Method

Solution

Dividing both sides of the equation by 2 will give us $x^2+3x+1=0$.

Then adding -1on both sides give us $x^2+3x=-1$.

Add $\left(\frac{3}{2}\right)^2$ on both sides so that

$$x^2+3x+\frac{9}{4}=\frac{9}{4}-1=\frac{5}{4}\Rightarrow\left(x+\frac{3}{2}\right)^2=\frac{5}{4}$$

$$\Rightarrow x=\frac{-3}{2}\pm\sqrt{\frac{5}{4}}=\frac{-3}{2}\pm\frac{\sqrt{5}}{2}.$$

$$\Rightarrow x=\frac{-3}{2}\pm\frac{\sqrt{5}}{2}$$

Therefore, the solution set is $\left\{\frac{-3}{2}+\frac{\sqrt{5}}{2},\frac{-3}{2}-\frac{\sqrt{5}}{2}\right\}$.

Solving Quadratic Equations Using the Quadratic Formula

Given a quadratic equation, first reduce it in the form $ax^2 + bx + c = 0$. Then in the previous case, we have seen that:

$$x_1 = \frac{-b-\sqrt{b^2-4ac}}{2a} \text{ and } x_2 = \frac{-b+\sqrt{b^2-4ac}}{2a}$$

are the two roots (solutions) of the given equation.

For the quadratic equation $ax^2+bx+c=0$, the number $D=b^2-4ac$ is called discriminant of the given equation.

1. If $D = 0$, then the equation has only one solution, $x=\frac{-b}{2a}$.
2. If $D > 0$, the equation has two solutions, $x_1 = \frac{-b-\sqrt{b^2-4ac}}{2a}$ and $x_2 = \frac{-b+\sqrt{b^2-4ac}}{2a}$
3. If $D < 0$, the equation has no real solution. That is, the equation will not have a solution in the set of real numbers.

Example 47

Solve each of the following quadratic equations using the quadratic formula.

(a) $x^2+4x+4=0$ (b) $4x^2+5x+3=0$ (c) $x \ \ -5x+6=0$

Solution

For the give equation: $a=1, b=4, c=4$. This implies: $\mathrm{D}=b^2-4ac=4^2-4\times4=0$.

Then the equation has one solution, $x=\frac{-4}{2}=-2$.

We have: $a=4, b=-5, c=3$ Then:

$$D=b^2-4ac=(-5)^2-4(4)\,(3)=25-48=-23<0.$$

Then the system has no solution.

Since $a=1, b=-5, c=6$ we have:

$$D=(-5)^2-4(1)(6)=25-24=1>0.$$

Then the given quadratic equation has two solutions: $x_1=\frac{5-1}{2}=2$ and $x_2=\frac{5+1}{2}=3$.

Note:

If $r_1 = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $r_2 = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$. are roots (or solutions) of the quadratic equation $ax^2+bx+c=0$ then :

1. $r_1 + r_2 = \dfrac{-b}{a}$.
2. $r_1 r_2 = \dfrac{c}{a}$.

EXAMPLE 48

Give the quadratic equation $x^2+2x-4=0$ find:

(a) The sum of the roots,

(b) The product of the roots.

Solution

$a=1, b=2$ and $c=-2$. If r_1 and r_2 are the roots of the given equation, then:

(a) $r_1 + r_2 = \dfrac{-b}{a} = \dfrac{-2}{1} = -2$ and

(b) $r_1 r_2 = \dfrac{c}{a} = \dfrac{-4}{1} = -4$.

EXERCISES

1. Find the solution of each of the following equations, if any.
   (f) $x^2 + 7x - 10 = 0$
   (g) $x^2 + 4x + 1 = 0$
   (h) $2x^2 - 4x + 3 = 0$
   (i) $4x^2 + 2x + 4 = 0$
   (j) $-x^2 - 4x = 0$
   (k) $-6 - x^2 - 4x = 0$
   (l) $2x^2 + 4x + 1 = 0$
   (m) $(x+8)(x-3) = 3x$
2. If $x^2 + px - 8 = 0$ has roots r and s such that $r = -2s$, then determine roots.
3. If the solutions of the equation $2x^2 + (k-3)x - 7 = 0$ are $\dfrac{7}{2}$ and -1, then what is the value of k?

Key Terms

- Algebraic expression
- Binomials
- Formula
- Highest common factor
- Linear Equation
- Monomial
- Quadratic Equation
- System of linear equations
- Term
- Variables

Summary

- Algebraic Expression are any combination of numbers and variables.
- Term is a part of an algebraic expression (along with it sign) which is linked or joined to other part of the algebraic expression by addition (+).
- An algebraic expression may contain one term is called monomial and that which contain two terms is called a Binomial.
- When more than one operation is involved we perform with the order: bracket, then power (exponents) and then multiplication or division and then addition or subtraction. When there is more than one bracket, we begin from the inner most one.
- The common factor of two algebraic expressions is any expression which is the factor of both algebraic expressions and the Highest Common Factor of two algebraic expressions is the algebraic expression that is the common factor and contains other common factors as its factor. The Highest common factor is written in short as GCD.
- We use GCD of two or more algebraic expression to factorize algebraic expression and simplify the algebraic expressions given as quotient (numerator over denominator) form.
- An equation in one variable, say x, that can be written in the form $ax+b=0$, where a and b are specified numbers such that $a \neq 0$, is called **linear equation**.
- An inequality in one variable, say x, that can be written in the form

  $ax+b<0$ or $ax+b\leq 0$ or $ax+b>0$ or $ax+b\geq 0$.
- where a and b are specified numbers such that $a\neq 0$, is called **linear inequality**.
  - A set of two or more linear equations is called a system of linear equations.

- A system of two linear equations in two variables x and y is a set of equations that can be written in the form:

  $a_1x+b_1y=c_1$

  $a_2x+b_2y=c_2$

  where a_1, a_2, b_1, b_2, c_1, and c_2, are constants.

- A solution to a system of linear equations $\begin{cases} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{cases}$ is an ordered pair (x_0, y_0) of real numbers such that $\begin{cases} a_1x_0+b_1y_0=c_1 \\ a_2x_0+b_2y_0=c_2 \end{cases}$

- Given a system of linear equations: $\begin{cases} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{cases}$,

  where $a_2b_2c_2 \neq 0$ (i.e. $a_2 \neq 0,\ b_2 \neq 0,\ c_2 \neq 0$), we have the following points about solutions of the system equations.

  1. If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then the given system has only one solution.
  2. If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, then the given system has no solution
  3. If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then the system has infinitely many solutions. That is, an ordered pair that satisfies one of the two equations also satisfies the other.

- An equation that can be form reduced to the form: $ax^2+bx+c=0$,
- where a, b and c are constants and $a \neq 0$ is called quadratic equation.
- For the quadratic equation $ax^2+bx+c=0$, the number $D=b^2-4ac$ is called discriminant of the given equation.

  1. If $D = 0$, then the equation has only one solution, $x=\frac{-b}{2a}$.
  2. If $D > 0$, the equation has two solutions, $x_1=\frac{-b-\sqrt{b^2-4ac}}{2a}$ and $x_2=\frac{-b+\sqrt{b^2-4ac}}{2a}$

  If $D < 0$, the equation has no real solution. That is, the equation will not have a solution in the set of real numbers.

EXERCISES

1. Determine the solution set (if any) of each of the following systems of equations.

   (a) $\begin{cases} x+y=6 \\ 2x-5y=0 \end{cases}$

   (b) $\begin{cases} x-2y=5 \\ -3x+6y=1 \end{cases}$

   (c) $\begin{cases} 3x+y=-1 \\ -x-\frac{1}{3}y=\frac{1}{3} \end{cases}$

   (d) $\begin{cases} x+3y=2 \\ 2x-4y=1 \end{cases}$

2. Suppose the quadratic equation $ax^2+x-c=0$ has two real roots r_1 and r_2. If $r_1+r_2=r_1\times r_2=-\frac{1}{2}$, then what is the value of C?
3. If one of the roots of the equation $x^2-x+k=0$ exceeds the other root by 5, then what is the value of k?
4. What is the product of roots of $5x^2-7x-6=0$?
5. If $-1 < a < 1$, then what is the number of solutions of the quadratic equation $ax^2 + 2x + a = 0$.

WHAT IS CYBERCRIME?

Cybercrime is criminal activity that either targets or uses a computer, a computer network or a networked device. Most cybercrime is committed by cybercriminals or hackers who want to make money or take advantage of a person.

Types of Cybercrime

- Email and internet fraud.
- Identity fraud (where personal information is stolen and used).
- Theft of financial or card payment data.
- Theft and sale of corporate data.
- Cyber extortion (demanding money to prevent a threatened attack).
- Ransomware attacks (a type of cyberextortion).
- Cryptojacking (where hackers mine cryptocurrency using resources they do not own).
- Cyberespionage (where hackers access government or company data).
- Interfering with systems in a way that compromises a network.
- Infringing copyright.
- Illegal gambling.
- Selling illegal items online.
- Soliciting, producing, or possessing child pornography.

How to Prevent Cyber Crimes?

- Enforce concrete security and keep it up-to-date.
- Never give out personal information to a stranger.
- Check security settings to prevent cybercrime.
- Using an antivirus software helps to recognize any threat or malware before it infects the computer system.
- When visiting unauthorized websites, keep your information secure.
- Restriction on access to your most valuable data.
- Backup all data, system, and considerations.
- Don't use free USB sticks.

Source: Teacher's Diary on *Cyber-Crime Awareness* by UNODC, Cybercrime and MoE, Republic of Liberia

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TEXTBOOK MANAGEMENT
Student Textbook Assignment Form

School : ____

District : ________________________Country: ____________________

Teacher : __________________________Subject: ________________

Grade : ____

No.	Name of the Student	Name of the Textbook	Barcode Number	Condition of the TB when received	Date Received	Date Returned	Condition of TB when returned	Signature of Student
1								
2								
3								
4								
5								
6								
7								
8								
9								
10								
11								
12								
13								
14								
15								